

## On factorizations of limited solubly $\omega$ -saturated formations

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ABSTRACT. If  $\mathfrak{F} = \mathfrak{F}_1 \dots \mathfrak{F}_t$  is the product of the formations  $\mathfrak{F}_1, \dots, \mathfrak{F}_t$  and  $\mathfrak{F} \neq \mathfrak{F}_1 \dots \mathfrak{F}_{i-1} \mathfrak{F}_{i+1} \dots \mathfrak{F}_t$  for all  $i = 1, \dots, t$ , then we call this product a non-cancellative factorization of the formation  $\mathfrak{F}$ . In this paper we give a description of factorizable limited solubly  $\omega$ -saturated formations.

### Introduction

All groups considered are finite.

We will use  $C^p(G)$  to denote the intersection of all centralizers of Abelian  $p$ -chief factors of the group  $G$  [1] (we note that  $C^p(G) = G$  if  $G$  has no such chief factors). Let  $\mathfrak{X}$  be a set of groups. Then we use  $\text{Com}(\mathfrak{X})$  to denote the class of all Abelian groups  $A$  such that  $A \simeq H/K$  for some composition factor  $H/K$  of some group  $G \in \mathfrak{X}$ . Also, we write  $\text{Com}(G)$  for the set  $\text{Com}(\{G\})$ .

Let  $\emptyset \neq \omega \subseteq \mathbb{P}$ . For every function  $f$  of the form

$$f : \omega \cup \{\omega'\} \rightarrow \{\text{group formations}\} \quad (1)$$

we put

$$CF_\omega(f) = \{G \text{ is a group} \mid G/R(G) \cap O_\omega(G) \in f(\omega')\}$$

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and

$$G/C^p(G) \in f(p) \text{ for any prime } p \in \omega \cap \pi(\text{Com}(G)).$$

Here  $R(G)$  denotes the radical of  $G$  (i.e.  $R(G)$  is the largest normal soluble subgroup of  $G$ ).

We call  $\mathfrak{F}$  a *solubly  $\omega$ -saturated formation* [2] if  $\mathfrak{F} = CF_\omega(f)$  for some function  $f$  of the form (1). In this case, we call  $f$  a *composition  $\omega$ -satellite* of the formation  $\mathfrak{F}$ .

If

$$\mathfrak{F} = \mathfrak{F}_1 \dots \mathfrak{F}_t \tag{2}$$

is the product of the formations  $\mathfrak{F}_1, \dots, \mathfrak{F}_t$  and  $\mathfrak{F} \neq \mathfrak{F}_1 \dots \mathfrak{F}_{i-1} \mathfrak{F}_{i+1} \dots \mathfrak{F}_t$  for all  $i = 1, \dots, t$ , then we call (2) a *non-cancellative factorization* of the formation  $\mathfrak{F}$ .

A formation  $\mathfrak{F}$  is called a *one-generated formation* if there is a group  $G$  such that  $\mathfrak{F}$  is the intersection of all formations containing  $G$ . A formation  $\mathfrak{F}$  is called a *one-generated solubly  $\omega$ -saturated formation* if there is a group  $G$  such that  $\mathfrak{F}$  is the intersection of all solubly  $\omega$ -saturated formations containing  $G$ .

A formation  $\mathfrak{F}$  is called *limited* if  $\mathfrak{F}$  is a subformation of some one-generated formation. Analogously, a solubly  $\omega$ -saturated formation  $\mathfrak{F}$  is called a *limited solubly  $\omega$ -saturated formation* if it is a subformation of some one-generation solubly  $\omega$ -saturated formation. Let  $\mathfrak{H}$  be a class of groups.

We use  $\mathfrak{H}(\omega')$  to denote the class  $\text{form}(A/(R(A) \cap O_\omega(R)) \mid A \in \mathfrak{H})$ . and use  $\mathfrak{H}(p)$  to denote the class  $\text{form}(A/O_p(A) \mid A \in \mathfrak{H})$ .

In this paper we prove the following theorem which gives the answer to Problem 21 in [2].

**Theorem 1.** *The product*

$$\mathfrak{F}_1 \mathfrak{F}_2 \dots \mathfrak{F}_t \tag{*}$$

is a *non-cancellative factorization of some limited solubly  $\omega$ -saturated formation  $\mathfrak{F}$  if and only if the following conditions hold:*

- (1)  $t \leq 3$  and every factor in (\*) is a *nonidentity formation*;
- (2)  $\mathfrak{F}_1$  is a *one-generated  $\omega$ -saturated subformation in  $\mathfrak{N}_\omega \mathfrak{N}$  and  $\pi(\text{Com}(\mathfrak{F})) \cap \omega \subseteq \pi(\mathfrak{F}_1)$* ;
- (3) *If  $\mathfrak{F}_1 \not\subseteq \mathfrak{N}_\omega$ , then  $t = 2$ ,  $\mathfrak{F}_2$  is an Abelian one-generated formation and for all groups  $A \in \mathfrak{F}_1$  and  $B \in \mathfrak{F}_2$ ,  $(|A/F_\omega(A)|, |B|) = 1$  and  $(|A/O_\omega(A)|, |B|) = 1$ ;*

- (4) If  $\mathfrak{F}_1 \subseteq \mathfrak{N}_\omega$  and  $t = 3$ , then  $|\pi(\mathfrak{F}_1)| > 1$ ,  $\mathfrak{F}_3$  is a one-generated Abelian formation and for every  $p \in \pi(\mathfrak{F}_1)$ , the formation  $\mathfrak{F}_2(p)$  is one-generated nilpotent and for all groups  $A \in \mathfrak{F}_2$  and  $B \in \mathfrak{F}_3$ ,  $\pi(A/O_p(A)) \cap \pi(B) = \emptyset$ ;
- (5) If  $\mathfrak{F}_1 \subseteq \mathfrak{N}_\omega$ ,  $t = 2$  and  $|\pi(\mathfrak{F}_1)| > 1$ , then  $\mathfrak{F}_2(\omega')$ ,  $\mathfrak{F}_2$  are limited formations;
- (6) If  $\mathfrak{F}_1 = \mathfrak{N}_p$  for some prime  $p$ , then  $\mathfrak{F}_2(\omega')$  and  $\mathfrak{F}_2(p)$  (if  $p \in \omega$ ) are limited formations,  $\mathfrak{F}_2 \not\subseteq \mathfrak{F}_1$ , and there is a group  $B \in \mathfrak{F}_2$  such that for all groups  $A \in \mathfrak{F}_1$ , the  $\mathfrak{F}_2$ -residual  $T^{\mathfrak{F}_2}$  of the wreath product  $T = A \wr B$  is contained subdirectly in the base group of  $T$ .

All unexplained notation and terminology are standard. The reader is referred to [3], [4], [5] if necessary.

### 1. Proof of Theorem

*Proof.* Suppose that  $\mathfrak{F} = \mathfrak{F}_1\mathfrak{F}_2 \dots \mathfrak{F}_t$  and the conditions (1)–(6) hold. Our first step is to show that  $\mathfrak{F}$  is a limited solubly  $\omega$ -saturated formation. In view of Lemma 5 [2] and Theorem 1 [6], we only need to show that  $f(a)$  is a limited formation for all  $a \in \pi(\text{Com}(\mathfrak{F})) \cap \omega \cup \{\omega'\}$ , where  $f$  is the smallest composition  $\omega$ -satellite of the formation  $\mathfrak{F}$ , and  $|\pi(\text{Com}(\mathfrak{F})) \cap \omega| < \infty$ .

By (2), we see that  $|\pi(\text{Com}(\mathfrak{F})) \cap \omega| < \infty$ . Let  $\mathfrak{H} = \mathfrak{F}_2 \dots \mathfrak{F}_t$ , and  $m$  be the smallest composition  $\omega$ -satellite of the formation  $\mathfrak{F}_1$ . Then by Lemma 4.5 [7], we have  $\mathfrak{F} = CF_\omega(t)$ , where

$$t(a) = \begin{cases} m(p)\mathfrak{H}, & \text{if } a = p \in \pi(\text{Com}(\mathfrak{F}_1)) \cap \omega \\ \emptyset, & \text{if } a = p \in \omega \setminus \pi(\text{Com}(\mathfrak{F}_1)) \\ m(\omega')\mathfrak{H}, & \text{if } a = \omega'. \end{cases}$$

Let  $\mathfrak{F}_1 \not\subseteq \mathfrak{N}_\omega$ . Then by hypothesis,  $\mathfrak{H} = \mathfrak{F}_2$  is a one-generated Abelian formation. Let  $p \in \pi(\text{Com}(\mathfrak{F}_1)) \cap \omega$ . Since  $\mathfrak{F}_1$  is a one-generated solubly  $\omega$ -saturated formation and  $\mathfrak{F}_1 \subseteq \mathfrak{N}_\omega\mathfrak{N}$ , then  $m(p)$  is a nilpotent one-generated formation. Since for any groups  $A \in \mathfrak{F}_1$  and  $B \in \mathfrak{F}_2$  we have  $(|A/F_p(A)|, |B|) = 1$ , then by Lemma 5 [2],  $m(p) \cap \mathfrak{H} = (1)$ . Then  $m(p)\mathfrak{H}$  is soluble formations and it is is a one-generated formation. But  $f(p) \subseteq t(p) = m(p)\mathfrak{H}$ . Hence  $f(p)$  is a limited formation. Analogously, we can show that  $f(\omega')$  is a limited formation. Hence,  $\mathfrak{F}$  is indeed a limited solubly  $\omega$ -saturated formation.

Let  $\mathfrak{F}_1 \subseteq \mathfrak{N}_\omega$ . In this case, by Theorem 2 [6], we only need to show that if  $t = 3$ , then  $\mathfrak{H}(\omega')$  and  $\mathfrak{H}(p)$  (for all  $p \in \pi(\mathfrak{F}_1)$ ) are limited formations and  $\mathfrak{H}$  is a one-generated formation if  $|\pi(\mathfrak{F})| > 1$ .

Let  $p \in \pi(\mathfrak{F}_1) \cap \omega$ . Consider the formation  $\mathfrak{F}_2(p)\mathfrak{F}_3$ . In view of Condition (4) we have  $\pi(\mathfrak{F}_2(p)) \cap \pi(\mathfrak{F}_3) = \emptyset$ . Besides  $\mathfrak{F}_2(p)$  is a nilpotent one-generated formation. In these cases, the product  $\mathfrak{F}_2(p)\mathfrak{F}_3$  is a one-generated formation. But evidently  $\mathfrak{F}_2(p)\mathfrak{F}_3$  is a soluble formation, and every subformation of  $\mathfrak{F}_2(p)\mathfrak{F}_3$  is also one-generated. Thus, in order to prove that  $(\mathfrak{F}_2\mathfrak{F}_3)(p)$  is a limited formation, we only need to show that it is a subformation in  $\mathfrak{F}_2(p)\mathfrak{F}_3$ . Let  $A \in \mathfrak{F}_2\mathfrak{F}_3$ . Then  $A^{\mathfrak{F}_3} \in \mathfrak{F}_2$ . Hence  $O_p(A^{\mathfrak{F}_3})$  is a characteristic subgroup of  $A^{\mathfrak{F}_3}$  such that

$$A^{\mathfrak{F}_3}/O_p(A^{\mathfrak{F}_3}) \in \mathfrak{F}_2(p).$$

But  $A^{\mathfrak{F}_3}/O_p(A^{\mathfrak{F}_3}) = (A/O_p(A^{\mathfrak{F}_3}))^{\mathfrak{F}_3}$ , we have  $A/O_p(A^{\mathfrak{F}_3}) \in \mathfrak{F}_2(p)\mathfrak{F}_3$ , and so  $A/O_p(A) \in \mathfrak{F}_2(p)\mathfrak{F}_3$ . Thus  $(\mathfrak{F}_2\mathfrak{F}_3)(p) \subseteq \mathfrak{F}_2(p)\mathfrak{F}_3$ . This shows that the formation  $(\mathfrak{F}_2\mathfrak{F}_3)(p)$  is limited.

Assume that  $|\pi(\mathfrak{F})| > 1$  and let  $p, q \in \pi(\mathfrak{F}_1)$ . Let  $A \in \mathfrak{F}_2$  and  $B \in \mathfrak{F}_3$ . Since  $|A/O_p(A), |B|| = 1$  and  $(|A/O_q(A), |B|| = 1$ , we have  $(|A|, |B|) = 1$ . This shows that the exponents of the formations  $\mathfrak{F}_2$  and  $\mathfrak{F}_3$  are coprime. Same as above, one can show that  $\mathfrak{F}_2$  is a nilpotent formation. Hence by Theorem 1 [8],  $\mathfrak{F}_2\mathfrak{F}_3$  is a limited formation.

Consider the formation  $\mathfrak{F}_2(\omega')\mathfrak{F}_3$ . Clearly,  $\mathfrak{F}_2(\omega')$  is a one-generated nilpotent formation and  $\pi(\mathfrak{F}_2(\omega')) \cap \pi(\mathfrak{F}_3) = \emptyset$ . Hence  $\mathfrak{F}_2(\omega')\mathfrak{F}_3$  is a soluble one-generated formation. Now, in order to prove that  $(\mathfrak{F}_2\mathfrak{F}_3)(\omega')$  is a limited formation, we only need to show that  $(\mathfrak{F}_2\mathfrak{F}_3)(\omega') \subseteq \mathfrak{F}_2(\omega')\mathfrak{F}_3$ . Let  $A \in \mathfrak{F}_2\mathfrak{F}_3$ . Since  $O_\omega(A^{\mathfrak{F}_3})$  is a characteristic subgroup of  $A^{\mathfrak{F}_3}$  such that  $A^{\mathfrak{F}_3}/O_\omega(A^{\mathfrak{F}_3}) \in \mathfrak{F}_2(\omega')$ , and so  $A/O_\omega(A) \in \mathfrak{F}_2(\omega')\mathfrak{F}_3$ . Hence  $(\mathfrak{F}_2\mathfrak{F}_3)(\omega') \subseteq \mathfrak{F}_2(\omega')\mathfrak{F}_3$ .

We still need to show that the factorization (\*) is non-cancellative. For this purpose, we first take  $t = 2$ . Assume that  $\mathfrak{F}_1 \not\subseteq \mathfrak{N}_\omega$ . Then by Conditions (3),  $\mathfrak{F}_2$  is an Abelian formation, and so by Lemma 5.1 [7],  $\mathfrak{F} \neq \mathfrak{F}_2$ . Suppose that  $\mathfrak{F} = \mathfrak{F}_1$ . And let  $A$  be a group with minimal order in  $\mathfrak{F}_1 \setminus \mathfrak{N}_\omega$ . Let  $R$  be the monolith of  $A$ . Then  $R = A^{\mathfrak{N}_\omega}$ . Clearly, we have  $R \not\subseteq \Phi(A)$ . Let  $B$  be a simple group in  $\mathfrak{F}_2$  and  $T = A \wr B = [K]B$  where  $K$  is the base group of  $T$ . Since  $\mathfrak{F}_2$  is Abelian, we have  $B = C_p$ . Assume that the  $\mathfrak{F}_2$ -residual  $T^{\mathfrak{F}_2}$  of the wreath product  $T = A \wr B$  is not contained subdirectly in the base group of  $T$ . Let  $\pi(T^{\mathfrak{F}_2})$  is a projection of  $T^{\mathfrak{F}_2}$  in  $A_1$ , where  $A_1$  is the first copy of  $A$  in  $K$ . Then  $N/\pi(T^{\mathfrak{F}_2}) \wr B$  is a homomorphic image of the group  $T/T^{\mathfrak{F}_2}$ . By (3),  $|N/O_\omega(N), |A|| = 1$ . This contradiction

shows that the  $\mathfrak{F}_2$ -residual  $T^{\mathfrak{F}_2}$  of the wreath product  $T = A \wr B$  is contained subdirectly in the base group of  $T$ . Hence  $T^{\mathfrak{F}_2} \simeq A \in \mathfrak{F}_1$ . So

$$T \in \mathfrak{F} = \mathfrak{F}_1 \subseteq \mathfrak{N}_\omega \mathfrak{N}.$$

It is clear that  $R = F(A)$  and by Lemma 3.1.9 [5], we deduce that

$$L = R^\natural = \prod_{b \in B} R_1^b = F(T)$$

is the monolith of  $T$ , where  $R_1$  is the monolith of the first copy  $A$  in  $K$ . Since  $T \in \mathfrak{N}_\omega \mathfrak{N}$ , we have  $T^{\mathfrak{N}} \in \mathfrak{N}_\omega$ , i.e.  $T^{\mathfrak{N}} \subseteq L$ . Let  $R$  be an  $\omega'$ -group. Hence  $L$  is an  $\omega'$ -group, and so  $O_\omega(T) = 1$ . Since  $T \in \mathfrak{N}_\omega \mathfrak{N}$ ,  $T$  must be a nilpotent group. But  $F(T) = L \neq T$ , which is a contradiction. Hence  $R$  is a  $p$ -group, for some  $p \in \omega$ . However, since  $A \notin \mathfrak{N}_\omega$ , we have  $R = F_\omega(A)$  and consequently,  $(|A/R|, |B|) = 1$ . Let  $B$  be a  $q$ -group. Then  $B$  is a Sylow  $q$ -subgroup of  $T_1 = (A/R) \wr B$ . By [[1], A, (18.2)], we have  $T_1 \simeq T/L$ . This proves that  $T_1$  is a nilpotent group. Thus,  $B \trianglelefteq T$ , and so  $B \cap K_1 \neq 1$ , where  $K_1$  is the base group of  $T_1$ , a contradiction. This shows that  $\mathfrak{F}_1 \neq \mathfrak{F} \neq \mathfrak{F}_2$ .

Now we assume that  $\mathfrak{F}_1 \subseteq \mathfrak{N}_\omega$ . Let  $|\pi(\mathfrak{F}_1)| > 1$  and  $p, q$  be different primes in  $\pi(\mathfrak{F}_1)$ . Also we let  $B$  be a group such that  $\mathfrak{F}_2 \subseteq \text{form} B$ . Since  $\mathfrak{F}_1$  is an  $\omega$ -local formation, we have  $\mathfrak{N}_{\{p,q\}} \subseteq \mathfrak{F}_1$ . Hence  $\mathfrak{F} \neq \mathfrak{F}_2$  by Lemma 3.1.5 [5]. In view of Lemma 5.1 [7], we conclude that  $\mathfrak{F} \neq \mathfrak{F}_1$ .

Let  $\pi(\mathfrak{F}_1) = \{p\}$  for some  $p \in \omega$ . Then  $\mathfrak{F}_1 = \mathfrak{N}_p$ . Let  $B$  be a group in  $\mathfrak{F}_2$  such that for every group  $A \in \mathfrak{F}_1$  the  $\mathfrak{F}_2$ -residual  $T^{\mathfrak{F}_2}$  of the wreath product  $T = A \wr B$  is contained subdirectly in the base group of  $T$ . Assume that  $\mathfrak{F} = \mathfrak{F}_2 = \mathfrak{N}_p \mathfrak{F}_2$  and let  $A$  be a non-identity group in  $\mathfrak{N}_p$ . If  $T = A \wr B$ , then  $T \in \mathfrak{F} = \mathfrak{F}_2$ , and so  $T^{\mathfrak{F}_2} = 1$  is not contained subdirectly in the base group of  $T$ . This contradiction shows that  $\mathfrak{F} \neq \mathfrak{F}_2$ . And since, by Condition (6),  $\mathfrak{F}_2 \not\subseteq \mathfrak{F}_1$ , then we have  $\mathfrak{F} \neq \mathfrak{F}_1$ . This shows that the factorization (\*) is indeed non-cancellative.

Now let  $t = 3$ . Consider the case  $\pi(\mathfrak{F}_1) = \{p\}$ . By (6)  $\mathfrak{F}_2 \not\subseteq \mathfrak{N}_p$ . Let  $A$  be a group of minimal order in  $\mathfrak{F}_2 \setminus \mathfrak{N}_p$ . Then  $O_p(A) = 1$ . Thus, if  $B \in \mathfrak{F}_3$ , we have  $(|A|, |B|) = 1$ . Let  $T = Z_p \wr (A \wr B)$ . Evidently,  $T \in \mathfrak{F}$  and  $T$  is not a metanilpotent group. Hence  $\mathfrak{F} \not\subseteq \mathfrak{N}^2$ . Since the formations  $\mathfrak{F}_1 \mathfrak{F}_2$ ,  $\mathfrak{F}_1 \mathfrak{F}_3$  are all metanilpotent,  $\mathfrak{F} \neq \mathfrak{F}_1 \mathfrak{F}_2, \mathfrak{F}_1 \mathfrak{F}_3$ . By (6) we can let  $B$  be a group in  $\mathfrak{F}_2$  such that for all non-identity groups  $A \in \mathfrak{F}_1$  the  $\mathfrak{F}_2$ -residual  $D^{\mathfrak{F}_2}$  of the wreath product  $D = A \wr B$  is contained subdirectly in the base group of the wreath product  $D$ . Let  $C$  be a non-identity group in  $\mathfrak{F}_3$ . Assume that  $\mathfrak{F}_2 \mathfrak{F}_3 = \mathfrak{F}_1 \mathfrak{F}_2 \mathfrak{F}_3$  and  $T = D \wr C = [K]C$ , where  $K$  is the base group

of  $D$ . Then since  $\mathfrak{F}_3$  is Abelian formation, we see that  $T^{\mathfrak{F}_3}$  is contained subdirectly in  $K$ , and so  $D \in \mathfrak{F}_2$ , that is,  $D^{\mathfrak{F}_2} = 1$ , a contradiction. Hence  $\mathfrak{F} \neq \mathfrak{F}_2\mathfrak{F}_3$ . Now assume  $|\pi(\mathfrak{F}_1)| > 1$ . Let  $\{p, q\} \subseteq \pi(\mathfrak{F}_1)$ . By (4),  $\mathfrak{F}_2(p)$  and  $\mathfrak{F}_2(q)$  are a one-generated nilpotent formations. Hence,  $\mathfrak{F}_2$  is also a one-generated nilpotent formation by Lemma 4.6 [7]. Therefore  $\mathfrak{F} \neq \mathfrak{F}_2\mathfrak{F}_3$  by Lemma 5.1 [7]. For the case  $|\pi(\mathfrak{F}_1)| > 1$ , the proof is similar.

Assume that

$$\mathfrak{F} = \mathfrak{F}_1\mathfrak{F}_2 \dots \mathfrak{F}_t \subseteq c_\omega \text{form}(G) = \mathfrak{F}^*$$

for some group  $G$ . Let  $f$  be the smallest composition  $\omega$ -satelitte of the formation  $\mathfrak{F}$ ,  $f^*$  be the smallest composition  $\omega$ -satelitte of the formation  $\mathfrak{F}^*$ . We will show that the factors of the non-cancellative factorization (\*) all satisfy Conditions (1)–(6).

It is clear that every factor in (\*) is a non-identity formation. In additions, by Lemma 5.3 [7], we have  $t \leq 3$ . Thus, (1) is true.

By Theorem 1 [9] we have  $\mathfrak{F}_1 \subseteq \mathfrak{N}_\omega \mathfrak{N}$ , hence  $\mathfrak{F}_1$  is a hereditary formation. Then  $\mathfrak{F}_1 \subseteq \mathfrak{F} \subseteq \mathfrak{F}^*$ . By Lemma 3.5 [7] and Proposition A [7] we see that  $\mathfrak{F}_1$  is an  $\omega$ -saturated soluble formation. But by Lemma 4 [10] the set of all  $\omega$ -saturated subformations of the formation  $\mathfrak{F}_1$  is finite. Hence, there is a chain

$$(1) = \mathfrak{M}_0 \subseteq \mathfrak{M}_1 \subseteq \dots \subseteq \mathfrak{M}_{n-1} \subseteq \mathfrak{M}_n = \mathfrak{F}_1,$$

where  $\mathfrak{M}_i$  is a maximal  $\omega$ -saturated subformation of  $\mathfrak{M}_{i+1}$ ,  $i = 0, 1, \dots, n-1$ . Let  $A_i \in \mathfrak{M}_i \setminus \mathfrak{M}_{i-1}$ ,  $i = 1, 2, \dots, n$ . Assume that  $l_\omega \text{form}(\mathfrak{M}_{i-1} \cup \{A_i\}) \neq \mathfrak{M}_i$ . Then

$$\mathfrak{M}_{i-1} \subseteq l_\omega \text{form}(\mathfrak{M}_{i-1} \cup \{A_i\}) \subseteq \mathfrak{M}_i.$$

But  $\mathfrak{M}_{i-1}$  is a maximal  $\omega$ -saturated subformation of  $\mathfrak{M}_i$ . This contradiction shows that  $l_\omega \text{form}(\mathfrak{M}_{i-1} \cup \{A_i\}) = \mathfrak{M}_i$ , and so

$$\mathfrak{F}_1 = l_\omega \text{form}(A_1, \dots, A_n) = l_\omega \text{form}(A_1 \times \dots \times A_n)$$

is a one-generated  $\omega$ -saturated formation.

Assume that  $p$  is prime such that  $p \in \pi(\text{Com}(\mathfrak{F})) \cap \omega$  and  $p \notin \pi(\mathfrak{F}_1)$ . Then  $\mathfrak{N}_p \subseteq \mathfrak{F}$  and  $Z_p \notin \mathfrak{F}_1$ . Let  $Z_q$  be a simple group in  $\mathfrak{F}_1$  such that  $q \in \pi(\text{Com}(\mathfrak{F}_1)) \cap \omega$ . Then  $q \neq p$ . Let  $B$  be a cycle group of order  $p^m$ , where  $m = |G|$ . And let  $D = Z_q \wr B = [K]B$ , where  $K$  is the base of the wraeth product  $D$ . In is clear that  $D \in \mathfrak{F}$ , and hence

$$D/C^q(D) = D/K \simeq B \in f(q).$$

By Lemma 5 [2] we see

$$f(q) \subseteq f^*(q) = \text{form}(G/C^q(G)),$$

which is a contradiction to Lemma 3.1.5 [5]. Hence  $\pi(\text{Com}(\mathfrak{F})) \cap \omega \subseteq \pi(\mathfrak{F}_1)$ . This shows that (2) holds.

Assume that  $\mathfrak{F}_1 \not\subseteq \mathfrak{N}_\omega$ . Let  $\mathfrak{H} = \mathfrak{F}_2 \dots \mathfrak{F}_t$  and  $A$  be a group of minimal order in  $\mathfrak{F}_1 \setminus \mathfrak{N}_\omega$ . Let  $R$  be the monolith of  $A$ . Then  $R = A^{\mathfrak{N}_\omega} = C_A(R)$  and  $R \not\subseteq \Phi(A)$ . If  $R$  is a  $p$ -group, then  $R = O_p(A) = F_p(A)$ . By (2),  $A/R$  is a nilpotent group. But by Lemma 1.7.11 [11], we have  $O_p(A/C_A(R)) = 1$ , and so  $p \notin \pi(A/R)$ . Assume that  $R = A$ . Then, we have  $|R| = p$  and  $R = A^{\mathfrak{N}_\omega}$ . We can deduce that  $p \notin \omega$ . Hence  $\mathfrak{H}$  is an Abelian formation by Lemma 3.1 [7]. Let  $R \neq A$  and let  $R \leq M \leq A$ , where  $M$  is a maximal subgroup of  $A$ . Then  $A/M$  is a simple group with  $|A/M| \neq p$ . Let  $m$  be the smallest local  $\omega$ -satellite of  $\mathfrak{F}_1$ . Since  $A \in \mathfrak{F}_1$ , we have  $A/F_p(A) = A/R \in m(p)$ , and so  $A/M \in m(p)$ . Now, by using Lemma 5.2 [7], we see that  $t = 2$  and that  $\mathfrak{F}_2$  is an Abelian formation. It follows that  $\mathfrak{F} = \mathfrak{F}_1\mathfrak{F}_2$  is a solubly  $\omega$ -saturated soluble formation and hence, in this case,  $\mathfrak{F}$  is an  $\omega$ -saturated formation.

Now we prove that  $\mathfrak{F}_2$  is a one-generated formation. Assume that  $\pi(\mathfrak{F}_1) \cap \omega = \{p\}$ . Since  $\mathfrak{F}_1 \not\subseteq \mathfrak{N}_\omega$ , we may choose in  $\mathfrak{F}_1$  a group  $A$  of prime order  $q \notin \omega$ . Let  $B \in \mathfrak{F}_2$  and  $T = A \wr B$ . Then, it is clear that  $T \in \mathfrak{F}$  and that  $O_\omega(T) = O_p(T) = 1$ . Hence by Lemma 5 [2], we have

$$T \simeq T/O_\omega(T) \in f(\omega') \subseteq f^*(\omega') = \text{form}(G/O_\omega(G)).$$

Hence  $\mathfrak{F}_2 \subseteq \text{form}(G/O_\omega(G))$ . Since the formation  $\mathfrak{F}_2$  is soluble,  $\mathfrak{F}_2$  is a one-generated formation. Now let  $|\pi(\mathfrak{F}_1) \cap \omega| > 1$  and let  $p, q$  be two different primes in  $\pi(\mathfrak{F}_1) \cap \omega$ . Let  $B \in \mathfrak{F}_2$ . Then by Lemma 4.5 [7], we have  $\mathfrak{F} = CF(t)$ , where

$$t(a) = \begin{cases} m(p)\mathfrak{F}_2 & \text{if } a = p \in \pi(\text{Com}(\mathfrak{F}_1)) \cap \omega; \\ \emptyset & \text{if } a = p \in \omega \setminus \pi(\text{Com}(\mathfrak{F}_1)); \\ m(\omega')\mathfrak{F}_2 & \text{if } a = \omega'. \end{cases}$$

We note that the formation function  $m$  is an inner composition  $\omega$ -satellite of  $\mathfrak{F}_1$ . Now, using Lemma 5 [2], we deduce that

$$\begin{aligned} B/O_p(B) \in f(p) \subseteq f^*(p) &= \\ &= \text{form}(G/C^p(G)) \subseteq \text{form}(G/C^p(G) \times (G/C^q(G))) \end{aligned}$$



and

$$B/O_q(B) \in f(q) \subseteq f^*(q) = \text{form}(G/C^q(G)) \subseteq \text{form}(G/C^p(G) \times (G/C^q(G))).$$

Hence

$$B \simeq B/(O_p(B) \cap O_q(B)) \in \text{form}((G/C^p(G)) \times (G/C^q(G))).$$

This shows that  $\mathfrak{F}_2 \subseteq \text{form}(G/C^p(G) \times (G/C^q(G)))$  and so  $\mathfrak{F}_2$  is a one-generated formation.

Let  $A \in \mathfrak{F}_1$  and  $B \in \mathfrak{F}_2$ . Then by Lemma 3.1.5 [5] and Lemma 3.1.7 [5] we can show that  $(|A/F_\omega(A)|, |B|) = 1$  and  $(|A/O_\omega(A)|, |B|) = 1$ .

This proves the condition (3).

Next we assume that  $\mathfrak{F}_1 \subseteq \mathfrak{N}_\omega$  and  $t = 3$ . First consider the case where  $|\pi(\mathfrak{F}_1)| > 1$ . We claim that  $\mathfrak{F}_2(p)$  is a nilpotent formation, for all  $p \in \omega$ . In fact, if the claim is not true, then we can let  $A$  be a non-nilpotent group of smallest order in  $\mathfrak{F}_2(p)$ . Let  $R$  be the monolith of  $A$ . And let  $q$  be a prime in  $\pi(\mathfrak{F}_1)$  such that  $R \not\subseteq O_q(A)$ . Let  $T = Z_q \wr A$ , where  $Z_q$  is a group of order  $q$ . Then

$$T \in \mathfrak{F}_1(\mathfrak{F}_2(p)) \subseteq \mathfrak{F}_1\mathfrak{F}_2 \subseteq \mathfrak{N}_\omega\mathfrak{N}.$$

Also, it is not difficult to show that  $F(T) = K$ , where  $K$  is the base group of  $T$ . Clearly,  $T/K \simeq A$  is not nilpotent, and so  $T \notin \mathfrak{N}_\omega\mathfrak{N}$ . This contradiction shows that  $\mathfrak{F}_2(p)$  is a nilpotent formation, and our claim is established. If  $\mathfrak{F}_1\mathfrak{F}_2 \subseteq \mathfrak{N}_\omega$ , then by Lemma 5.1 [7], we have  $\mathfrak{F}_1\mathfrak{F}_2 = \mathfrak{F}_1 = \mathfrak{N}_p$  which is impossible. Hence  $\mathfrak{F}_1\mathfrak{F}_2 \not\subseteq \mathfrak{N}_\omega$  and so by (3),  $\mathfrak{F}_3$  is an Abelian one-generated formation. Let  $p \in \pi(\mathfrak{F}_1)$ ,  $A \in \mathfrak{F}_2$ ,  $T_0 = A/O_p(A)$  and  $B \in \mathfrak{F}_3$ . Assume that there is prime  $q$  such that  $q \in \pi(T_0) \cap \pi(B)$ . Let  $Z_q$  be a group of order  $q$ . Let  $T = Z_q \wr (Z_1 \times \dots \times Z_n)$ , where  $Z_1 \simeq Z_2 \simeq \dots \simeq Z_n \simeq Z_q$  and  $n = |G|$ . Form  $Y = T_0 \wr (B_1 \times \dots \times B_n)$ , where  $B_i \simeq B$ . Then, by Lemma 3.1.7 [5], the nilpotent class  $c(T)$  of the group  $T$  is at least  $n + 1$ . If  $q \notin \omega$ , then by [[1], A, (18,2)] and by Lemma 5 [2],

$$T/O_\omega(T) \simeq T \simeq E \leq Y/O_\omega(Y) \in \text{form}(G/O_\omega(G)).$$

This clearly contradicts to Lemma 3.1.5 [5]. Let  $q \in \omega$ . Then, in view of (2), we have  $q \in \pi(\mathfrak{F}_1) \setminus \{p\}$ . Let  $Z_p$  be a group of order  $p$  and  $D = Z_p \wr T = [K]T$ , where  $K$  is the base group of  $D$ . Clearly  $D \in \mathfrak{F}_1\mathfrak{F}_2\mathfrak{F}_3 = \mathfrak{F}$ , and so by Lemma 5 [2], we have

$$D/C^p(D) = D/K \simeq T \in \text{form}(G/C^p(G)),$$



again, this contradicts to Lemma 3.1.5 [5]. Thus, for all groups  $A \in \mathfrak{F}_2$  and  $B \in \mathfrak{F}_3$ , we have  $\pi(A/O_p(A)) \cap \pi(B) = \emptyset$ .

Now we claim that  $\mathfrak{F}_2(p)$  is a one-generated nilpotent formation. Indeed, by (2),  $\mathfrak{F}_1\mathfrak{F}_2$  is a one-generated  $\omega$ -saturated formation. By using Proposition 4.7 [7] we see that  $\mathfrak{F}_2$  or  $\mathfrak{F}_2(p)$  is a one-generated formation. But  $\mathfrak{F}_2$  is a soluble formation and  $\mathfrak{F}_2(p) \subseteq \mathfrak{F}_2$ . Hence  $\mathfrak{F}_2(p)$  is a one-generated formation in view of Theorem VII.1.7 [1].

Now consider  $\mathfrak{F}_1 = \mathfrak{N}_p$ , for some  $p \in \omega$ . By Proposition 4.7 [7],  $\mathfrak{F}_2(p)$  is a one-generated formation since  $\mathfrak{F}_1\mathfrak{F}_2$  is a one-generated solubly  $\omega$ -saturated formation. Thus, condition (4) holds.

Condition (5) and the first two conditions of (6) follow directly from Proposition 4.7 [7]. It is clear that  $\mathfrak{F}_2 \not\subseteq \mathfrak{F}_1$ . Now we assume that for every group  $B \in \mathfrak{F}_2$  there is a group  $A \in \mathfrak{F}_1$  such that the  $\mathfrak{F}_2$ -residual  $T^{\mathfrak{F}_2}$  of the wreath product  $T = A \wr B$  is not contained subdirectly in the base group of  $T$ . And let  $B$  be a group of minimal order in  $\mathfrak{N}_p\mathfrak{F}_2 \setminus \mathfrak{F}_2$ . Then the group  $B$  is monolithic and its monolith  $R = B^{\mathfrak{F}_2}$ . Now let  $T = A \wr (B/R)$ , where  $A$  is a group in  $\mathfrak{F}_1$  such that the  $\mathfrak{F}_2$ -residual  $T^{\mathfrak{F}_2}$  of  $T$  is not contained subdirectly in the base group  $K$  of  $T$ . But then the formation  $\mathfrak{F}_2$  contains the group  $Z_p \wr (B/R)$ , where  $|Z_p| = p$ . Evidently,  $R$  is an elementary Abelian  $p$ -group, so by Lemma 3.5.2 [5],  $B \in \mathfrak{F}_2$ . This contradiction shows that  $\mathfrak{N}_p\mathfrak{F}_2 \subseteq \mathfrak{F}_2$ . Hence  $\mathfrak{N}_p\mathfrak{F}_2 = \mathfrak{F}_2 = \mathfrak{F}$ . This contradiction shows that condition (6) holds. Thus, the theorem is proved.  $\square$

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