Algebra and Discrete Mathematics SURVEY ARTICLE
Volume 13 **(2012)**. Number 2. pp. 220 – 236

Survey of generalized pregroups and a question of Reinhold Baer

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Communicated by I. Ya. Subbotin

Abstract. There has been recent interest in Stallings' Pregroups. (See [2] and [12].) This paper gives a survey of generalized pregroups. We also answer a question of Reinhold Baer [1] on pregroups and answer a generalization of this question for generalized pregroups.

1. Preliminary results

There has been recent interest in Stallings' Pregroups. For example:

- [12] Pregroups and the Big Powers Condition: Kvaschuk, Miasnikov, Serbin, Algebra and Logic, Vol. 48, No. 3, 2009
- [2] Geodesic Rewriting Systems and Pregroups, Diekert, Duncan, Miasnikov, 2009, Preprint

First we give some preliminary results.

Let P be a nonempty set with a partial operation, called an "add" by Baer [1] (1950). Formally, a partial operation on P is a mapping m: $D \to P$ where $D \subseteq P \times P$. If (p, q) belongs to D, we denote $m(p, q)$ by pq and say that pq is defined or exists. (Baer denoted $m(p, q)$ by $p + q$.)

An add P will be called a BS-pree or simply a *pree* (term invented by Rimlinger [15]) if it satisfies the following three axioms of Stallings:

2000 MSC: Primary 20E06.
Key words and phrases: Pregroups, Kushner Axiom K. small cancellation.

[P1] (Identity) There exists $1 \in P$ such that for all a, we have $1a$ and $a1$ are defined and $1a = a1 = a$.

[P2] (Inverses) For each $a \in P$, there exists $a^{-1} \in P$ such that aa^{-1} and $a^{-1}a$ are defined, and $aa^{-1} = a^{-1}a = 1$

[P4]=[A] (Weak Associative Law) If ab and bc are defined, then $(ab)c$ is defined if and only if $a(bc)$ is defined, in which case $(ab)c = a(bc)$. (We then say the triple abc is defined.)

Remark 1.1. Stallings also gave the axiom:

[P3] If ab is defined, then $b^{-1}a^{-1}$ is defined and $(ab)^{-1} = b^{-1}a^{-1}$.

However, one can show that **[P3]** follows from **[P1]**, **[P2]**, and **[P4]**.

It is not difficult to show that: (i) inverses are unique in a pree, (ii) if ab is defined, then $(ab)b^{-1} = a$ and $a^{-1}(ab) = b$.

A sequence $X = [a_1, a_2, ..., a_n]$ of n elements of P is called a *word with length* $|X| = n$. The word $X = [a_1, a_2, ..., a_n]$ is said to be *defined* if each pair

$$a_1a_2, a_2a_3, ..., a_{n-1}a_n$$

is defined. A *triple* in X is a subsequence $a_ia_{i+1}a_{i+2}$.

A prouct $ab = c$ in a pree may be viewed as a triangle as shown in Fig. 1-1. Bob Gilman [4] noted that the associative law is equivalent to the statement that if three triangles in a pree P fit around a common vertex then the perimeter is also a valid triangle in P. Figure 1-2 illustrates the associative law; that is, the side X is equal to $a(bc)$ and also $(ab)c$.

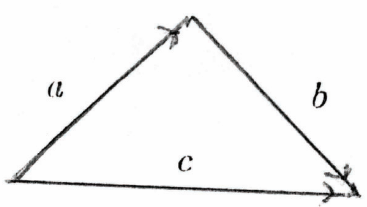

Fig. 1-1 Product $ab = c$

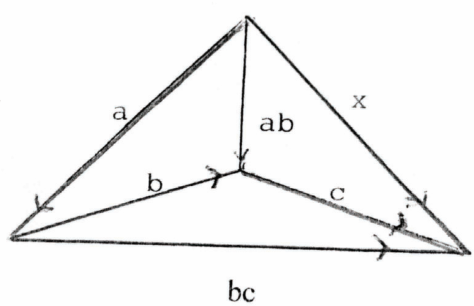

Fig.1-2 Associative law

Definition 1.2. The *universal group* $G(P)$ of a pree P is the group with presentation $G(P) = gp(P;$ operation $m)$

That is, P is the set of generators for $G(P)$ and the defining relations of $G(P)$ are of the form $z = xy$ where $m(x, y) = z$.

Definition 1.3. A pree P is said to be *group-embeddable* or simply *embeddable* if P can be embedded in its universal group $G(P)$.

Theorem 1.4. *The question of whether or not a finite pree P embeds in its universal group $G(P)$ is undecidable.*

Bob Gilman [4] noted that this theorem is a special case of a result of Trevor Evans [Embeddable and the word problem] which says that if the embedding problem is solvable for a class of finite partial algebras, then the word problem is solvable for the corresponding class of algebras.

Next follows classical examples of embeddable prees.

Example 1.5. Let K and L be groups with isomorphic subgroups A, pictured in Fig. 1-3. Then the amalgam $P = K \cup_A L$ is a pree which is embeddable in $G(P) = K *_A L$, the free product of K and L with A amalgamated. A typical element w in $G(P)$ is of the form $w = a$ in A or $w = x_1 y_1 \cdots x_n y_n$ where x_i and y_i come from different factors in $G(P)$ outside of A.

$K \overset{A}{\text{———}} L$ $\qquad$ $K \overset{A}{\text{———}} H \overset{B}{\text{———}} L$

Fig 1-3 $\qquad$ Fig 1-4

Example 1.6. Let K, H, L be groups. Suppose K and H have isomorphic groups A, and suppose H and L have isomorphic groups B, pictured in Fig. 1-4. Then the amalgam $P = K \cup_A H \cup_B L$ is a pree which is embeddable in $G(P) = K *_A H *_B L$ the free product of K,H, L with subgroups A and B amalgamated.

Example 1.7. Let $T = (K_i; A_{rs})$ be a *tree* graph of groups with vertex groups K_i, and with edge groups A_{rs}. Here A_{rs} is a subgroup of vertex groups K_r and K_s. Let $P = \bigcup_i (K_i; A_{rs})$, the amalgam of the groups in T. Then P is a pree which is embeddable in $G(P) = *(K_i; A_{rs})$, the *tree product* of the vertex groups K_i with the subgroups A_{rs} amalgamated.

Example 1.8. Let $G = (K_i; A_{rs})$ be a graph of groups with vertex groups K_i and with edge groups A_{rs}. Again A_{rs} is a subgroup of vertex groups K_r and K_s. Let $P = \bigcup_i (K_i; A_{rs})$. Then P is a pree but P may not be embeddable in $G(P) = *(K_i; A_{rs})$, the free product of groups K_i with the subgroups A_{rs} amalgamated. In fact, there are cases where $G(P) = \{e\}$.

2. Stallings' pregroup

Overall Problem: Find additional axioms so that a pree P is embeddable.

Notation: If X is a set of axioms, then an X-pree will be a pree which also satisfies the axioms in X.

Stallings [16] (1971) invented the name "pregroup" for a pree P and the following axiom:

[P5]=**[T1]** If ab, bc, and cd are defined, then abc or bcd is defined.

[The reason for the 1 in [**T1**] is explained in Remark 6.3.]

Theorem 2.1. *(Stallings): A pregroup P is embedded in $G(P)$.*

[**Note:** A pregroup P is a **T1**-pree.]

We quickly outline Stallings' proof of the theorem. A word $w = (x_1, x_2, ..., x_n)$ is *reduced* if no $x_i x_{i+1}$ is defined. Suppose w is reduced and suppose $x_i a$ and $a^{-1}x_{i+1}$ are defined. Then one can show that

$$w * a = (x_1, x_2, ..., x_i a, a^{-1}x_{i+1}, ..., x_n)$$

is also reduced. Stallings called $w * a$ an *interleaving* of w by a.

Define $w \approx v$ if v can be obtained from w by a sequence of interleavings.

Lemma 2.2. *$w \approx v$ is an equivalence relation on the set of reduced words.*

Lemma 2.3. *For any $a \in P$, we define f_a on reduced words by:*

$$f_a(x_1, x_2, ..., x_n) = \begin{cases} (a, x_1, x_2, ..., x_n) & \textit{if } ax_1 \textit{is not defined,} \\ (ax_1, x_2, ..., x_n) & \textit{if } ax_1 \textit{ is defined,} \\ & \textit{but } ax_1x_2 \textit{ is not defined,} \\ (ax_1x_2, x_3 ..., x_n) & \textit{if } ax_1x_2 \textit{ is defined.} \end{cases}$$

Lemma 2.4. *f_a is a permutation on the equivalence classes of reduced words.*

Lemma 2.5. ***(Main Lemma):*** *If ab is defined then $f_{ab} = f_a f_b$.*

The proof of the main lemma consists of the nine possibilities of f_{ab}.

Theorem 2.6. *$G(P) = \{permuations\ f_a\}$and P is embedded in $G(P)$ by*

$$a \mapsto f_a.$$

Remark 2.7. The pree $P = K \cup_A L$ in Example 1.5 is an example of a pregroup.

3. Baer's question

Reinhold Baer ["Free sums of groups and their generalizations", 1950, [1]] also considered the embedding of prees. In particular, the following appears in his paper:

Postulate XI: (Consists of three parts)

(a) If ab, bc, cd exist, then $a(bc)$ or $(bc)d$ exist.

(b) If bc, cd and $a(bc)$ exist, then ab or $(bc)d$ exist.

(c) If ab, bc and $(bc)d$ exist, then $a(bc)$ or cd exist.

Baer then states:

"In certain instances it is possible to deduce properties (b), (c) from (a); but whether or not this is true in general, the author does not know."

The following theorem (L. and Shi, [14]) answers Baer's question:

Theorem 3.1. *The following conditions on a pree P are equivalent.*

(i) [**P5**] = [**T1**]: *If ab, bc, cd are defined, then $a(bc)$ or $(bc)d$ is defined.*

(ii) [**A1**]: *If $ab, (ab)c, ((ab)c)d$ are defined then bc or cd is defined.*

(iii) [**A2**]: *If $cd, b(cd), a(b(cd))$ are defined, then ab or bc is defined.*

(iv) [**A3**]: *If $bc, cd, a(bc)$ are defined, then ab or $(bc)d$ is defined.*

(v) [**A4**]: *If $ab, bc, (bc)d$ are defined, then $a(bc)$ or cd is defined.*

Note: [**P5**] = [**T1**] is Baer's (a), [**A3**] is Baer's (b) and [**A4**] is Baer's (c).

Corollary 3.2. *Let P be a pree which satisfies one of the axioms in Theorem 3.1. Then P is embeddable in its universal group $G(P)$.*

4. Kushner's generalization of a pregroup. T2-prees

Note again that $G = K *_A L$ in Example 1.5 is a pregroup since [**P5**] = [**T1**] does hold in G. However, $G = K *_A H *_B L$ in Example 1.6 is not a pregroup since[**P5**] = [**T1**] does not hold in G. For example, let $x \in K\backslash A, y \in L\backslash B, a \in A, b \in B$, as pictured in Fig. 4-1. Then $xa \in K, ab \in H$and $by \in L$ are defined, but xab and aby need not be defined.

$$\begin{array}{ccccc} x \to K & \text{———} & H & \text{———} & L \leftarrow y \\ A & & & & B \\ \uparrow & & & & \uparrow \\ a & & & & b \end{array}$$

Fig.4-1

On the other hand, $G = K *_A H *_B L$ does satisfy the axiom:

[**T2**] If ab,bc, cd, de are defined, then abc, bcd, or cde is defined. That is, if $X = [a, b, c, d, e]$ is defined, then a triple in X is defined.

Theorem 4.1 (Kushner)**.** *Let P be a* **T2***-pree. Then P is embeddable in $G(P)$.*

We outline the proof of Kushner's theorem.

Recall that in a pregroup, a reduced word is still reduced under an interleaving. This is not true for a **T2**-pree. For example, let $x \in K\backslash A, y \in L\backslash B, a \in A, b \in B$,as pictured in Fig.4-1. The word $w = [x, ab, y]$ is reduced in $G = K *_A H *_B L$. But

$$w * a = [xa, a^{-1}(ab), y] = [xa, b, y]$$

is not reduced since by is defined. Thus a reduced word in a **T2**-pree may not be reduced by an interleaving.

The following definitions are new.

Definition 4.2. The word $w = (x_1, x_2, ..., x_n)$ is *fully reduced* if w is reduced and w is reduced under any sequence of interleavings.

Definition 4.3. Suppose $w = (x_1, x_2, ..., x_n)$ is reduced and suppose $x_i = ab$ where $x_{i-1}a$ and bx_{i+1} are defined. Then x_i is said to *split* in w, and w is *reducible* to $v = (x_1, ..., x_{i-1}a, bx_{i+1}, ...x_n)$.

Note first that if w is reducible to v then $|v| < |w|$. Note also that in the above reduced word $w = [x, ab, y]$, the element ab splits in w, and w is reducible to $v = [xa, by]$.

Lemma 4.4. *(Main Lemma) If $w = (x_1, x_2, ..., x_n)$ is reduced in a* **T2***-pree P, but not fully reduced, then some x_i in w splits.*

That is, w is fully reduced if and only if w is *nonsplitable.*

Define $w \approx v$ if v can be obtained from w by a sequence of interleavings.

Lemma 4.5. *$w \approx v$ is an equivalence relation on the set of fully reduced words.*

If $w = (x_1, x_2, ..., x_n)$ is fully reduced, then $f_a(w) = f_a(x_1, x_2, ..., x_n)$ has 5 possible cases (rather the 3 in a pregroup). Thus then following lemma requires 25 cases (not 9).

Lemma 4.6. $f_{ab} = f_a f_b$.

Theorem 4.7. *$G(P) = \{permuations \;\; f_a\}$ and P is embedded in $G(P)$ by*

$$a \mapsto f_a.$$

5. Baer's question for T2-prees. Open questions for T2-prees

The following theorem generalizes Bair's question for the axiom [**T2**].

Theorem 5.1 (Gaglione, L, Spellman, 2010). *The following are equivalent in a pree P where a, b, c, d, e are elements in P.*

1) [**T2**] *If ab, bc, cd, de are defined, then $a(bc), b(cd)$, or $c(de)$ is defined.*
2) [**B1**] *If $bc, cd, a(bc), (cd)e$ are defined, then $ab, (bc)d$, or de is defined.*
3) [**B2**] *If $ab, (ab)c, de, c(de)$ are defined, then bc, cd, or $(ab)c(de)$ is defined.*

We Prove Theorem 5.1 in Section 9.

5.1. Transitive order in a pree

The following transitive order relation on a pree P is due to Stallings:

Definition 5.2. Let $L(x) = \{a \in P : ax \text{ is defined}\}$. Put $x \leq y$ if $L(y) \subseteq L(x)$ and $x < y$ if $L(y) \subseteq L(x)$ and $L(y) \neq L(x)$. Also, we let $x \sim y$ if $L(x) = L(y)$.

Example 5.3. Let $P = K \cup_A L$ as in Fig.1-3. Let $x \in K \backslash A$, $y \in K \backslash A$, and $a \in A$. Then $L(x) = K,\ L(y) = K,\ L(a) = P$. Thus, $a < x$ and $a < y$. Also, $x \sim y$.

Theorem 5.4 (Rimlinger, Hoare). *The following conditions on a pree P are equivalent.*

(i) [**P5**] = [**T1**]: *If ab, bc, cd are defined, then $a(bc)$ or $(bc)d$ is defined.*

(ii) *If* $x^{-1}a$*and* $a^{-1}y$ *are defined but* $x^{-1}y$ *is not defined, then* $a < x$ *and* $a < y$.

(iii) *If* $x^{-1}y$ *is defined, then* $x \leq y$ *or* $y \leq x$.

Problem (1): Find analogous conditions which are equivalent to [**T2**].

Theorem 5.5. *(Hoare, Chiswell) The universal group* $G(P)$ *of a pregroup* P *admits an integer-valued length function in the sense of Lyndon.*

Problem (2): Prove that an integer-valued length function (in the sense of Lyndon) exists for the universal group $G(P)$ for a **T2**-pree P.

6. Kushner's axiom K, generalizing [T2]

The proof by Kushner (in his doctoral thesis) that a **T2**-pree is embeddable was very long and involved (for example, the proof of $f_{ab} = f_a f_b$ required 25 cases instead of 9 cases). Thus the following localization axiom was added in order to shorten the proof:

[**K**] If ab, bc, cd and $(ab)(cd)$ are defined, then abc or bcd is defined.

Theorem 6.1 (Kushner-L)**.** *Let* P *be a* **KT2**-*pree. Then* P *is embeddable in* $G(P)$.

After the paper appeared, Hoare independently obtained Kushner's original result with a considerably shorter and less involved proof (by reducing the proof of $f_{ab} = f_a f_b$ to only 9 cases):

Theorem 6.2 (Hoare)**.** *Let* P *be a* **T2**-*pree. Then* P *is embeddable in* $G(P)$.

Consider the following axioms for $n \geq 1$.

[**Tn**] If $X = [a_1, a_2, ..., a_{n+3}]$ is defined, then some triple in X is defined.

That is, if $a_1a_2, a_2a_3, ..., a_{n+2}a_{n+3}$ are defined, then $(a_1a_2)a_3$, $(a_2a_3)a_4, \ldots$, or $(a_{n+1}a_{n+2})a_{n+3}$ is defined.

Remark 6.3. We emphasize that [**Tn**] holds for a tree pree P in Example 1.7 when the diameter of the tree does not exceed n.

Theorem 6.4 (Kushner-L, 1993)**.** *Let* P *be a* **KT3**-*pree. Then* P *is embeddable in* $G(P)$.

We note that Theorem 6.4 requires Axiom [**K**]. The proof of the above theorem again requires:

Lemma 6.5. *(Main Lemma) Let P be a* **KT3**-*pree. If $w = (x_1, x_2, ..., x_n)$ is reduced but not fully reduced, then some x_i in w splits.*

Problem (3): Prove that if P is a **T3**-pree, then P is embeddable in $G(P)$.

7. Further generalization

We extend the above Theorem 6.4 to all tree products of groups with finite diameters.

Theorem 7.1. *(L) Let P be a* **KTn**-*pree. Then P is embeddable in $G(P)$.*

The above theorem requires a generalizing of the notion of a splitting. Specifically:

Definition 7.2. Let $w = (x_1, x_2, ..., x_n) = (x_1, a_2b_2, a_3b_3, a_4b_4, ..., a_{n-1}b_{n-1}, x_n)$ where $x_1a_2, b_2a_3, b_3a_4, ..., b_{n-1}x_n$ are defined. Then we say w is *reducible* to

$$v = (x_1a_2, b_2a_3, b_3a_4, ..., b_{n-1}x_n)$$

and the factorization $a_2b_2, a_3b_3, a_4b_4, ..., a_{n-1}b_{n-1}$ is called a *general splitting* of w.

Remark 7.3. We note that in the above general splitting, $|v| < |w|$.

Example 7.4. Figure 7-1 illustrates a general splitting. Specifically, $w = [x, ab, cd, y]$ need not be reduced where $x \in K_1$, $y \in K_5$, $a \in A$, $b \in B$, $c \in C$, $d \in D$. Also, ab need not split and cd need not split. However, xa, bc and dy are defined. Accordingly, $w = [x, ab, cd, y]$ reduces, by a general splitting, to $v = [xa, bc, dy]$.

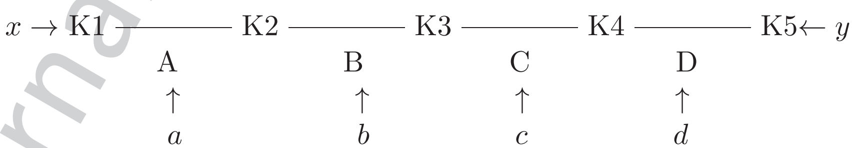

Fig. 7-1

The following Lemma is essential in the proof of Theorem 7.1.

Lemma 7.5. *Suppose w is reduced but not fully reduced in a* **KTn***-pree. Then w contains a general splitting.*

We would like to find a theorem which generalizes Bair's question for Axiom [**Tn**]. Theorem5.1 answers Baire's question for axiom [**T2**]. We do have an answer to Baer's question for Axiom [**T6**] which we prove in Section 10. Specifically:

Theorem 7.6. *The following axioms,* [**T6**], [**C6-1**], *and* [**C6-2**], *are equivalent in a pree P:*

[**T6**] *Suppose $X = [a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9]$ is defined, that is, each $a_i a_{i+1}$ is defined. Then a triple in X is defined.*

[**C6-1**] *Suppose all the following are defined:*

$$\begin{aligned}&(1)\ b_2b_3,\ \ b_3b_4,\ \ b_1(b_2b_3),\ \ (b_3b_4)b_5,\\&(2)\ b_6b_7,\ \ b_7b_8,\ \ b_5(b_6b_7),\ \ (b_7b_8)b_9.\end{aligned}$$

Then one of the following is defined:

$$b_1b_2,\ \ (b_2b_3)b_4,\ \ b_4b_5,\ \ (b_3b_4)b_5(b_6b_7),\ \ b_5b_6,\ \ (b_6b_7)b_8,\ or\ b_8b_9.$$

[**C6-2**] *Suppose all the following are defined:*

$$\begin{aligned}&(1)\ b_1b_2, (b_1b_2)b_3, b_4b_5, b_3(b_4b_5),\\&(2)\ b_5b_6,\ \ (b_5b_6)b_7,\ \ b_8b_9,\ \ b_7(b_8b_9).\end{aligned}$$

Then one of the following is defined:

$$b_2b_3,\ (b_1b_2)b_3(b_4b_5),\ b_3b_4,\ (b_4b_5)b_6,\ b_6b_7,\ (b_5b_6)b_7(b_8b_9),\ or\ b_7b_8.$$

Remark 7.7. Note that (2) in both cases [**C6-1**] and [**C6-2**] can be obtained from (1) by adding 4 to each subscript.

Remark 7.8. The proof of Theorem 7.6 for [**T6**] is very similar to the proof of Theorem 5.1 for [**T2**] by mainly adding 4 to various subscripts. Likely one can prove an analogous theorem for [**Tm**] where $m \equiv 2(\mod 4)$.

Problem (4): Find a theorem which generalizes Bair's question for axioms [**T3**], [**T4**] and/or [**T5**].

8. Further, further generalizations

Consider Baer's (1953) axioms:

[$\mathbf{S_n}$, $n \geq 4$] Suppose $a_1^{-1}a_2 = b_1$, $a_2^{-1}a_3 = b_2, \ldots, a_{n-1}^{-1}a_n = b_{n-1}$, $a_n^{-1}a_1 = b_n$ are defined in a pree P. Then at least one of the products b_ib_{i+1} is also defined. (The product may be b_nb_1.) In other words, for some $i, a_i^{-1}a_{i+2}(mod\ n)$ is defined.

Definition 8.1. An **S**-pree is a pree P which satisfies all axioms $\mathbf{S_n}$ for $n \geq 4$.

Axiom $\mathbf{S_n}$ is illustrated in Fig. 8-1.

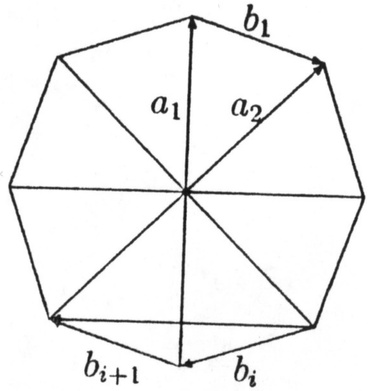

Fig. 8-1

Theorem 8.2. *(Baer) Let P be an* **S***-pree. Then P is embeddable in $G(P)$.*

Consider two other axioms:

[L] Suppose ab, bc, cd are defined, but $[ab, cd]$ and $[a, bc, d]$ are reduced. If $(ab)z$ and $z^{-1}(cd)$ are defined, then bz and $z^{-1}c$ are defined.

[M] Equivalent fully reduced words have the same length.

Axiom **[M]**, which we call Baer's axiom, is analogous to his axiom: "Similar irreducible vectors have the same length"

Theorem 8.3. *(L, 1996) Let P be a* **KLM***-pree. Then P is embeddable in $G(P)$.*

The theorem requires the following proposition which is due to Hoare:

Proposition 8.4 (Hoare)**.** *In a KLM-pree, X is fully reduced if and only if X is nonsplittable.*

Remark 8.5. A **KLM**-pree includes all tree products of groups, even those without finite diameter.

Theorem 8.6 (Gilman (preprint), Hoare 1998)**.** *Let P be a KL-pree = S_4S_5-pree. Then P is embeddable in $G(P)$.*

Hoare proved the theorem by showing that axiom [**M**] follows from [**K**] and [**L**].

Gilman proved the theorem using small-cancellation. In particular, Gilman's preprint ["Generalized small cancelation presentations"] indicates an intimate relationship between pregroups and small cancellation theory.

9. Proof of Theorem 5.1

First we restate Theorem 5.1 using different letters for axioms [**T2**], [**B1**], and [**B2**].

Theorem 9.1. *The following are equivalent in a pree P:*

[**T2**] *If $X = [a_1, a_2, a_3, a_4, a_5]$ is defined, then a triple in X is defined.*

[**B1**] *If $b_2b_3,\ b_3b_4,\ b_1(b_2b_3),\ (b_3b_4)b_5$ are defined, then one of the following is defined:*

$$b_1b_2,\quad (b_2b_3)b_4 \quad or \quad b_4b_5 \quad .$$

[**B2**] *If $b_1b_2,\ (b_1b_2)b_3,\ b_4b_5,\ b_3(b_4b_5)$ are defined, then one of the following is defined:*

$$b_2b_3,\ b_3b_4,\ or\ (b_1b_2)b_3(b_4b_5).$$

Lemma 9.2. [**T2**] *and* [**B1**] *are equivalent.*

(1) Assume [**T2**] holds. Suppose the hypothesis of [**B1**] holds, that is, suppose $b_2b_3,\ b_3b_4,\ b_1(b_2b_3),\ (b_3b_4)b_5$ are defined. Let

$$a_1 = b_1,\ a_2 = b_2b_3,\ a_3 = b_3^{-1},\ a_4 = b_3b_4,\ a_5 = b_5.$$

Then the hypothesis of [**T2**] holds, that is, $[a_1, a_2, a_3, a_4, a_5]$ is defined. By [**T2**], one of the following is defined:

$$a_1a_2a_3 = b_1b_2, \quad a_2a_3a_4 = (b_2b_3)b_4, \text{ or } a_3a_4a_5 = b_4b_5.$$

This is the conclusion of [**B1**]. Thus [**T2**] implies [**B1**].

(2) Assume [**B1**] holds. Suppose the hypothesis of [**T2**] holds, that is, suppose $[a_1, a_2, a_3, a_4, a_5]$ is defined. Let

$$b_1 = a_1, \; b_2 = a_2a_3, \; b_3 = a_3^{-1}, \; b_4 = a_3a_4, b_5 = a_5.$$

Then the hypothesis of [**B1**] holds, that is, $b_2b_3, \; b_3b_4, \; b_1(b_2b_3), \; (b_3b_4)b_5$ are defined. By [**B1**], one of the following is defined:

$$b_1b_2 = a_1a_2a_3, \; (b_2b_3)b_4 = a_2a_3a_4, \text{ or } b_4b_5 = a_3a_4a_5.$$

This is the conclusion of [**T2**]. Thus [**B1**] implies [**T2**].

By (1) and (2), [**T2**] and [**B1**] are equivalent in a pree P.

Lemma 9.3. [**T2**] *and* [**B2**] *are equivalent.*

(1) Assume [**T2**] holds. Suppose the hypothesis of [**B2**] holds, that is, suppose $b_1b_2, \; (b_1b_2)b_3, \; b_4b_5, \; b_3(b_4b_5)$ are defined. Let

$$a_1 = b_1^{-1}, \; a_2 = b_1b_2, \; a_3 = b_3, \; a_4 = b_4b_5, \; a_5 = b_5^{-1}.$$

Then the hypothesis of [**T2**] holds, that is, $[a_1, a_2, a_3, a_4, a_5]$ is defined. By [**T2**], one of the following is defined:

$$a_1a_2a_3 = b_2b_3, \; a_2a_3a_4 = (b_1b_2)b_3(b_4b_5), \text{ or } a_3a_4a_5 = b_3b_4.$$

This is the conclusion of [**B2**]. Thus [**T2**] implies [**B2**].

(2) Assume [**B2**] holds. Suppose the hypothesis of [T2] holds, that is, suppose $[a_1, a_2, a_3, a_4, a_5]$ is defined. Let

$$b_1 = a_1^{-1}, \; b_2 = a_1a_2, \; b_3 = a_3, \; b_4 = a_4a_5, \; b_5 = a_5^{-1}.$$

Then the hypothesis of [**B2**] holds, that is, $b_1b_2, \; (b_1b_2)b_3, \; b_4b_5, \; b_3(b_4b_5)$ are defined. By [**B2**], one of the following is defined:

$$b_2b_3, = a_1a_2a_3, \; b_3b_4 = a_3a_4a_5, \text{ or } (b_1b_2)b_3(b_4b_5) = a_2a_3a_4.$$

This is the conclusion of [**T2**]. Thus [**B2**] implies [**T2**].

By (1) and (2), [**T2**] and [**B2**] are equivalent in a pree P.

Lemma 9.2 and Lemma 9.3 prove Theorem 5.1.

10. Proof of Theorem 7.6.

First we restate Theorem 7.6.

Theorem 10.1. *The following are equivalent in a pree P, where a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9 are elements in P.*

[T6] *Suppose $X = [a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9]$ is defined, that is, each $a_i a_{i+1}$ is defined. Then a triple in X is defined.*

[C6-1] *Suppose all the following are defined:*

$$(1)\quad b_2b_3,\ b_3b_4,\ b_1(b_2b_3),\ (b_3b_4)b_5,$$
$$(2)\quad b_6b_7,\ b_7b_8,\ b_5(b_6b_7),\ (b_7b_8)b_9.$$

Then one of the following is defined:

$$b_1b_2,\ (b_2b_3)b_4,\ b_4b_5,\ (b_3b_4)b_5(b_6b_7),\ b_5b_6, (b_6b_7)b_8,\ \text{or } b_8b_9.$$

[C6-2] *Suppose all the following are defined:*

$$(1)\quad b_1b_2,\ (b_1b_2)b_3,\ b_4b_5,\ b_3(b_4b_5),$$
$$(2)\quad b_5b_6,\ (b_5b_6)b_7,\ b_8b_9,\ b_7(b_8b_9).$$

Then one of the following is defined:

$$b_2b_3,\ (b_1b_2)b_3(b_4b_5),\ b_3b_4,\ (b_4b_5)b_6,\ b_6b_7,\ (b_5b_6b_7(b_8b_9),\ \text{or } b_7b_8.$$

Remark 10.2. Note that (2) in **[C6-1]** and (2) in **[C6-2]** can each be obtained from (1) by adding 4 to each subscript.

Lemma 10.3. *In a pree P, axiom* **[T6]** *is equivalent to* **[C6-1]**.

(1) Proof that **[T6]** implies **[C6-1]**.

Assume **[T6]** holds. Suppose the hypothesis of **[C6-1]** holds, that is, the following are defined:

$$(1)\quad b_2b_3,\ b_3b_4,\ b_1(b_2b_3),\ (b_3b_4)b_5,$$
$$(2)\quad b_6b_7,\ b_7b_8, b_5(b_6b_7),\ (b_7b_8)b_9.$$

Let

$$a_1 = b_1,\quad a_2 = b_2b_3,\quad a_3 = b_3^{-1},\quad a_4 = b_3b_4,$$
$$a_5 = b_5,\quad a_6 = b_6b_7,\quad a_7 = b_7^{-1},\quad a_8 = b_7b_8,\quad a_9 = b_9.$$

Then each a_ia_{i+1} is defined, that is, the hypothesis of **[T6]** holds. By **[T6]**, one of the following is defined:

$$\begin{array}{lll} a_1a_2a_3 = b_1b_2, & a_2a_3a_4 = (b_2b_3)b_4, & a_3a_4a_5 = b_4b_5, \\ a_4a_5a_6 = (b_3b_4)b_5(b_6b_7), & a_5a_6a_7 = b_5b_6, & a_6a_7a_8 = (b_6b_7)b_8, \\ \text{or } a_7a_8a_9 = b_8b_9. & & \end{array}$$

This is the conclusion of **[C6-1]**. Thus **[T6]** implies **[C6-1]**.

(2) Proof that **[C6-1]** implies [**T6**].

Assume **[C6-1]** holds. Suppose the hypothesis of [T6] holds, that is, suppose $a_1a_2, a_2a_3, \cdots, a_8a_9$ are defined. Let

$$\begin{array}{lll} b_1 & = & a_1,\ b_2 = a_2a_3,\ b_3 = a_3^{-1},\ b_4 = a_3a_4, \\ b_5 & = & a_5,\ b_6 = a_6a_7,\ b_7 = a_7^{-1},\ b_8 = a_7a_8, b_9 = a_9. \end{array}$$

Then the hypothesis of **[C6-1]** holds, that is, the following are defined:

$$\begin{array}{c} b_2b_3,\ b_3b_4,\ b_1(b_2b_3),\ (b_3b_4)b_5, b_6b_7, \\ b_7b_8,\ b_5(b_6b_7),\ (b_7b_8)b_9. \end{array}$$

By **[C6-1]**, one of the following is defined:

$$\begin{array}{rrr} b_1b_2 = a_1a_2a_3, & (b_2b_3)b_4 = a_2a_3a_4, & b_4b_5 = a_3a_4a_5, \\ (b_3b_4)b_5(b_6b_7) = a_4a_5a_6, & b_5b_6 = a_5a_6a_7, & (b_6b_7)b_8 = a_6a_7a_8, \\ \text{or } b_8b_9 = a_7a_8a_9. & & \end{array}$$

This is the conclusion of **[T6]**. Thus **[C6-1]** implies **[T6]**.

By (1) and (2), Lemma 10.3 is proved.

Lemma 10.4. *In a pree P, axiom* **[T6]** *is equivalent to* **[C6-2]**.

1) Proof that **[T6]** implies **[C6-2]**

Assume **[T6]** holds. Suppose the hypothesis of **[C6-2]** holds, that is, that the following are defined:

$$b_1b_2,\ (b_1b_2)b_3,\ b_4b_5,\ b_3(b_4b_5),\ b_5b_6,\ (b_5b_6)b_7,\ b_8b_9,\ b_7(b_8b_9).$$

Let:

$$\begin{array}{lll} a_1 & = & b_1^{-1},\ a_2 = b_1b_2,\ a_3 = b_3,\ a_4 = b_4b_5,\ a_5 = b_5^{-1}, \\ a_6 & = & b_5b_6,\ a_7 = b_7,\ a_8 = b_8b_9,\ a_9 = b_9^{-1}. \end{array}$$

Then each $a_i a_{i+1}$ is defined, that is, the hypothesis of **[T6]** holds. By **[T6]**, one of the following is defined:

$$\begin{aligned} a_1a_2a_3 &= b_2b_3,\ a_2a_3a_4 = (b_1b_2)b_3(b_4b_5),\ a_3a_4a_5 = b_3b_4,\ a_4a_5a_6 = (b_4b_5)b_6, \\ a_5a_6a_7 &= b_6b_7,\ a_6a_7a_8 = (b_5b_6)b_7(b_8b_9),\ \text{or } a_7a_8a_9 = b_7b_8. \end{aligned}$$

This is the conclusion of **[C6-2]**. Thus **[T6]** implies **[C6-2]**.

(2) Proof that **[C6-2]** implies **[T6]**.

Assume **[C6-2]** holds. Suppose the hypothesis of **[T6]** holds, that is, suppose $a_1a_2, a_2a_3, \cdots, a_8a_9$ are defined. Let:

$$\begin{aligned} b_1 &= a_1^{-1},\ b_2 = a_1a_2,\ b_3 = a_3,\ b_4 = a_4a_5,\ b_5 = a_5^{-1}, \\ b_6 &= a_5a_6,\ b_7 = a_7,\ b_8 = a_8a_9,\ b_9 = a_9^{-1}. \end{aligned}$$

Then the hypothesis of **[C6-2]** holds, that is, the following are defined:

$$b_1b_2,\ (b_1b_2)b_3,\ b_4b_5,\ b_3(b_4b_5),\ b_5b_6,\ (b_5b_6)b_7,\ b_8b_9, b_7(b_8b_9).$$

By **[C6-2]**, one of the following is defined:

$$\begin{aligned} b_2b_3 &= a_1a_2a_3,\ (b_1b_2)b_3(b_4b_5) = a_2a_3a_4,\ b_3b_4 = a_3a_4a_5,\ (b_4b_5)b_6 = a_4a_5a_6, \\ b_6b_7 &= a_5a_6a_7,\ (b_5b_6)b_7(b_8b_9) = a_6a_7a_8,\ \text{or } b_7b_8 = a_7a_8a_9. \end{aligned}$$

This is the conclusion of **[T6]**. Thus **[C6-2]** implies **[T6]**.

By (1) and (2), Lemma 10.4 is proved.

Lemma 10.3 and Lemma 10.4, prove Theorem 7.6.

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Received by the editors: 07.02.2012
and in final form 03.04.2012.