Algebra and Discrete Mathematics
Volume 13 **(2012)**. Number 2. pp. 139 – 146

SURVEY ARTICLE

S-Embedded subgroups in finite groups

A. Ballester-Bolinches, J. C. Beidleman, A. D. Feldman, H. Heineken and M. F. Ragland

Communicated by L. A. Kurdachenko

Dedicated to 100th anniversary of Professor S. N. Chernikov

Abstract. In this survey paper several subgroup embedding properties related to permutability are introduced and studied.

All groups in this paper are finite.

The purpose of this survey paper is to show how the embedding of certain types of subgroups in a group G help determine some of the structural properties of G.

Among the types of subgroup embedding properties we consider include: S-permutability, S-semipermutability, semipermutability, seminormality, τ-semipermutability and τ-seminormality.

A subgroup H of a group G is said to permute with a subgroup K of G if HK is a subgroup of G. H is said to be permutable (resp. S-permutable) in G if H permutes with all the subgroups (resp. Sylow subgroups) of G. Kegel ([4, 1.2.14(3)]) showed that any S-permutable subgroup is subnormal. A group G is called a PST-group if S-permutability is a transitive relation in G, that is, if A and B are subgroups of G such that A is S-permutable in B and B is S-permutable in G, then A is S-permutable in G. By Kegel's result G is a PST-group if and only if every subnormal subgroup is S-permutable in G. Agrawal (see [4, 2.1.8]) characterised soluble PST-groups. He proved that a soluble group G is a

2010 MSC: 20D05, 20D10, 20F16.
Key words and phrases: finite groups, permutability, S-permutability, seminormality, semipermutability.

PST-group if and only if the nilpotent residual of G is an abelian Hall subgroup of G on which G acts by conjugation as power automorphisms. A number of research papers have been written on these groups. The reader is referred to [4, Chap. 2] for basic results about this class of groups.

A subgroup H of a group G is said to be semipermutable (resp. S-semipermutable) provided that it permutes with every subgroup (resp. Sylow subgroup) K of G such that $(|H|, |K|) = 1$. An S-semipermutable subgroup of a group need not be subnormal. For example, a Sylow 2-subgroup of the nonabelian group of order 6 is semipermutable and S-semipermutable but not subnormal. A group G is called a BT-group if semipermutability is a transitive relation in G. L. Wang, Y. Li, and Y. Wang [12] introduced soluble BT-groups and proved the following theorem which showed that they are a subclass of PST-groups.

Theorem 1. *Let G be a group with nilpotent residual L. The following statements are equivalent:*

(1) *G is a soluble* BT*-group.*

(2) *Every subgroup of G of prime power order is S-semipermutable.*

(3) *Every subgroup of G of prime power order is semipermutable.*

(4) *Every subgroup of G is semipermutable.*

(5) *G is a soluble* PST*-group and if p and q are distinct primes not dividing the order of L with $G_p \in \mathrm{Syl}_p(G)$ and $G_q \in \mathrm{Syl}_q(G)$, then $[G_p, G_q] = 1$.*

Research papers on BT-groups include [1, 3, 12].

We next present an example of a soluble PST-group which is not a BT-group.

Example 2. Let L be a cyclic group of order 7 and $A = C_3 \times C_2$ be the automorphism group of L. Here C_3 (resp. C_2) is the cyclic group of order 3 (resp. 2). Let $G = L \rtimes A$ be the semidirect product of L by A. Let $L = \langle x \rangle$, $C_3 = \langle y \rangle$ and $C_2 = \langle z \rangle$ and note that $[\langle y \rangle^x, \langle z \rangle] \neq 1$. Now G is a PST-group by Agrawal's theorem, but G is not a BT-group by Theorem 1.

Now we consider some different types of embedding subgroups of a group.

A subgroup X of a group G is said to be seminormal (resp. S-seminormal) if it is normalized by every subgroup (resp. Sylow subgroup)

H of G such that $(|X|, |H|) = 1$. By Theorem 1.2 of [6] a subgroup of a group is seminormal if and only if it is S-seminormal.

A group G is called an SN-group provided that every subnormal subgroup of G is seminormal in G.

In [6] the following theorem is established.

Theorem 3 ([6]). *Let G be a soluble group. Then the following are equivalent:*

(1) *G is an* SN-*group.*

(2) *All subnormal subgroups are semipermutable.*

(3) *All subnormal subgroups are S-semipermutable.*

(4) *G is a* PST-*group.*

Lukyanenko and Skiba [9, 10] introduced a somewhat different subgroup embedding property which is also used to characterise soluble PST-groups. A subgroup H of a group G is called τ-semipermutable or τ-quasinormal in G if it permutes with all Sylow subgroups Q of G such that $(|Q|, |H|) = 1$ and $(|H|, |Q^G|) \neq 1$, where Q^G denotes the normal closure of Q in G. In this paper we will use the term τ-semipermutable for such subgroups. It is clear that every S-semipermutable subgroup of a group G is τ-semipermutable. However, the converse does not hold in general as the following example illustrates.

Example 4 ([10]). Let $p < r < q$ be primes, R a cyclic r-group, Q a faithful irreducible R-module over $\mathbb{F}_q$ and $Q \rtimes R$ the semidirect product of Q by R. Let P be a faithful irreducible $Q \rtimes R$-module over $\mathbb{F}_p$ and let $G = P \rtimes (Q \rtimes R)$. Then $M = QR$ is a maximal subgroup of G such that $M_G = 1$ where M_G is the core of M in G. Assume that R is S-semipermutable in G. Then $RQ^x = Q^xR$ for any $x \in G$. By [7, Chapter IV, Satz 2.8] $R \leq N_G(Q^x)$ for all $x \in G$. Then $M = QR \leq N_G(Q) < G$. Since M is maximal in G, $M = N_G(Q)$ so that $M^x = N_G(Q)^x = N_G(Q^x)$ and $R \leq M^x$ for all $x \in G$. This means $R \leq M_G = 1$, a contradiction. Thus R is not S-semipermutable in G. Note that $Q^G \leq PQ$ so that R is τ-semipermutable in G.

In [10] the authors establish the following remarkable result.

Theorem 5 ([10]). *Let G be a group. The following are equivalent:*

(1) *Every subgroup of $F^*(G)$ is τ-semipermutable in G.*

(2) *G is a supersoluble group and every subgroup of $F(G)$ is τ-semipermutable in G.*

(3) *$G = L \rtimes M$, the semidirect product of L by M, where M is a nilpotent subgroup of G and L is a normal abelian Hall subgroup of G of odd order such that every subgroup of L is normal in G.*

(4) *G is a soluble group and every subnormal subgroup of G is S-permutable in G.*

We note that (3) and (4) are just Agrawal's characterisation of soluble PST-groups.

In Theorem 5, $F^*(G)$ denotes the generalized Fitting subgroup of G; that is, the set of all elements x of G which induce an inner automorphism on each chief factor of G. For a number of properties of $F^*(G)$, see [8, Chapter X]. For example, if G is a soluble group, then $F^*(G) = F(G)$, the Fitting subgroup of G.

A subgroup H of a group G is said to be τ-seminormal in G if H is normalized by every Sylow subgroup P of G such that $(|H|, |P|) = 1$ and $(|H|, |P^G|) \neq 1$. We note that every seminormal subgroup of G is τ-seminormal. We also note that a τ-seminormal subgroup is τ-semipermutable but the converse is false as the following example shows.

Example 6 ([2])**.** Let G be the semidirect product of a cyclic group of order 7 and its automorphism group, which is cyclic of order 6. Let H be a subgroup of G of order 14. Then if P is a Sylow subgroup of G whose order is relatively prime to $|H|$, the order of P will be 3, and its normal closure will be of order 21, which is not relatively prime to 14. P will permute with H but not normalize H. Hence H will be τ-semipermutable but not τ-seminormal in G.

Theorem 7 ([2])**.** *Let G be a group. The following statements are equivalent:*

(1) *Every subgroup of $F^*(G)$ is τ-seminormal in G.*

(2) *G is supersoluble and every subgroup of $F(G)$ is τ-seminormal in G.*

(3) *G is a soluble* PST-*group.*

(4) *G is soluble and every subgroup of $F(G)$ is seminormal in G.*

Note that because of part (4) of Theorem 7, only subgroups of $F(G)$ need to be seminormal for the soluble group to be a PST-group. This is an improvement of part (1) of Theorem 3.

We now consider the case when seminormality is a transitive relation, that is, if H and K are subgroups of a group G and H is a seminormal subgroup of K and K is seminormal in G, then H is seminormal in G. A group G is called an SNT-group (see [3]) if seminormality is a transitive relation in G.

Theorem 8 ([3]). *A soluble* SNT-*group is an* SN-*group. In particular, a soluble* SNT-*group is a* PST-*group.*

A soluble PST-group need not to be an SNT-group as the next example shows.

Example 9 ([3]). Let $L = \langle x \rangle$ be a cyclic group of order 49 and $A = \langle y \rangle$ be a cyclic group of order 6. Then we may assume A is a subgroup of the automorphism group of L. Put $G = L \rtimes A$, the semidirect product of L by A. Then G is a soluble PST-group. Let $K = \langle x^7, y \rangle$ and note that K is seminormal in G since $\pi(K) = \pi(G)$. Let $H = \langle x^7, y^2 \rangle$ and note that H is normal in K. Let $z = y^3 x$ and note that z has order 2 and does not normalize H. Thus H is seminormal in K but not seminormal in G. Therefore, G is a PST-group but not an SNT-group.

We now provide several properties of soluble SNT-groups.

Theorem 10 ([3]). *If H is a subgroup of a soluble* SNT-*group G, then H is an* SNT-*group.*

Let G be the soluble group in Example 2. Then G is an SNT-group which is not a BT-group. However, the following is true.

Theorem 11 ([3]). *If G is a soluble* BT-*group, then G is an* SNT-*group.*

Let L be the nilpotent residual of a PST-group G and let D be a system normalizer of G. By [4, 2.1.8], L is a normal abelian Hall subgroup of G and hence by [11, 9.2.7, p. 264], $G = L \rtimes D$ is a semidirect product of L by D and D is also a Hall subgroup of G.

Using the notation above, we now give a characterisation of soluble SNT-groups.

Theorem 12 ([3]). *Let G be a soluble group with nilpotent residual L and a system normalizer D. Let $\pi = \pi(L)$ and $\rho = \pi(D)$. Then G is an* SNT-*group if and only if G satisfies:*

(1) *G is a* PST-*group so that $G = L \rtimes D$.*

(2) *Let H be a seminormal subgroup of K and K a seminormal subgroup of G. For all primes $p \in \rho$ with $(|H|, p) = 1$ and a Sylow p-subgroup P, there is a Hall ρ-subgroup M of H such that $[M, P] = 1$.*

The theory of certain classes of groups has benefited greatly from local techniques. For a given group property Θ which determines a class of groups, we wish to find a somewhat weaker property Θ_p depending on a prime p such that a group G satisfies Θ if and only if it satisfies Θ_p for all primes p.

Let p be a prime. The first author and Esteban-Romero called a group G a $\mathcal{Y}_p$-group if whenever K is a p-subgroup of G, then every subgroup of K is S-permutable in $N_G(K)$ (see [4, 2.2.1]). They proved that a group G is a soluble PST-group if and only if G is a $\mathcal{Y}_p$-group for all primes p ([4, 2.2.9]). The first author, Esteban-Romero and the fifth author also proved in [5, Theorem B] the following result.

Theorem 13. *A group G is a soluble* PST*-group if and only if G is a $\mathcal{Y}_p$-group for all primes p dividing the order of $F^*(G)$.*

The following classes related to $\mathcal{Y}_p$ were introduced in [1, 2, 6].

Definition 14. Let p be a prime and G a group. G is called a

(1) $\widehat{Y}_p$-group if for every p-subgroup K of G every subgroup of K is semipermutable in $N_G(K)$.

(2) $\widetilde{Y}_p$-group if for every p-subgroup K of G every subgroup of K is S-semipermutable in $N_G(K)$.

(3) $\widetilde{\widetilde{Y}}_p$-group if for every p-subgroup K of G every subgroup of K is seminormal in $N_G(K)$.

(4) Y'_p-group if whenever K is a p-subgroup of G every subgroup of K is τ-semipermutable in $N_G(K)$.

(5) Y''_p-group if whenever K is a p-subgroup of G every subgroup of K is τ-seminormal in $N_G(K)$.

Note that the classes (1)-(5) presented above relate a number of subgroup embeddings discussed in this paper. The next two theorems were established in [2].

Theorem 15. $Y_p = \widehat{Y}_p = \widetilde{Y}_p = \widetilde{\widetilde{Y}}_p = Y'_p = Y''_p$.

Theorem 16. *Let G be a group. The following are equivalent:*

(1) *G is a soluble* PST*-group.*

(2) *G is a* Y_p*-group for all primes* $p \in \pi(F^*(G))$.

(3) *G is a* $\widehat{Y}_p$*-group for all primes* $p \in \pi(F^*(G))$.

(4) *G is a* $\widetilde{Y}_p$*-group for all primes* $p \in \pi(F^*(G))$.

(5) *G is a* $\widetilde{\widetilde{Y}}_p$*-group for all primes* $p \in \pi(F^*(G))$.

(6) *G is a* Y'_p*-group for all primes* $p \in \pi(F^*(G))$.

(7) *G is a* Y''_p*-group for all primes* $p \in \pi(F^*(G))$.

Acknowledgements

The first author has been supported by the research grant MTM2010-19938-C03-01 from MICINN (Spain).

References

[1] Kahled A. Al-Sharo, James C. Beidleman, Hermann Heineken, and Matthew F. Ragland. Some characterizations of finite groups in which semipermutability is a transitive relation. *Forum Math.*, 22(5):855–862, 2010.

[2] A. Ballester-Bolinches, J. C. Beidleman, and A. D. Feldman. Some new characterizations of solvable PST-groups. Accepted in *Ric. Mat.*

[3] A. Ballester-Bolinches, J. C. Beildleman, A. D. Feldman, H. Heineken, and M. F. Ragland. Finite solvable groups in which semi-normality is a transitive relation. *Beitr. Algebra Geom.*, in press. DOI 10.1007/s13366-012-0099-1.

[4] A. Ballester-Bolinches, R. Esteban-Romero, and M. Asaad. *Products of finite groups*, volume 53 of *de Gruyter Expositions in Mathematics*. Walter de Gruyter, Berlin, 2010.

[5] A. Ballester-Bolinches, R. Esteban-Romero, and M. Ragland. A note on finite PST-groups. *J. Group Theory*, 10(2):205–210, 2007.

[6] James C. Beidleman and Matthew F. Ragland. Subnormal, permutable, and embedded subgroups in finite groups. *Cent. Eur. J. Math.*, 9(4):915–921, 2011.

[7] B. Huppert. *Endliche Gruppen I*, volume 134 of *Grund. Math. Wiss.* Springer Verlag, Berlin, Heidelberg, New York, 1967.

[8] B. Huppert and N. Blackburn. *Finite groups III*, volume 243 of *Grund. Math. Wiss.* Springer-Verlag, Berlin, 1982.

[9] V. O. Lukyanenko and A. N. Skiba. On weakly τ-quasinormal subgroups of finite groups. *Acta Math. Hungar.*, 125(3):237–248, 2009.

[10] Vladimir O. Lukyanenko and Alexander N. Skiba. Finite groups in which τ-quasinormality is a transitive relation. *Rend. Semin. Mat. Univ. Padova*, 124:231–246, 2010.

[11] D. J. S. Robinson. *A course in the theory of groups.* Springer-Verlag, New-York, 1982.

[12] L. Wang, Y. Li, and Y. Wang. Finite groups in which (S-)permutability is a transitive relation. *Int. J. Algebra*, 2(1-4):143–152, 2008.

Contact information

A. Ballester-Bolinches Departament d'Àlgebra, Universitat de València; Dr. Moliner, 50; 46100, Burjassot, València, Spain
E-Mail: Adolfo.Ballester@uv.es

J. C. Beidleman Department of Mathematics, University of Kentucky, Lexington, KY 40506-0027, USA
E-Mail: clark@ms.uky.edu

A. D. Feldman Department of Mathematics, Franklin & Marshall College, Lancaster, PA 17604-3003, USA
E-Mail: afeldman@fandm.edu

H. Heineken Institut für Mathematik, Universität Würzburg, Emil-Fischer-Strasse 30, 97074 Würzburg, Germany
E-Mail: heineken@mathematik.uni-wuerzburg.de

M. F. Ragland Department of Mathematics, Auburn University Montgomery, P.O. Box 244023, Montgomery, AL 36124-4023 USA
E-Mail: mragland@aum.edu

Received by the editors: 02.04.2012
and in final form 19.05.2012.