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Automorphism groups of tetravalent Cayley graphs on minimal non-abelian groups

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Abstract. A Cayley graph X=Cay(G, S) is called *normal* for G if the right regular representation $R(G)$ of G is normal in the full automorphism group $\mathrm{Aut}(X)$ of X. In the present paper it is proved that all connected tetravalent Cayley graphs on a minimal non-abelian group G are normal when $(|G|, 2) = (|G|, 3) = 1$, and X is not isomorphic to either Cay(G, S), where $|G| = 5^n$, and $|\mathrm{Aut(X)}|=2^m.3.5^n$, where $m \in \{2, 3\}$ and $n \geq 3$, or Cay(G, S) where $|G| = 5q^n$ (q is prime) and $|\mathrm{Aut}(X)| = 2^m.3.5.q^n$, where $q \geq 7$, $m \in \{2, 3\}$ and $n \geq 1$.

1. Introduction

Throughout this paper, graphs are finite and simple. For a graph X, we denote by $\mathrm{V}(X)$, $\mathrm{E}(X)$ and $\mathrm{Aut}(X)$ the vertex set, the edge set and the automorphism group of X, respectively, and X is said to be *vertex-transitive* if $\mathrm{Aut}(X)$ is transitive on vertices.

A finite group G is called *minimal non-abelian* if G is not abelian, but every proper subgroup of G is abelian. For a finite group G and a subset S of G such that $1_G \notin S$, and S is symmetric, that is $S^{-1} = \{s^{-1} | s \in S\}$ is equal to S. The *Cayley graph* X=Cay(G, S) on G with respect to S is defined as the graph with vertex set G and edge set $\{[g, sg] | g \in G, s \in S\}$. Clearly the right regular representation $R(G)$ of G acts regularly on

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vertices, that is, $R(G)$ is transitive on vertices and only the identity element of $R(G)$ fixes a vertex. Furthermore, the group $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) | S^\alpha = S\}$ is also a subgroup of $\mathrm{Aut}(X)$. Actually, $\mathrm{Aut}(G, S)$ is a subgroup of $\mathrm{Aut}(X)_1$, the stabilizer of the vertex 1 in $\mathrm{Aut}(X)$. For two groups M and N, $N \rtimes M$ denotes a semidirect product of N by M.

A Cayley graph $\mathrm{Cay}(G, S)$ is said to be *normal* on G if the right regular representation $R(G)$ of G is normal in $\mathrm{Aut}(X)$. Let $N_{Aut(X)}(R(G))$ be the normalizer of $R(G)$ in $\mathrm{Aut}(X)$. By Godsil [8], or Xu [14], $N_{Aut(X)}(R(G))=R(G)\rtimes \mathrm{Aut}(G, S)$. Thus, $\mathrm{Cay}(G, S)$ is normal on G if and only if $\mathrm{Aut}(X)=R(G)\rtimes \mathrm{Aut}(G, S)$. If $N_{Aut(X)}(R(G))$ is transitive on edges then $\mathrm{Cay}(G, S)$ is called a *normal edge-transitive* Cayley graph by Praeger [13]. Clearly, if $\mathrm{Cay}(G, S)$ is normal on G and edge-transitive then it is normal edge-transitive. Note that being a normal Cayley graph is not invariant under graph isomorphism, and so strictly depends upon which group the graph is a Cayley graph on. For example, the three-dimensional hypercube Q_3 ia a Cayley graph on either the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or the group $\mathbb{Z}_4 \times \mathbb{Z}_2$. The Cayley graph on the first group is normal, but the Cayley graph on the second group is not.

A Cayley graph X of a group G is called a *graphical regular representation* if $\mathrm{Aut}(X)=G$. It implies that $\mathrm{Aut}(X)_1 = 1$ and so $\mathrm{Aut}(G, S) = 1$. Therefore, a necessary condition for X to be a graphical regular representation of G is $\mathrm{Aut}(G, S) = 1$. In some circumstance, this necessary condition is also sufficient for X to be a graphical regular representation. However, it is of course not the case in general, that is, there exist groups G and Cayley graphs $\mathrm{Cay}(G, S)$ such that $\mathrm{Aut}(G, S) = 1$ but $\mathrm{Cay}(G, S)$ is not a graphical regular representation of G. Hence the following natural problem was proposed by Godsil [9].

Problem A. Determine classes of groups G and Cayley graphs $\mathrm{Cay}(G, S)$ for which $\mathrm{Cay}(G, S)$ is a graphical regular representation of G if and only if $\mathrm{Aut}(G, S) = 1$.

The solution of Problem A is known only for some classes of groups. Godsil solved the problem in [8, Theorem 3.8] for p-groups with p prime which have no homomorphism onto $\mathbb{Z}_p \times \mathbb{Z}_p$, and Li [10], solved the problem A for arbitrary cubic Cayley graphs of 2-power order, also Li and Sim solved the problem A for metacyclic p-groups with prime p (see [11]). In the present paper, it is proved that for most finite minimal non-abelian groups G, a tetravalent Cayley graph $\mathrm{Cay}(G, S)$ is a graphical regular representation if and only if $\mathrm{Aut}(G, S) = 1$.

Fang et al. [5] proved that the vast majority of connected cubic

Cayley graphs on non-abelian simple groups are normal. It was shown in [1, 2, 6], that most connected Cayley graphs of small valency on abelian groups are normal, and that all connected tetravalent Cayley graphs on p-groups of nilpotency class 2 with p an odd prime are normal. Also the normality of tetravalent Cayley graphs on regular p-groups was done by Feng and Xu [7]. In the present paper, we shall prove that all connected tetravalent Cayley graphs on a minimal non-abelian group are normal when $(|G|, 2) = (|G|, 3) = 1$, and X is not isomorphic to either $\mathrm{Cay}(G, S)$, where $|G| = 5^n$, and $|\mathrm{Aut(X)}|{=}2^m.3.5^n$ where $m \in \{2, 3\}$ and $n \geq 3$, or $\mathrm{Cay}(G, S)$ where $|G| = 5q^n$ (q is prime) and $|\mathrm{Aut}(X)| = 2^m.3.5.q^n$, where $q \geq 7$, $m \in \{2, 3\}$ and $n \geq 1$.

2. Preliminaries

In this section we give some results which will be used later. The following fact is basic for Cayley graphs.

Proposition 2.1. [3, Lemma 16.3] *A graph X is a Cayley graph of a group G if and only if $Aut(X)$ contains a regular subgroup isomorphic to G.*

We have the following result for minimal non-abelian p-groups.

Proposition 2.2. [12] *Let G be a finite minimal non-abelian group. Then one of the following holds:*
(1) G is a minimal non-abelian p-group;
(2) G is a semi-direct product of an elementary abelian q-group Q of order q^α by a cyclic group $P = \langle b\rangle$ of order p^β, where p, q are distinct primes and the action of b on Q is an automorphism of Q of order p.

Let X=$\mathrm{Cay}(G, S)$ be a Cayley graph on a finite group G and H a normal subgroup of G. Let X_H be the quotient graph corresponding to the cosets of H in G, with two cosets adjacent in X_H, whenever there is an edge between those cosets in X.

Proposition 2.3 ([13]). *The quotient graph X_H is a Cayley graph and*

$$X_H = Cay(G/H, SH/H),$$

where $SH/H = \{sH|s \in S\}$.

The *socle* of a group G is the subgroup generated by the set of all minimal normal subgroups of G, and it is denoted by $\mathrm{soc}(G)$. By [4, Theorem 4.3], we have the following proposition:

Proposition 2.4. *Let G be a nontrivial finite group. Then $soc(G) = T_1 \times ... \times T_k$, where T_i $(1 \leq i \leq k)$ are simple normal subgroups and $T_i \cong T_j$ $(1 \leq i, j \leq k)$.*

3. Main result

The main purpose of this paper is to prove the following theorem.

Theorem 3.1. *Let p be a prime and G a minimal non-abelian group which $(|G|, 2) = (|G|, 3) = 1$. Let X=$Cay(G,S)$ be a connected tetravalent Cayley graph on G. Then we have $Aut(Cay(G, S)) = R(G) \rtimes Aut(G, S)$, except when X is not isomorphic to either $Cay(G, S)$, where $|G| = 5^n$, and $|Aut(X)|$=$2^m.3.5^n$, where $m \in \{2, 3\}$ and $n \geq 3$, or $Cay(G, S)$ where $|G| = 5q^n$ (q is prime) and $|\mathrm{Aut}(X)| = 2^m.3.5.q^n$, where $q \geq 7$, $m \in \{2, 3\}$ and $n \geq 1$.*

Let $Q_8 = \langle a, b | a^4 = 1, bab^{-1} = a^{-1}, a^2 = b^2\rangle$, be a quaternion group, and $S = \{a, a^3, b, b^3\}$ a subset of Q_3. Now $\sigma = (ab, a^3b) \in A_1$, but $\sigma \notin Aut(G, S)$ and hence $\mathrm{Cay}(Q_8, \{a, a^3, b, b^3\})$ is not normal. Also if $D_6 = \langle a, b | a^3 = b^2 = 1, b^{-1}ab = a^{-1}\rangle$, and $S = \{a, a^2, b, ab\}$, then $\delta = (a, b)(a^2, ab) \in A_1$, but $\delta \notin Aut(G, S)$ and hence $\mathrm{Cay}(D_6, \{a, a^2, b, ab\})$ is not normal. It follows that Theorem 3.1 is not true generally when $(|G|, 2) \neq 1$, and $(|G|, 3) \neq 1$. As a consequence of Theorem 3.1, we have the following result.

Corollary 3.2. *Let G be a minimal non-abelian group, and $(|G|, 2) = (|G|, 3) = 1$. Also let X is not isomorphic to either $Cay(G, S)$, where $|G| = 5^n$, and $|Aut(X)|$=$2^m.3.5^n$, where $m \in \{2, 3\}$ and $n \geq 3$, or $Cay(G, S)$ where $|G| = 5q^n$ (q is prime) and $|\mathrm{Aut}(X)| = 2^m.3.5.q^n$, where $q \geq 7$, $m \in \{2, 3\}$ and $n \geq 1$. Then a connected Cayley graph X=$Cay(G, S)$ of valency 4 is a graphical regular representation of G if and only if $Aut(G, S) = 1$.*

Proof of Theorem 3.1. Recall that $\mathrm{Aut}(Cay(G, S))$=$R(G) \rtimes Aut(G, S)$ if and only if $\mathrm{Cay}(G, S)$ is normal on G. Let G be a counterexample of least order, that is, G has the smallest order with the following properties: G is a minimal non-abelian group and there exists a subset $S = \{s_1, s_2, s_1^{-1}, s_2^{-1}\}$ of G such that X=$\mathrm{Cay}(G, S)$ is not a normal 4-valent connected Cayley graph. Let A=$\mathrm{Aut}(X)$ and let A_1 be the stabilizer of 1 in A. Since X has valency 4, the stabilizer A_1 of the vertex 1 in A is a $\{2, 3\}$-group. By Proposition 2.2, first assume that G is a p-group. By our assumption G is not 2- or 3-group. By Proposition 2.4, $\mathrm{soc}(A) = T_1 \times ... \times T_k$, where

$k \geq 1$ and $T_i \cong T_j$ $(1 \leq i, j \leq k)$. T is a $\{2,3,p\}$-group because A is $\{2,3,p\}$-group.

First let T be an abelian group. Then $\mathrm{soc}(A)$ is an elementary abelian q-group, where $q = 2$, 3, or p. If $H = \mathrm{soc}(A)$ is transitive on $\mathrm{V}(X)$, then $H = \mathrm{soc}(A)$ acts regularly on $\mathrm{V}(X)$. Thus $|\mathrm{soc}(A)| = |G|$, and hence $\mathrm{soc}(A) = G$, a contradiction. So $H = \mathrm{soc}(A)$ is not transitive on $\mathrm{V}(X)$. Now assume that $\Sigma = \{B_1, B_2, ..., B_t\}$ is the set of orbits of H on $\mathrm{V}(X)$. By the normality of H in A we have that t and $|B_i|$ $(1 \leq i \leq t)$ are powers of p. It follows that H is a p-group and $H \leq G$. Consider the quotient graph X_H defined by $\mathrm{V}(X_H)$=Σ, and $(B_i, B_j) \in E(X_H)$ if and only if there exists $v_i \in B_i$, $v_j \in B_j$ such that $(v_i, v_j) \in E(X)$. Since the quotient group G/H acts regularly on Σ, therefore by Proposition 2.1 X_H must be connected Cayley graph of G/H. So by Proposition 2.3 $X_H \cong \mathrm{Cay}(\overline{G}, \overline{S})$, where $\overline{G} = G/H$ and $\overline{S} = \{sH | s \in S\}$. Since $|V(X_H)|$ is odd, the valency $v(X_H)$ of X_H could not be 3 and hence $v(X_H) = 2$, or $v(X_H) = 4$. If $v(X_H) = 4$ and G/H be a non-abelian, then by the minimality of $|G|$, X_H is normal as a Cayley graph of G/H. Thus $G/H \trianglelefteq \mathrm{Aut}(X_H)$. Let K be the kernel of A on Σ. Also let K_1 be the stabilizer of the vertex 1 in K. Then K_1 fixes the neighborhood $N(1)$ of 1 in X pointwise. Since X is connected, K_1 fixes each vertex in X. Thus $K_1 = 1$, and hence $K = H$. Therefore $A/H \leq \mathrm{Aut}(X_H)$, and G/H is normal in A/H, which implies that $G \trianglelefteq \mathrm{Aut}(X)$, a contradiction. Now if $v(X_H)$=4 and G/H is an abelian, then by [2], X_H is normal except when X_H=$\mathrm{Cay}(G/H, SH/H)$, where $G/H \cong \mathbb{Z}_5$. If X_H is normal, then $G/H \trianglelefteq \mathrm{Aut}(X_H)$ and so $G \trianglelefteq \mathrm{Aut}(X)$, a contradiction. Now suppose that $X_H = K_5$, and $\mathrm{Aut}(X_H)$=S_5. Therefore $|A/H| \mid 2^3.3.5$ and hence $|\mathrm{Aut(X)}|$=$2^m.3.5^n$, where $m \in \{2,3\}$ and $n \geq 3$. If $3 \nmid |A|$, then clearly $\mathrm{R}(G) \trianglelefteq A$, a contradiction. Thus $3 \mid |A|$. Now X=$\mathrm{Cay}(G,S)$ is not normal. Suppose to the contrary that X is normal. Hence A_1=$\mathrm{Aut}(G,S)$, and so $3 \mid |\mathrm{Aut}(G,S)|$. Therefore there exists an element α of order 3 in $\mathrm{Aut}(G,S)$. Then $|S| = 4$ implies that α fixes an element in S, say s. Consequently, α fixes s^{-1}. Since $s, s^{-1} \in S$ and $s \neq s^{-1}$, we have that α fixes S pointwise and hence $\langle S \rangle = G$ implies $\alpha = 1$, a contradiction. Now assume that $v(X_H) = 2$. Then X_H is a cycle of length $p^m =\mid G/H \mid$, for some positive integer m. Also $\mathrm{Aut}(X_H) \cong D_{2p^m}$. We have $SH/H = \{Hs_1, Hs_2\}$. Clearly $Hs_1 \neq Hs_1^{-1}$ and $Hs_2 \neq Hs_2^{-1}$. Therefore $Hs_1 = Hs_2^{-1}$, and $s_1 s_2 \in H$. If $H \cap S \neq \emptyset$, then $G = H$, a contradiction. So $H \cap S = \emptyset$, and hence $G = \langle H, s_1 \rangle$. Now since G/H is a cyclic group and $\{Hs_1\}$ is a minimal generating set, by [14, Proposition 3.1], we have $\mathrm{Aut}(X)$=G/H, a contradiction.

Now let T is not an abelian group. Assume that H=$\mathrm{soc}(A) \cap G$. By considering the order of A, one has $H \neq 1$. Clearly H is not transitive on $V(X)$. Now assume that $\Sigma = \{B_1, B_2, ..., B_s\}$ is the set of orbits of H on $\mathrm{V}(X)$. By the normality of H in G we have that s and $|B_i|$ $(1 \leq i \leq s)$ are powers of p. Considering the quotient graph X_H defined by $\mathrm{V}(X_H)$=Σ, and $(B_i, B_j) \in E(X_H)$ if and only if there exists $v_i \in B_i$, $v_j \in B_j$ such that $(v_i, v_j) \in E(X)$. Since the quotient group G/H acts regularly on Σ, therefore X_H must be connected Cayley graph of G/H. So $X_H \cong$ Cay$(\overline{G}, \overline{S})$, where $\overline{G} = G/H$ and $\overline{S} = \{sH | s \in S\}$. Now with the same arguments as before we get same result.

Finally assume that $G = Q \rtimes P$, where P and Q are Sylow p-subgroup and Sylow q-subgroup of G, respectively such that P is a cyclic group and $Q \cong \mathbb{Z}_q \times \mathbb{Z}_q \times ... \times \mathbb{Z}_q$. Now considering $\mathrm{soc}(A) = T_1 \times ... \times T_k$, where $k \geq 1$ and $T_i \cong T_j$ $(1 \leq i, j \leq k)$. If soc(A) is $\{2, 3\}$-group, Then Soc(A) is abelian 2-group or abelian 3-group. Clearly Soc(A) is not transitive on $\mathrm{V}(X)$. Now assume that $\Sigma = \{B_1, B_2, ..., B_s\}$ is the set of orbits of soc(A) on $\mathrm{V}(X)$. By the normality of soc(A) in A, p, q or pq divides s and $|B_i|$ $(1 \leq i \leq s)$. This is a contradiction. Thus H=soc$(A)\cap G \neq 1$ and H is not transitive on $V(X)$. Now assume that $\Sigma = \{B_1, B_2, ..., B_s\}$ is the set of orbits of H on $\mathrm{V}(X)$. By the normality of H in G, p, q, or pq divides s and $|B_i|$ $(1 \leq i \leq s)$. Since the quotient group G/H acts regularly on Σ, therefore X_H must be connected Cayley graph of G/H. So $X_H \cong$ Cay$(\overline{G}, \overline{S})$, where $\overline{G} = G/H$ and $\overline{S} = \{sH | s \in S\}$. Since $|V(X_H)|$ is odd, the valency $v(X_H)$ of X_H could not be 3 and hence $v(X_H) = 2$, or $v(X_H) = 4$. If $v(X_H) = 4$ and G/H be a non-abelian, then by the minimality of $|G|$, X_H is normal as a Cayley graph of G/H. Thus $G/H \trianglelefteq$Aut(X_H). Let K be the kernel of A on Σ. Also let K_1 be the stabilizer of the vertex 1 in K. Then K_1 fixes the neighborhood $N(1)$ of 1 in X pointwise. Since X is connected, K_1 fixes each vertex in X. Thus $K_1 = 1$, and hence $K = H$. Therefore $A/H \leq$ Aut(X_H), and G/H is normal in A/H, which implies that $G \trianglelefteq$ Aut(X), a contradiction. Now if $v(X_H)$=4 and G/H is an abelian, then by [2], X_H is normal except when X_H=Cay$(G/H, SH/H)$, where $G/H \cong \mathbb{Z}_5$. If X_H is normal, then $G/H \trianglelefteq$Aut(X_H) and so $G \trianglelefteq$ Aut(X), a contradiction. Now suppose that $X_H = K_5$, and Aut(X_H)=S_5. Therefore $|$Aut(X)$|$=$2^m.3.5.q^n$, where $m \in \{2, 3\}$ and $n \geq 1$. If $3 \nmid |A|$, then $|A| = 2^m.5.q^n$. Clearly $PQ/Q \trianglelefteq A/Q$ and so $PQ \trianglelefteq A$. Thus $\mathrm{R}(G) \trianglelefteq A$, a contradiction. Thus $3 \mid |A|$. Now X=Cay(G, S) is not normal. Suppose to the contrary that X is normal. Hence A_1=Aut(G, S), and so $3 \mid |$Aut$(G, S)|$. Therefore there exists an element α of order 3 in Aut(G, S). Then $|S| = 4$ implies that α fixes an element in S, say s. Consequently, α fixes s^{-1}. Since $s, s^{-1} \in S$ and

$s \neq s^{-1}$, we have that α fixes S pointwise and hence $\langle S \rangle = G$ implies $\alpha = 1$, a contradiction. Now assume that $v(X_H) = 2$. Then X_H is a cycle of length $p^m q^n = \mid G/H \mid$, for some positive integer m, n. Also $\mathrm{Aut}(X_H) \cong D_{2p^m q^n}$. We have $SH/H = \{Hs_1, Hs_2\}$. Clearly $Hs_1 \neq Hs_1^{-1}$ and $Hs_2 \neq Hs_2^{-1}$. Therefore $Hs_1 = Hs_2^{-1}$, and $s_1 s_2 \in H$. If $H \cap S \neq \emptyset$, then $G = H$, a contradiction. So $H \cap S = \emptyset$, and hence $G = \langle H, s_1 \rangle$. Now since G/H is a cyclic group and $\{Hs_1\}$ is a minimal generating set, by [14, Proposition 3.1], we have $\mathrm{Aut}(X) = G/H$, a contradiction.

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