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Regular pairings of functors and weak (co)monads

Robert Wisbauer

Abstract. For functors $L : \mathbb{A} \to \mathbb{B}$ and $R : \mathbb{B} \to \mathbb{A}$ between any categories $\mathbb{A}$ and $\mathbb{B}$, a *pairing* is defined by maps, natural in $A \in \mathbb{A}$ and $B \in \mathbb{B}$,

$$\mathrm{Mor}_{\mathbb{B}}(L(A), B) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathrm{Mor}_{\mathbb{A}}(A, R(B)) \ .$$

(L, R) is an *adjoint pair* provided α (or β) is a bijection. In this case the composition RL defines a monad on the category $\mathbb{A}$, LR defines a comonad on the category $\mathbb{B}$, and there is a well-known correspondence between monads (or comonads) and adjoint pairs of functors.

For various applications it was observed that the conditions for a unit of a monad was too restrictive and weakening it still allowed for a useful generalised notion of a monad. This led to the introduction of *weak monads* and *weak comonads* and the definitions needed were made without referring to this kind of adjunction. The motivation for the present paper is to show that these notions can be naturally derived from pairings of functors (L, R, α, β) with $\alpha = \alpha \cdot \beta \cdot \alpha$ and $\beta = \beta \cdot \alpha \cdot \beta$. Following closely the constructions known for monads (and unital modules) and comonads (and counital comodules), we show that any weak (co)monad on $\mathbb{A}$ gives rise to a regular pairing between $\mathbb{A}$ and the category of *compatible (co)modules*.

Contents.

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1. Introduction

Similar to the unit of an algebra, the existence of a unit of a monad is essential for (most of) the interesting properties of the related structures. Yet, there are numerous applications for which the request for a unit of a monad is too restrictive. Dropping the unit completely makes the theory fairly poor and the question was how to weaken the conditions on a unit such that still an effective theory can be developed. The interest in these questions was revived, for example, by the study of *weak Hopf algebras* by G. Böhm et al. in [6] and *weak entwining structures* by S. Caenepeel et al. in [9] (see also [1], [8]). To handle this situation the theory of weak monads (and comonads) was developed and we refer to [5] for a recent account on this.

On any category, monads are induced by a pair of adjoint functors and, on the other hand, any monad (F, μ, η) induces an adjoint pair of functors, the free functor $\phi_F : \mathbb{A} \to \mathbb{A}_F$ and the forgetful functor $U_F : \mathbb{A}_F \to \mathbb{A}$, where $\mathbb{A}_F$ denotes the category of unital F-modules. This is all shown in Eilenberg-Moore [10].

In this correspondence the unitality of the monad is substantial and the purpose of the present paper is to exhibit a similar relationship between weak (co)monads and generalised forms of adjunctions. To this end, for functors $L : \mathbb{A} \to \mathbb{B}$ and $R : \mathbb{B} \to \mathbb{A}$ between categories $\mathbb{A}$ and $\mathbb{B}$, we consider maps

$$\mathrm{Mor}_{\mathbb{B}}(L(A), B) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathrm{Mor}_{\mathbb{A}}(A, R(B)) ,$$

required to be natural in $A \in \mathbb{A}$ and $B \in \mathbb{B}$. We call this a *pairing of functors*, or a *full pairing* if we want to stress that we have maps in both directions. Such a pairing is said to be *regular* provided α and β are regular maps, more precisely,

$$\alpha = \alpha \cdot \beta \cdot \alpha \quad \text{and} \quad \beta = \beta \cdot \alpha \cdot \beta.$$

In Section 2, regular pairings of functors are defined and some of their general properties are described.

Motivated by substructures showing up in pairings of functor, in Section 3.1, *q-unital monads* (F, μ, η) on $\mathbb{A}$ are defined as endofunctors $F : \mathbb{A} \to \mathbb{A}$ with natural transformations $\mu : FF \to F$ and $\eta : I_{\mathbb{A}} \to F$ (*quasi-unit*) and the sole condition that μ is associative. *(Non-unital) F-modules* are defined by morphisms $\varrho : F(A) \to A$ satisfying

$\varrho \circ \mu_A = \varrho \circ F(\varrho)$, and the category of all F-modules is denoted by $\underrightarrow{\mathbb{A}}_F$. For these data the free and forgetful functors,

$$\phi_F : \mathbb{A} \to \underrightarrow{\mathbb{A}}_F \quad \text{and} \quad U_F : \underrightarrow{\mathbb{A}}_F \to \mathbb{A}.$$

give rise to a full pairing. From this we define *regularity* of η and *compatibility* for the F-modules. The q-unital monad (F, μ, η) is said to be *r-unital* (short for *regular-unital*) provided η is regular and μ is compatible as an F-module. Now the free functor $\phi_F : \mathbb{A} \to \underline{\mathbb{A}}_F$ with the forgetful functor $U_F : \underline{\mathbb{A}}_F \to \mathbb{A}$ form a regular pairing, where $\underline{\mathbb{A}}_F$ denotes the (sub)category of compatible F-modules.

The dual notions for (non-counital) comonads are outlined in Section 4 and at the end of the section the comparison functors for a regular pairing (L, R, α, β) are considered (see 4.10).

In Section 5 we study the lifting of functors between categories to the corresponding categories of compatible modules or compatible comodules, respectively. This is described by generalising Beck's *distributive laws* (see [2]), also called *entwinings*, and it turns out that most of the diagrams are the same as for the lifting to unital modules (e.g. [22]) but to compensate the missing unitality extra conditions are imposed on the entwining natural transformation (e.g. Proposition 5.2). In this context we obtain a generalisation of Applegate's lifting theorem for (co)monads to weak (co)monads (Theorem 5.4, 5.8).

Lifting an endofunctor T of $\mathbb{A}$ to an endofunctor $\overline{T}$ of $\underline{\mathbb{A}}_F$ leads to the question when $\overline{T}$ is a weak monad (TF allows for the structure of a weak monad) and in Section 6 we provide conditions to make this happen.

The final Section 7 is concerned with weak monads (F, μ, η) and weak comonads (G, δ, ε) on any category $\mathbb{A}$ and the interplay between the respective lifting properties. Hereby properties of the lifting $\overline{G}$ to $\underline{\mathbb{A}}_F$ and the lifting $\widehat{F}$ to $\underline{\mathbb{A}}^G$ are investigated (see Theorems 7.9 and 7.10) which generalise observations known for weak bi-algebras (and weak Hopf algebras).

In our setting, notions like *pre-units*, *pre-monads*, *weak monads*, *demimonads*, *pre-A-corings*, *weak corings*, *weak Hopf algebras* from the literature (e.g. [1], [3], [7], [4], [21]) find their natural environment.

In the framework of 2-categories weak structures are investigated by Böhm et al. in [3], [4], [5] and an extensive list of examples of weak structures is given there.

2. Pairings of functors

Throughout $\mathbb{A}$ and $\mathbb{B}$ will denote arbitrary categories. By I_A, A or just by I, we denote the identity morphism of an object $A \in \mathbb{A}$, I_F or F stands for the identity natural transformation on the functor F, and $I_{\mathbb{A}}$ means the identity functor of a category $\mathbb{A}$. We write $F_{-,-}$ for the natural transformation of bifunctors determined by the maps $F_{A,A'}$: $\mathrm{Mor}_{\mathbb{A}}(A, A') \to \mathrm{Mor}_{\mathbb{B}}(F(A), F(A'))$ for $A, A' \in \mathbb{A}$.

Before considering regularity for natural transformations we recall basic properties of

2.1. Regular morphisms. Let A, A' be any objects in a category $\mathbb{A}$. Then a morphism $f : A \to A'$ is called *regular* provided there is a morphism $g : A' \to A$ with $fgf = f$. Clearly, in this case $gf : A \to A$ and $fg : A' \to A'$ are idempotent endomorphisms.

Such a morphism g is not necessarily unique. In particular, for gfg we also have $f(gfg)f = fgf = f$, and the identity $(gfg)f(gfg) = gfg$ shows that gfg is again a regular morphism.

If idempotents split in $\mathbb{A}$, then every idempotent morphism $e : A \to A$ determines a subobject of A, we denote it by eA.

If f is regular with $fgf = f$, then the restriction of fg is the identity morphism on fgA' and gf is the identity on gfA.

Examples for regular morphisms are retractions, coretractions, and isomorphisms. For modules M, N over any ring, a morphism $f : M \to N$ is regular if and only if the image and the kernel of f are direct summands in N and M, respectively.

This notion of regularity is derived from von Neumann regularity of rings. For modules (and in preadditive categories) it was considered by Nicholson, Kasch, Mader and others (see [14]). We use the terminology also for natural transformations and functors with obvious interpretations.

2.2. Pairing of functors. (e.g. [19, 2.1]) Let $L : \mathbb{A} \to \mathbb{B}$ and $R : \mathbb{B} \to \mathbb{A}$ be covariant functors. Assume there are morphisms, natural in $A \in \mathbb{A}$ and $B \in \mathbb{B}$,

$$\alpha : \mathrm{Mor}_{\mathbb{B}}(L(A), B) \to \mathrm{Mor}_{\mathbb{A}}(A, R(B)),$$
$$\beta : \mathrm{Mor}_{\mathbb{A}}(A, R(B)) \to \mathrm{Mor}_{\mathbb{B}}(L(A), B).$$

These maps correspond to two natural transformations between functors $\mathbb{A}^{op} \times \mathbb{B} \to \mathsf{Set}$. The quadruple (L, R, α, β) is called a *(full) pairing (of functors).*

Given such a pairing, the morphisms, for $A \in \mathbb{A}$, $B \in \mathbb{B}$,

$$\eta_A := \alpha_{A,L(A)}(I) : A \to RL(A) \quad \text{and} \quad \varepsilon_B := \beta_{R(B),B}(I) : LR(B) \to B$$

correspond to natural transformations

$$\eta : I_{\mathbb{A}} \to RL, \quad \varepsilon : LR \to I_{\mathbb{B}},$$

which we call *quasi-unit* and *quasi-counit* of (L, R, α, β), respectively.

From these the transformations α and β are obtained by

$$\begin{array}{rlcl} \alpha_{A,B} : & L(A) \xrightarrow{f} B & \longmapsto & A \xrightarrow{\eta_A} RL(A) \xrightarrow{R(f)} R(B), \\ \beta_{A,B} : & A \xrightarrow{g} R(B) & \longmapsto & L(A) \xrightarrow{L(g)} LR(B) \xrightarrow{\varepsilon_B} B. \end{array}$$

Thus a pairing (L, R, α, β) is also described by a quadruple $(L, R, \eta, \varepsilon)$.

Naturality of ε and η induces an associative product and a quasi-unit for the endofunctor $RL : \mathbb{A} \to \mathbb{A}$,

$$R\varepsilon L : RLRL \to RL, \quad \eta : I_{\mathbb{A}} \to RL,$$

and a coassociative coproduct and a quasi-counit for the endofunctor $LR : \mathbb{B} \to \mathbb{B}$,

$$L\eta R : LR \to LRLR, \quad \varepsilon : LR \to I_{\mathbb{B}}.$$

By the Yoneda Lemma we can describe compositions of α and β by the images of the identity transformations of the respective functors.

2.3. Composing α and β. Let (L, R, α, β) be a pairing with quasi-unit η and quasi-counit ε. The descriptions of α and β in 2.2 yield, for the identity transformations $I_L : L \to L$, $I_R : R \to R$,

$$\begin{array}{rcl} \alpha(I_L) & = & I_{\mathbb{A}} \xrightarrow{\eta} RL, \\ \beta \cdot \alpha(I_L) & = & L \xrightarrow{L\eta} LRL \xrightarrow{\varepsilon L} L, \\ \alpha \cdot \beta \cdot \alpha(I_L) & = & I_{\mathbb{A}} \xrightarrow{\eta} RL \xrightarrow{RL\eta} RLRL \xrightarrow{R\varepsilon L} RL, \end{array}$$

$$\begin{array}{rcl} \beta(I_R) & = & LR \xrightarrow{\varepsilon} I_{\mathbb{B}}, \\ \alpha \cdot \beta(I_R) & = & R \xrightarrow{\eta R} RLR \xrightarrow{R\varepsilon} R, \\ \beta \cdot \alpha \cdot \beta(I_R) & = & LR \xrightarrow{L\eta R} LRLR \xrightarrow{LR\varepsilon} LR \xrightarrow{\varepsilon} I_{\mathbb{B}}. \end{array}$$

The following morphisms will play a special role in what follows.

2.4. Natural endomorphisms. With the notions from 2.2, we define the natural transformations

$$\begin{aligned}
\vartheta &:= R(\beta\alpha(I_L)): && RL \xrightarrow{RL\eta} RLRL \xrightarrow{R\varepsilon L} RL,\\
\underline{\vartheta} &:= \alpha\beta(R(I_L)): && RL \xrightarrow{\eta RL} RLRL \xrightarrow{R\varepsilon L} RL,\\
\gamma &:= L(\alpha\beta(I_R)): && LR \xrightarrow{L\eta R} LRLR \xrightarrow{LR\varepsilon} LR,\\
\underline{\gamma} &:= \beta\alpha(L(I_R)): && LR \xrightarrow{L\eta R} LRLR \xrightarrow{\varepsilon LR} LR,
\end{aligned}$$

which have the properties

$$\begin{aligned}
&R\varepsilon L \cdot RL\vartheta = \vartheta \cdot R\varepsilon L, \quad R\varepsilon L \cdot \underline{\vartheta} RL = \underline{\vartheta} \cdot R\varepsilon L, \quad \underline{\vartheta} \cdot \vartheta = \vartheta \cdot \underline{\vartheta};\\
&LR\gamma \cdot L\eta R = L\eta R \cdot \gamma, \quad \underline{\gamma} LR \cdot L\eta R = L\eta R \cdot \underline{\gamma}, \quad \underline{\gamma} \cdot \gamma = \gamma \cdot \underline{\gamma}.
\end{aligned}$$

2.5. Definitions. Let (L, R, α, β) be a pairing (see 2.2). We call

$$\begin{array}{rll}
\alpha \textit{ regular} & \text{if} & \alpha \cdot \beta \cdot \alpha = \alpha;\\
\alpha \textit{ symmetric} & \text{if} & \vartheta = \underline{\vartheta}.\\
\beta \textit{ regular} & \text{if} & \beta \cdot \alpha \cdot \beta = \beta;\\
\beta \textit{ symmetric} & \text{if} & \gamma = \underline{\gamma};\\
(L, R, \alpha, \beta) \textit{ regular} & \text{if} & \alpha = \alpha \cdot \beta \cdot \alpha \text{ and } \beta = \beta \cdot \alpha \cdot \beta.
\end{array}$$

The following properties are easy to verify:

(i) If α is regular, then $\beta \cdot \alpha(I_L)$, ϑ, and $\underline{\vartheta}$ are idempotent and
$\vartheta \cdot \eta = \eta = \underline{\vartheta} \cdot \eta$;
furthermore, for $\beta' := \beta \cdot \alpha \cdot \beta$, (L, R, α, β') is a regular pairing.

(ii) If β is regular, then $\alpha \cdot \beta(I_R)$, γ, and $\underline{\gamma}$ are idempotent and
$\varepsilon \cdot \gamma = \varepsilon = \varepsilon \cdot \underline{\gamma}$;
furthermore, for $\alpha' := \alpha \cdot \beta \cdot \alpha$, (L, R, α', β) is a regular pairing.

Any pairing (L, R, α, β) with $\beta \cdot \alpha = I$ or $\alpha \cdot \beta = I$ is regular. The second condition defines the *semiadjoint functors* in Medvedev [16].

With manipulations known from ring theory one can show how pairings with regular components can be related with adjunctions provided idempotents split.

2.6. Related adjunctions. Let (L, R, α, β) be a pairing (with quasi-unit η, quasi-counit ε) and assume α to be regular.

If the idempotent $h := \beta \cdot \alpha(I_L) : L \xrightarrow{L\eta} LRL \xrightarrow{\varepsilon L} L$ splits, that is, there are a functor $\underline{L} : \mathbb{A} \to \mathbb{B}$ and natural transformations

$$p : L \to \underline{L}, \quad i : \underline{L} \to L \quad \text{with} \quad i \cdot p = h \quad \text{and} \quad p \cdot i = I_{\underline{L}},$$

then the natural transformations

$$\underline{\eta} : \ I_{\mathbb{A}} \xrightarrow{\ \eta\ } RL \xrightarrow{\ Rp\ } R\underline{L}\ , \quad \underline{\varepsilon} : \ \underline{L}R \xrightarrow{\ iR\ } LR \xrightarrow{\ \varepsilon\ } I_{\mathbb{B}}\ ,$$

as quasi-unit and quasi-counit, define a pairing $(\underline{L}, R, \underline{\alpha}, \underline{\beta})$ with $\underline{\beta} \cdot \underline{\alpha} = I$.

If $\alpha \cdot \beta = I$, then $(\widehat{L}, R, \underline{\alpha}, \underline{\beta})$ is an adjunction.

In case the natural transformation β is regular, similar constructions apply if we assume that the idempotent $\alpha \cdot \beta(I_R) : R \xrightarrow{\eta R} RLR \xrightarrow{R\varepsilon} R$ splits.

The properties of the $(RL, R\varepsilon R, \eta)$ and $(LR, L\eta R, \varepsilon)$ mentioned in 2.2 motivate the definitions in the next section.

3. Monads and modules

3.1. q-unital monads and their modules. We call (F, μ) a *functor with product* (or *non-unital monad*) provided $F : \mathbb{A} \to \mathbb{A}$ is an endofunctor on a category $\mathbb{A}$ and $\mu : FF \to F$ is a natural transformation satisfying the associativity condition $\mu \cdot F\mu = \mu \cdot \mu F$.

For (F, μ), a *(non-unital) F-module* is defined as an object $A \in \mathbb{A}$ with a morphism $\varrho : F(A) \to A$ in $\mathbb{A}$ satisfying $\varrho \cdot F\varrho = \varrho \cdot \mu_A$.

Morphisms between F-modules (A, ϱ), (A', ϱ') are morphisms $f : A \to A'$ in $\mathbb{A}$ with $\varrho' \cdot F(f) = f \cdot \varrho$. The set of all these is denoted by $\mathrm{Mor}_F(A, A')$. With these morphisms, (non-unital) F-modules form a category which we denote by $\underrightarrow{\mathbb{A}}_F$.

By the associativity condition on μ, for every $A \in \mathbb{A}$, $(F(A), \mu_A)$ is an F-module and this leads to the free functor and the forgetful functor,

$$\phi_F : \mathbb{A} \to \underrightarrow{\mathbb{A}}_F, \quad A \mapsto (F(A), \mu_A), \qquad U_F : \underrightarrow{\mathbb{A}}_F \to \mathbb{A}, \quad (A, \varrho) \mapsto A.$$

A triple (F, μ, η) is said to be a *q-unital monad* on $\mathbb{A}$ provided (F, μ) is a functor with product and $\eta : I_{\mathbb{A}} \to F$ is any natural transformation, called a *quasi-unit* (no additional properties are required). One always can define natural transformations

$$\vartheta : F \xrightarrow{F\eta} FF \xrightarrow{\ \mu\ } F, \quad \underline{\vartheta} : F \xrightarrow{\eta F} FF \xrightarrow{\ \mu\ } F.$$

Note that for any $A \in \mathbb{A}$, ϑ_A is an F-module morphism whereas $\underline{\vartheta}_A$ is not necessarily so.

Given q-unital monads (F, μ, η), (F', μ', η') on $\mathbb{A}$, a natural transformation $h : F \to F'$ is called a *morphism of q-unital monads* if

$$\mu' \cdot hh = h \cdot \mu \quad \text{and} \quad \eta' = h \cdot \eta.$$

The existence of a quasi-unit allows the following generalisation of the Eilenberg-Moore construction for (unital) monads.

3.2. q-unital monads and pairings. For a q-unital monad (F, μ, η) we obtain a pairing $(\phi_F, U_F, \alpha_F, \beta_F)$ with the maps, for $A \in \mathbb{A}$, $(B, \varrho) \in \underrightarrow{\mathbb{A}}_F$,

$$\begin{aligned} \alpha_F : \mathrm{Mor}_F(\phi_F(A), B) &\to \mathrm{Mor}_{\mathbb{A}}(A, U_F(B)), \quad f \mapsto f \cdot \eta_A, \\ \beta_F : \mathrm{Mor}_{\mathbb{A}}(A, U_F(B)) &\to \mathrm{Mor}_F(\phi_F(A), B), \quad g \mapsto \varrho \cdot F(g). \end{aligned}$$

The quasi-unit η is called *regular* if α_F is regular, that is,

$$I_{\mathbb{A}} \xrightarrow{\eta} F = I_{\mathbb{A}} \xrightarrow{\eta} F \xrightarrow{F\eta} FF \xrightarrow{\mu} F,$$

and we say η is *symmetric* if α_F is so, that is, $\vartheta = \underline{\vartheta}$.

An F-module $\varrho : F(A) \to A$ in $\underrightarrow{\mathbb{A}}_F$ is said to be *compatible (with η)* if $\beta_F \alpha_F(\varrho) = \varrho$, that is

$$F(A) \xrightarrow{\varrho} A = F(A) \xrightarrow{F\eta_A} FF(A) \xrightarrow{\mu_A} F(A) \xrightarrow{\varrho} A.$$

In particular, the natural transformation $\mu : FF \to F$ is compatible if

$$FF \xrightarrow{\mu} F = FF \xrightarrow{F\eta F} FFF \xrightarrow{\mu F} FF \xrightarrow{\mu} F.$$

It is easy to see that this implies

$$FF \xrightarrow{\vartheta\underline{\vartheta}} FF \xrightarrow{\mu} F = FF \xrightarrow{\mu} F.$$

Let $\underline{\mathbb{A}}_F$ denote the full subcategory of $\underrightarrow{\mathbb{A}}_F$ made up by the compatible F-modules. If μ is compatible, the image of the free functor ϕ_F lies in $\underline{\mathbb{A}}_F$ and (by restriction or corestriction) we get the functor pair (keeping the notation for the functors)

$$\phi_F : \mathbb{A} \to \underline{\mathbb{A}}_F, \quad U_F : \underline{\mathbb{A}}_F \to \mathbb{A},$$

and a pairing $(\phi_F, U_F, \alpha_F, \beta_F)$ between $\mathbb{A}$ and $\underline{\mathbb{A}}_F$.

Since for (A, ϱ) in $\underrightarrow{\mathbb{A}}_F$, $\beta_F(I_{U_F(A)}) = \varrho$, the compatibility condition on ϱ implies that $\beta \cdot \alpha \cdot \beta(\varrho) = \beta(\varrho)$, i.e., $\underline{\beta}$ is regular in $(\phi_F, U_F, \alpha_F, \beta_F)$ when restricted to $\underline{\mathbb{A}}_F$.

3.3. Definition. A q-unital monad (F, η, μ) is called

$$\begin{array}{rll} r\textit{-unital} & \text{if} & \eta \text{ is regular and } \mu \text{ is compatible;} \\ \textit{weak monad} & \text{if} & (F,\eta,\mu) \text{ is } r\text{-unital and } \eta \text{ is symmetric.} \end{array}$$

Summarising the observations from 3.2 we have:

3.4. Proposition. *Let (F, μ, η) be a q-unital monad.*

(1) *The following are equivalent:*

   (a) *(F, μ, η) is an r-unital monad;*

   (b) *$(\phi_F, U_F, \alpha_F, \beta_F)$ is a regular pairing of functors between $\mathbb{A}$ and $\underline{\mathbb{A}}_F$.*

(2) *The following are equivalent:*

   (a) *(F, μ, η) is a weak monad;*

   (b) *$(\phi_F, U_F, \alpha_F, \beta_F)$ is a regular pairing between $\mathbb{A}$ and $\underline{\mathbb{A}}_F$ with α_F symmetric.*

A quasi-unit η that is regular and symmetric is named *pre-unit* in the literature (e.g. [11, Definition 2.3]); for the notion of a weak monad (also called *demimonad*) see e.g. [5]. In case η is a unit, q-unital monads, r-unital monads and weak monads all are (unital) monads. In (non-unital) algebras over commutative rings, r-unital monads are obtained from idempotents while weak monads correspond to central idempotents (see 3.7).

3.5. Properties of weak monads. *Let (F, μ, η) be a weak monad.*

(i) *$\vartheta : F \to F$ is a morphism of q-unital monads;*

(ii) *for any $(A, \varphi) \in \underline{\mathbb{A}}_F$,*

$$F(A) \xrightarrow{\varphi} A = F(A) \xrightarrow{\varphi} A \xrightarrow{\eta_A} F(A) \xrightarrow{\varphi} A$$

and $\quad A \xrightarrow{\eta_A} F(A) \xrightarrow{\varphi} A \quad$ *is an idempotent F-morphism.*

In a q-unital monad (F, μ, η), if η is regular, a compatible multiplication for F can be found. More precisely one can easily show:

3.6. Proposition. *Let (F, μ, η) be a q-unital monad.*

(1) *If η is regular, then, for $\widetilde{\mu} := \mu \cdot F\mu \cdot \mu F\eta F : FF \to F$, $(F, \widetilde{\mu}, \eta)$ is an r-unital monad.*

(2) *If μ is compatible, then, for $\widetilde{\eta} := \mu \cdot F\eta \cdot \eta : I_A \to F$, $(F, \mu, \widetilde{\eta})$ is an r-unital monad.*

(3) *If (F,μ,η) is an r-unital monad, then for*

$$\widehat{\mu} : FF \xrightarrow{\eta FF\eta} FFFF \xrightarrow{\mu FF} FFF \xrightarrow{\mu F} FF \xrightarrow{\mu} F,$$

$(F,\widehat{\mu},\eta)$ is a weak monad.

As a special case, we consider q-unital monads on the category ${}_R\mathbb{M}$ of modules over a commutative ring R with unit. In the terminology used here this comes out as follows.

3.7. Non-unital algebras. A *q-unital R-algebra* (A,m,u) is a non-unital R-algebra (A,m) with some R-linear map $u : R \to A$. Put $e := u(1_R) \in A$. Then:

(1) u is regular if and only if e is an idempotent in A.

(2) u is regular and symmetric if and only if e is a central idempotent (then Ae is a unital R-subalgebra of A).

(3) μ is compatible if and only if $ab = aeb$ for all $a,b \in A$.

(4) If u is regular, then $\widetilde{m}(a\otimes b) := aeb$, for $a,b \in A$, defines an r-unital algebra $(A,\widetilde{m},u)$ ($\widetilde{m}$ and u are regular).

(5) If u is regular, then $\widehat{m}(a\otimes b) := eaebe$, for $a,b \in A$, defines an r-unital algebra $(A,\widehat{m},u)$ with u symmetric.

Clearly, the q-unital algebras (A,m,u) over R correspond to the q-unital monads given by $(A\otimes_R -, m\otimes -, u\otimes -)$ on ${}_R\mathbb{M}$.

For an A-module $\varrho : A\otimes M \to M$, writing as usual $\varrho(a\otimes m) = am$, the compatibility condition comes out as $am = aem$ for all $a \in A$, $m \in M$.

3.8. Monads acting on functors. Let $T : \mathbb{A} \to \mathbb{B}$ be a functor and (G,μ',η') a q-unital monad on $\mathbb{B}$. We call T a left *G-module* if there exists a natural transformation $\varrho : GT \to T$ such that

$$GGT \xrightarrow{G\varrho} GT \xrightarrow{\varrho} T = GGT \xrightarrow{\mu' T} GT \xrightarrow{\varrho} T,$$

and we call it a *compatible G-module* if in addition

$$GT \xrightarrow{\varrho} T = GT \xrightarrow{G\eta'} GGT \xrightarrow{\mu' T} GT \xrightarrow{\varrho} T.$$

3.9. Proposition. *Let $T : \mathbb{A} \to \mathbb{B}$ be a functor and (G,μ',η') a weak monad on $\mathbb{B}$. Then the following are equivalent:*

(a) *there is a functor $\overline{T} : \mathbb{A} \to \mathbb{B}_G$ with $T = U_G\overline{T}$;*

(b) *T is a compatible G-module.*

Proof. (b)⇒(a) Given T as a compatible G-module with $\varrho : GT \to T$, a functor with the required properties is

$$\overline{T} : \mathbb{A} \to \underline{\mathbb{B}}_G, \quad A \ \mapsto \ (T(A), \varrho_A : GT(A) \to T(A)).$$

(a)⇒(b) For any $A \in \mathbb{A}$, there are morphisms $\rho_A : GT(A) \to T(A)$ and we claim that these define a natural transformation $\rho : GT \to T$. For this we have to show that, for any morphism $f : A \to \widehat{A}$, the middle rectangle is commutative in the diagram

$$\begin{tikzcd}
GGT(A) \arrow[rr, "\mu'_T(A)"] & & GT(A) \arrow[dl, "\rho_A"'] \arrow[ddd, "GT(f)"] \\
GT(A) \arrow[u, "G\eta' T_A"] \arrow[r, "\rho_A"] \arrow[d, "GT(f)"'] & T(A) \arrow[d, "T(f)"] & \\
GT(\widehat{A}) \arrow[r, "\rho_{\widehat{A}}"] \arrow[d, "G\eta' T_{\widehat{A}}"'] & T(\widehat{A}) & \\
GGT(\widehat{A}) \arrow[rr, "\mu'_{T(\widehat{A})}"] & & GT(\widehat{A}). \arrow[ul, "\rho_{\widehat{A}}"']
\end{tikzcd}$$

The top and bottom diagrams are commutative by compatibility of the G-modules, the right trapezium is commutative since $T(f)$ is a G-morphism, and the outer paths commute by symmetry of η'. Thus the inner diagram is commutative showing naturality of ρ. □

4. Comonads and comodules

In this section we sketch the transfer of the constructions for monads to comonads.

4.1. q-counital comonads and their comodules. A *functor with coproduct* (or *non-counital comonad*) is a pair (G, δ) where $G : \mathbb{A} \to \mathbb{A}$ is an endofunctor and $\delta : G \to GG$ is a natural transformation subject to the coassociativity condition $G\delta \cdot \delta = \delta G \cdot \delta$.

For (G, δ), a *(non-counital) G-comodule* is defined as an object $A \in \mathbb{A}$ with a morphism $\upsilon : A \to G(A)$ in $\mathbb{A}$ such that $G\upsilon \cdot \upsilon = \delta_A \cdot \upsilon$.

Morphisms between G-comodules (A, υ), (A', υ') are morphisms $g : A \to A'$ in $\mathbb{A}$ satisfying $\upsilon' \cdot g = G(g) \cdot \upsilon$, and the set of all these is denoted by $\mathrm{Mor}^G(A, A')$. With these morphisms, (non-counital) G-comodules form a category which we denote by $\underrightarrow{\mathbb{A}}^G$. For this there are the obvious free and forgetful functors

$$\phi^G : \mathbb{A} \to \underrightarrow{\mathbb{A}}^G, \quad U^G : \underrightarrow{\mathbb{A}}^G \to \mathbb{A}.$$

A triple (G, δ, ε) is said to be a *q-counital comonad* provided (G, δ) is a functor with coproduct and $\varepsilon : G \to I_{\mathbb{A}}$ is any natural transformation, called a *quasi-counit*. One can always define natural transformations

$$\gamma : G \xrightarrow{\delta} GG \xrightarrow{G\varepsilon} G, \quad \underline{\gamma} : G \xrightarrow{\delta} GG \xrightarrow{\varepsilon G} G.$$

Morphisms of q-counital comonads are defined in an obvious way (dual to 3.1).

4.2. q-counital comonads and pairings. For (G, δ, ε) as above, the functors ϕ^G and U^G allow for a pairing $(U^G, \phi^G, \alpha^G, \beta^G)$ where, for $A \in \mathbb{A}$ and $(B, \upsilon) \in \underrightarrow{\mathbb{A}}^G$,

$$\begin{aligned} \alpha^G &: \mathrm{Mor}_{\mathbb{A}}(U^G(B), A) \to \mathrm{Mor}^G(B, \phi^G(A)), \quad f \ \mapsto \ G(f) \cdot \upsilon, \\ \beta^G &: \mathrm{Mor}^G(B, \phi^G(A)) \to \mathrm{Mor}_{\mathbb{A}}(U^G(B), A), \quad g \ \mapsto \ \varepsilon_A \cdot g. \end{aligned}$$

The quasi-counit ε is called *regular* if β^G is regular, that is,

$$G \xrightarrow{\varepsilon} I_{\mathbb{A}} = G \xrightarrow{\delta} GG \xrightarrow{G\varepsilon} G \xrightarrow{\varepsilon} I_{\mathbb{A}},$$

and we say η is *symmetric* provided ϕ^G is so, that is $\gamma = \underline{\gamma}$.

A (non-counital) G-comodule (B, υ) is said to be *compatible (with ε)* provided $\alpha^G \beta^G(\upsilon) = \upsilon$, that is

$$B \xrightarrow{\upsilon} G(B) = B \xrightarrow{\upsilon} G(B) \xrightarrow{\delta_B} GG(B) \xrightarrow{G\varepsilon_B} G(B).$$

In particular, δ is compatible if

$$G \xrightarrow{\delta} GG = G \xrightarrow{\delta} GG \xrightarrow{\delta G} GGG \xrightarrow{G\varepsilon G} GG.$$

This obviously implies

$$G \xrightarrow{\delta} GG = G \xrightarrow{\delta} GG \xrightarrow{\gamma\underline{\gamma}} GG.$$

By $\underline{\mathbb{A}}^G$ we denote the full subcategory of $\underrightarrow{\mathbb{A}}^G$ whose objects are compatible G-comodules.

If δ is compatible, the image of the free functor ϕ^G lies in $\underline{\mathbb{A}}^G$ and (by restriction and corestriction) we obtain the functor pairing (keeping the notation for the functors)

$$\phi^G : \mathbb{A} \to \underline{\mathbb{A}}^G, \quad U^G : \underline{\mathbb{A}}^G \to \mathbb{A},$$

leading to a pairing $(U^G, \phi^G, \alpha^G, \beta^G)$ between $\mathbb{A}$ and $\underline{\mathbb{A}}^G$.

Since for (B, υ) in $\underset{\to}{\mathbb{A}}^G$, $\alpha^G(I_{U^G(B)}) = \upsilon$, the compatibility condition on υ implies that $\alpha^G \cdot \beta^G \cdot \alpha^G(\upsilon) = \alpha^G(\upsilon)$, i.e., α is regular in $(U^G, \phi^G, \alpha^G, \beta^G)$ when restricted to $\underline{\mathbb{A}}^G$.

4.3. Definition. A q-counital comonad (G, δ, ε) is called

r-counital	if	ε is regular and δ is compatible;
weak comonad	if	it is r-counital and ε is symmetric.

From the constructions above we obtain:

4.4. Proposition. *Let (G, δ, ε) be a q-counital comonad.*

(1) *The following are equivalent:*

  (a) *(G, δ, ε) is an r-counital comonad;*

  (b) *$(U^G, \phi^G, \alpha^G, \beta^G)$ is a regular pairing of functors between $\mathbb{A}$ and $\underline{\mathbb{A}}^G$.*

(2) *The following are equivalent:*

  (a) *(G, δ, ε) is a weak comonad;*

  (b) *$(U^G, \phi^G, \alpha^G, \beta^G)$ is a regular pairing of functors between $\mathbb{A}$ and $\underline{\mathbb{A}}^G$ with β^G symmetric.*

Similar to the situation for modules, for any (counital) comonad (G, δ, ε), all non-counital G-comodules are compatible (i.e., $\underset{\to}{\mathbb{A}}^G = \underline{\mathbb{A}}^G$).

4.5. Properties of weak comonads. *Let (G, δ, ε) be a weak comonad.*

(i) *$\gamma : G \to G$ is an idempotent morphism of q-counital comonads;*

(ii) *for any $(B, \upsilon) \in \underline{\mathbb{A}}^G$,*

$$B \xrightarrow{\upsilon} G(B) = B \xrightarrow{\upsilon} G(B) \xrightarrow{\varepsilon_B} B \xrightarrow{\upsilon} G(B)$$

and $B \xrightarrow{\upsilon} G(B) \xrightarrow{\varepsilon_B} B$ is an idempotent G-morphism.

Properties of pairings can improved in the following sense.

4.6. Proposition. *Let (G, δ, ε) be a q-counital comonad.*

(1) *If ε is regular, then, for $\widetilde{\delta} : G \xrightarrow{\delta} GG \xrightarrow{G\delta} GGG \xrightarrow{G\varepsilon G} GG$, $(G, \widetilde{\delta}, \varepsilon)$ is an r-counital comonad.*

(2) *If δ is compatible, then, for $\widetilde{\varepsilon} : G \xrightarrow{\delta} GG \xrightarrow{G\varepsilon} G \xrightarrow{\varepsilon} I_{\mathbb{A}}$, $(G, \delta, \widetilde{\varepsilon})$ is an r-counital comonad.*

(3) *If (G, δ, ε) is a regular quasi-comonad, then, for*

$$\widehat{\delta} : G \xrightarrow{\delta} GG \xrightarrow{G\delta} GGG \xrightarrow{GG\delta} GGGG \xrightarrow{\varepsilon GG\varepsilon} GG,$$

$(G, \widehat{\delta}, \varepsilon)$ is a weak comonad.

As a special case, consider non-counital comonads on the category ${}_R\mathbb{M}$ of modules over a commutative ring R with unit. In our terminology this comes out as follows.

4.7. Non-counital coalgebras. A *q-counital coalgebra* (C, Δ, ε) is a non-counital R-coalgebra (C, Δ) with some R-linear map $\varepsilon : C \to R$. Writing $\Delta(c) = \sum c_{\underline{1}} \otimes c_{\underline{2}}$ for $c \in C$, we have:

(1) ε is regular if and only if for any $c \in C$, $\varepsilon(c) = \sum \varepsilon(c_{\underline{1}})\varepsilon(c_{\underline{2}})$.

(2) ε is symmetric if and only if $\sum c_{\underline{1}}\varepsilon(c_{\underline{2}}) = \sum \varepsilon(c_{\underline{1}})c_{\underline{2}}$.

(3) Δ is compatible if and only if $\Delta(c) = \sum c_{\underline{1}} \otimes c_{\underline{2}}\varepsilon(c_{\underline{3}})$.

(4) If ε is regular, then $\widetilde{\Delta}(c) := \sum c_{\underline{1}} \otimes \varepsilon(c_{\underline{2}})c_{\underline{3}}$ defines an r-counital coalgebra $(C, \widetilde{\Delta}, \varepsilon)$.

(5) If (C, Δ, ε) is an r-counital comonad, then $\widehat{\Delta}(c) := \sum \varepsilon(c_{\underline{1}})c_{\underline{2}} \otimes c_{\underline{3}}\varepsilon(c_{\underline{4}})$ defines an r-counital coalgebra $(C, \widehat{\Delta}, \varepsilon)$ with ε symmetric.

Clearly, the q-counital coalgebras (C, Δ, ε) over R correspond to the q-counital comonads given by $(C \otimes_R -, \Delta \otimes -, \varepsilon \otimes -)$ on ${}_R\mathbb{M}$. From this the compatibility conditions for C-comodules are derived (see 4.2).

4.8. Weak corings and pre-A-corings. Let A be a ring with unit 1_A and $\mathcal{C}$ a non-unital (A, A)-bimodule which is unital as right A-module. Assume there are (A, A)-bilinear maps

$$\underline{\Delta} : \mathcal{C} \to \mathcal{C} \otimes_A \mathcal{C}, \quad \underline{\varepsilon} : \mathcal{C} \to A,$$

where $\underline{\Delta}$ is coassociative. $(\mathcal{C}, \underline{\Delta}, \underline{\varepsilon})$ is called a *right unital weak A-coring* in [21], provided for all $c \in \mathcal{C}$,

$$(\underline{\varepsilon} \otimes I_{\mathcal{C}}) \cdot \underline{\Delta}(c) = 1_A \cdot c = (I_{\mathcal{C}} \otimes \underline{\varepsilon}) \cdot \underline{\Delta}(c),$$

which reads in (obvious) Sweedler notation as $\sum \underline{\varepsilon}(c_{\underline{1}})c_{\underline{2}} = 1_A \cdot c = \sum c_{\underline{1}}\underline{\varepsilon}(c_{\underline{2}})$. From the equations

$$\begin{aligned}(I_{\mathcal{C}} \otimes \underline{\varepsilon} \otimes I_{\mathcal{C}}) \cdot (I_{\mathcal{C}} \otimes \underline{\Delta}) \cdot \underline{\Delta}(c) &= \sum c_{\underline{1}} \otimes 1_A \cdot c_{\underline{2}} = \sum c_{\underline{1}} \otimes c_{\underline{2}} = \underline{\Delta}(c),\\ (I_{\mathcal{C}} \otimes \underline{\varepsilon} \otimes I_{\mathcal{C}}) \cdot (\underline{\Delta} \otimes I_{\mathcal{C}}) \cdot \underline{\Delta}(c) &= \sum 1_A \cdot c_{\underline{1}} \otimes c_{\underline{2}} = 1_A \cdot \underline{\Delta}(c),\end{aligned}$$

it follows by coassociativity that $1_A \cdot \underline{\Delta}(c) = \underline{\Delta}(c)$. Summarising we see that, in this case, $(\mathcal{C}, \underline{\Delta}, \underline{\varepsilon})$ induces a weak comonad on the category ${}_A\underrightarrow{\mathbb{M}}$ of left non-unital A-modules ($= {}_A\underline{\mathbb{M}}$ since A has a unit).

$(\mathcal{C}, \underline{\Delta}, \underline{\varepsilon})$ is called an *A-pre-coring* in [7, Section 6], if

$$(\underline{\varepsilon} \otimes I_{\mathcal{C}}) \cdot \underline{\Delta}(c) = c, \quad (I_{\mathcal{C}} \otimes \underline{\varepsilon}) \cdot \underline{\Delta}(c) = 1_A \cdot c,$$

which reads (in Sweedler notation) as $c = \sum \underline{\varepsilon}(c_{\underline{1}})c_{\underline{2}}$, $1_A \cdot c = \sum c_{\underline{1}}\underline{\varepsilon}(c_{\underline{2}})$.

Similar to the computation above we obtain that $1_A \cdot \underline{\Delta}(c) = \underline{\Delta}(c)$. Now $(\mathcal{C}, \underline{\Delta}, \underline{\varepsilon})$ induces an r-counital comonad on ${}_A\underrightarrow{\mathbb{M}}$ but $\underline{\varepsilon}$ is not symmetric.

Notice that in both cases considered above, restriction and corestriction of $\underline{\Delta}$ and $\underline{\varepsilon}$ yield an *A-coring* $(A\mathcal{C}, \underline{\Delta}, \underline{\varepsilon})$ (e.g. [21, Proposition 1.3]).

4.9. Comonads acting on functors. Let $T : \mathbb{A} \to \mathbb{B}$ be a functor and (G, δ, ε) a weak comonad on $\mathbb{B}$. We call T a left *(non-counital) G-comodule* if there exists a natural transformation $\upsilon : T \to GT$ such that

$$T \xrightarrow{\upsilon} GT \xrightarrow{\upsilon G} GGT = T \xrightarrow{\upsilon T} GT \xrightarrow{\delta} GGT,$$

and we call it a *compatible G-comodule* if, in addition,

$$T \xrightarrow{\upsilon} GT = T \xrightarrow{\upsilon} GT \xrightarrow{\delta} GGT \xrightarrow{G\varepsilon} GT.$$

Dual to Proposition 3.9, given a weak comonad (G, δ, ε) on $\mathbb{B}$, a functor $T : \mathbb{A} \to \mathbb{B}$ is a compatible G-comodule if and only if there is a functor $\overline{T} : \mathbb{A} \to \underline{\mathbb{B}}^G$ with $T = U^G\overline{T}$.

The motivation for considering generalised monads and comonads came from structures observed while handling full pairings of functors (see end of Section 2). Now we want to reconsider the pairings in view of these constructions.

For any pairing (L, R, α, β) between categories $\mathbb{A}$ and $\mathbb{B}$, $(RL, R\varepsilon L, \eta)$ is a q-unital monad and $(LR, L\eta R, \varepsilon)$ is a q-counital comonad. It is easy to see that

(i) *if β is regular, then for any $B \in \mathbb{B}$, $R\varepsilon : RLR(B) \to R(B)$ is a compatible RL-module.*

(ii) *if α is regular, then for any $A \in \mathbb{A}$, $L(A), L\eta : L(A) \to LRL(A)$ is a compatible LR-comodules.*

4.10. Comparison functors. *For a regular pairing (L, R, α, β) between $\mathbb{A}$ and $\mathbb{B}$,*

$(RL, R\varepsilon L, \eta)$ is an r-unital monad on $\mathbb{A}$ with a (comparison) functor

$$\widehat{R} : \mathbb{B} \to \underline{\mathbb{A}}_{RL}, \quad B \mapsto (R(B), R\varepsilon : RLR(B) \to R(B)),$$

$(LR, L\eta R, \varepsilon)$ is an r-counital comonad on $\mathbb{B}$ with a (comparison) functor

$$\widetilde{L} : \mathbb{A} \to \underrightarrow{\mathbb{B}}^{LR}, \quad A \mapsto (L(A), L\eta : L(A) \to LRL(A)),$$

inducing commutativity of the diagrams

$$\begin{tikzcd} \mathbb{A} \arrow[r, "L"] \arrow[dr, "\phi_{RL}"'] & \mathbb{B} \arrow[r, "R"] \arrow[d, "\widehat{R}"] & \mathbb{A} \\ & \underline{\mathbb{A}}_{RL} \arrow[ur, "U_{RL}"'] & \end{tikzcd}, \qquad \begin{tikzcd} \mathbb{B} \arrow[r, "R"] \arrow[dr, "\phi^{LR}"'] & \mathbb{A} \arrow[r, "L"] \arrow[d, "\widetilde{L}"] & \mathbb{B} \\ & \underline{\mathbb{B}}^{LR} \arrow[ur, "U^{LR}"'] & \end{tikzcd}.$$

It follows from 3.2 that for the r-unital monad $(RL, R\varepsilon L, \eta)$, we have a regular pairing $(\phi_{RL}, U_{RL}, \alpha_{RL}, \beta_{RL})$ between $\mathbb{A}$ and $\underline{\mathbb{A}}_{RL}$. Similarly, by 4.2, for the R-counital comonad $(LR, L\eta R, \varepsilon)$, $(U^{LR}, \phi^{LR}, \alpha^{LR}, \beta^{LR})$ is a regular pairing between $\mathbb{B}$ and $\underline{\mathbb{B}}^{LR}$.

4.11. Relating (L, R) with (ϕ_{RL}, U_{RL}) and (U^{LR}, ϕ^{LR}). With the above notions we form the diagram

$$\begin{tikzcd} \mathrm{Mor}_{\mathbb{B}}(L(A), B) \arrow[r, "\alpha"] \arrow[d, "\widehat{R}_{-,-}"'] & \mathrm{Mor}_{\mathbb{A}}(A, R(B)) \arrow[r, "\beta"] & \mathrm{Mor}_{\mathbb{B}}(L(A), B) \arrow[d, "\widehat{R}_{-,-}"] \\ \mathrm{Mor}_{RL}(\phi_{RL}(A), R(B)) \arrow[r, "\alpha_{RL}"] & \mathrm{Mor}_{\mathbb{A}}(A, U_{RL}R(B)) \arrow[r, "\beta_{RL}"] & \mathrm{Mor}_{RL}(\phi_{RL}(A), R(B)). \end{tikzcd}$$

This diagram is commutative if and only if α is symmetric (see Definitions 2.5).

Similar constructions apply for (L, R), (U^{LR}, ϕ^{LR}) and $\widetilde{L}_{-,-}$, and β is symmetric if and only if $\widetilde{K}_{-,-} \cdot \alpha \cdot \beta = \alpha^{LR} \cdot \beta^{LR} \cdot \widetilde{L}_{-,-}$.

4.12. Corollary. *Consider a pairing (L, R, α, β) (see 2.2).*

(1) *The following are equivalent:*

(a) (L, R, α, β) *is a regular pairing;*

(b) $(RL, R\varepsilon L, \eta)$ *is an r-unital monad on $\mathbb{A}$ and* $(LR, L\eta R, \varepsilon)$ *is an r-counital comonad on $\mathbb{B}$.*

(2) *The following are equivalent:*

(a) (L, R, α, β) *is a regular pairing with α and β symmetric;*

(b) $(RL, R\varepsilon L, \eta)$ *is a weak monad on $\mathbb{A}$ and* $(LR, L\eta R, \varepsilon)$ *is a weak comonad on $\mathbb{B}$.*

5. Entwining monads and comonads

5.1. Lifting of functors to module categories. Let (F, μ, η) and (L, μ', η') be r-unital monads on the categories $\mathbb{A}$ and $\mathbb{B}$, respectively, and $\underline{\mathbb{A}}_F$, $\underline{\mathbb{B}}_L$ the categories of the corresponding compatible modules (see 3.2). Given functors $T : \mathbb{A} \to \mathbb{B}$ and $\overline{T} : \underline{\mathbb{A}}_F \to \underline{\mathbb{B}}_L$, we say that $\overline{T}$ *is a lifting of T* provided the diagram

$$\begin{array}{ccc} \underline{\mathbb{A}}_F & \xrightarrow{\overline{T}} & \underline{\mathbb{B}}_L \\ {\scriptstyle U_F}\downarrow & & \downarrow{\scriptstyle U_L} \\ \mathbb{A} & \xrightarrow{T} & \mathbb{B} \end{array} \tag{5.1}$$

is commutative, where the U's denote the forgetful functors.

5.2. Proposition. *With the data given in* 5.1, *consider the functors TF, $LT : \mathbb{A} \to \mathbb{B}$ and a natural transformation $\lambda : LT \to TF$. The non-unital F-module (F, μ) induces an L-action on TF,*

$$\chi : LTF \xrightarrow{\lambda F} TFF \xrightarrow{T\mu} TF.$$

(1) *If (TF, χ) is a (non-unital) L-module, then we get the commutative diagram*

$$\begin{array}{ccccccc} LLT & \xrightarrow{L\lambda} & LTF & \xrightarrow{LT\vartheta} & LTF & \xrightarrow{\lambda F} & TFF \\ {\scriptstyle \mu' T}\downarrow & & & & & & \downarrow{\scriptstyle T\mu} \\ LT & & \xrightarrow{\quad\quad\lambda\quad\quad} & & TF & \xrightarrow{T\vartheta} & TF. \end{array} \tag{5.2}$$

(2) *If (TF, χ) is a compatible L-module, then (with $\vartheta' = \mu' \cdot F\eta'$)*

$$LT \xrightarrow{\vartheta' T} LT \xrightarrow{\lambda} TF \xrightarrow{T\vartheta} TF \;=\; LT \xrightarrow{\lambda} TF \xrightarrow{T\vartheta} TF. \tag{5.3}$$

(3) *If η is symmetric in (F, μ, η) and (A, φ) is a compatible F-module, then*

$$T\varphi \cdot \lambda_A = T\varphi \cdot \lambda_A \cdot LT\varphi \cdot LT\eta_A. \tag{5.4}$$

Proof. The proof follows essentially as in the monad case replacing the identity on F at some places by $\vartheta = \mu \cdot F\eta$ (see 3.1).

To show (3), Proposition 3.5 is needed. □

5.3. Proposition. *Let (F, μ, η) and (L, μ', η') be r-unital monads on $\mathbb{A}$ and $\mathbb{B}$, respectively, and $T : \mathbb{A} \to \mathbb{B}$ any functor. Then a natural transformation $\lambda : LT \to TF$ induces a lifting to the compatible modules,*

$$\overline{T} : \underline{\mathbb{A}}_F \to \underline{\mathbb{B}}_L, \quad (A, \varphi) \mapsto (T(A), T\varphi \cdot \lambda_A : LT(A) \to T(A)),$$

if and only if the diagram (5.2) *is commutative and equation* (5.3) *holds.*

Proof. One direction follows from Proposition 5.2, the other one by a slight modification of the proof in the monad case. □

To show that the lifting property implies the existence of a natural transformation $\lambda : LT \to TF$ we need the symmetry of the units, that is, we require the r-unital monads to be weak monads. Then we can extend Applegate's lifting theorem for monads (and unital modules) (e.g. [13, Lemma 1], [22, 3.3]) to weak monads (and compatible modules).

5.4. Theorem. *Let (F, μ, η) and (L, μ', η') be weak monads on $\mathbb{A}$ and $\mathbb{B}$, respectively. For any functor $T : \mathbb{A} \to \mathbb{B}$, there are bijective correspondences between*

(i) *liftings of T to $\overline{T} : \underline{\mathbb{A}}_F \to \underline{\mathbb{B}}_L$;*

(ii) *compatible L-module structures ϱ on $TU_F : \mathbb{A}_F \to \mathbb{B}$;*

(iii) *natural transformations $\lambda : LT \to TF$ with commuting diagrams*

$$\begin{tikzcd} LLT \arrow[r, "L\lambda"] \arrow[d, "\mu' T"'] & LTF \arrow[r, "\lambda F"] & TFF \arrow[d, "T\mu"] \\ LT \arrow[rr, "\lambda"] & & TF, \end{tikzcd} \qquad \begin{tikzcd} LT \arrow[r, "\vartheta' T"] \arrow[d, "\lambda"'] \arrow[dr, "\lambda"] & LT \arrow[d, "\lambda"] \\ TF \arrow[r, "T\vartheta"'] & TF. \end{tikzcd} \tag{5.5}$$

Proof. (i)⇔(ii) follows by Proposition 3.9.

(ii)⇒(iii) Given the compatible L-module structure map ϱ, put

$$\lambda := \varrho F \cdot LT\eta : LT \xrightarrow{LT\eta} LTF \xrightarrow{\varrho F} TF.$$

Notice that for λ we can take $T\vartheta \cdot \lambda$ from Proposition 5.2.

(iii)⇒(i) Given λ with the commutative diagram in (iii), it follows by Propositions 5.3 that $\varrho_A := T\varphi \cdot \lambda_A$ induces a lifting. □

5.5. Lifting of functors to comodules. Let (G, δ, ε) and $(H, \delta', \varepsilon')$ be r-unital comonads on the categories $\mathbb{A}$ and $\mathbb{B}$, respectively, and $\underline{\mathbb{A}}^G$, $\underline{\mathbb{B}}^H$ the corresponding categories of the compatible comodules (see 4.2). Given a functor $T : \mathbb{A} \to \mathbb{B}$, a functor $\widehat{T} : \underline{\mathbb{A}}^G \to \underline{\mathbb{B}}^H$, is said to be *a lifting of* T if the diagram

$$\begin{tikzcd} \underline{\mathbb{A}}^G \arrow[r, "\widehat{T}"] \arrow[d, "U^G"'] & \underline{\mathbb{B}}^H \arrow[d, "U^H"] \\ \mathbb{A} \arrow[r, "T"] & \mathbb{B} \end{tikzcd} \tag{5.6}$$

is commutative where the U's denote the forgetful functors.

5.6. Proposition. *With the data given in* 5.5*, consider the functors* TG, $HT : \mathbb{A} \to \mathbb{B}$ *and a natural transformation* $\psi : TG \to HT$*. The (non-counital)* G*-comodule* (G, δ) *induces an* H*-coaction on* TG*,*

$$\zeta : TG \xrightarrow{T\delta} TGG \xrightarrow{\psi G} HTG.$$

(1) *If* (TG, ζ) *is a (non-counital)* H*-comodule, we get the commutative diagram*

$$\begin{tikzcd} TG \arrow[r, "T\gamma"] \arrow[d, "T\delta"'] & TG \arrow[rr, "\psi"] & & HT \arrow[d, "\delta' T"] \\ TGG \arrow[r, "\psi G"] & HTG \arrow[r, "HT\gamma"] & HTG \arrow[r, "H\psi"] & HHT. \end{tikzcd} \tag{5.7}$$

(2) *If* H (TG, ζ) *is a compatible* H*-module, then*

$$TG \xrightarrow{T\gamma} TG \xrightarrow{\psi} HT \xrightarrow{\gamma' T} HT \; = \; TG \xrightarrow{T\gamma} TG \xrightarrow{\psi} HT. \tag{5.8}$$

(3) *If* ε *is symmetric and* (A, υ) *is a compatible* G*-comodule, then*

$$\psi \cdot T\upsilon = HT\varepsilon \cdot HT\upsilon \cdot \psi \cdot T\upsilon.$$

Proof. The situation is dual to that of Proposition 5.2. □

5.7. Proposition. *Let (G, δ, ε) and $(H, \delta', \varepsilon')$ be r-counital comonads on the categories $\mathbb{A}$ and $\mathbb{B}$, respectively, and $T : \mathbb{A} \to \mathbb{B}$ any functor. A natural transformation $\psi : TG \to HT$ induces a lifting*

$$\widehat{T} : \underline{\mathbb{A}}^G \to \underline{\mathbb{B}}^H, \quad (A, \upsilon) \mapsto (T(A), \psi \cdot T\upsilon : T(A) \to HT(A)),$$

if and only if the diagram (5.7) *is commutative and equation* (5.8) *holds.*

Proof. The proof is dual to that of Proposition 5.3. □

Dualising Theorem 5.4, we obtain an extension of Applegate's lifting theorem for comonads (and comodules) (e.g. [22, 3.5]) to weak comonads (and compatible comodules).

5.8. Theorem. *Let (G, δ, ε) and $(H, \delta', \varepsilon')$ be weak comonads on $\mathbb{A}$ and $\mathbb{B}$, respectively. For any functor $T : \mathbb{A} \to \mathbb{B}$, there are bijective correspondences between*

(i) *liftings of T to $\widehat{T} : \underline{\mathbb{A}}^G \to \underline{\mathbb{B}}^H$;*
(ii) *compatible H-comodule structures $\upsilon : TU^G \to HTU^G$;*
(iii) *natural transformations $\psi : TG \to HT$ with commutative diagrams*

$$\begin{tikzcd}
TG \arrow[rr, "\psi"] \arrow[d, "T\delta"'] & & HT \arrow[d, "\delta' T"] \\
TGG \arrow[r, "\psi G"] & HTG \arrow[r, "H\psi"] & HHT,
\end{tikzcd}
\qquad
\begin{tikzcd}
TG \arrow[r, "T\gamma"] \arrow[d, "\psi"'] \arrow[dr, "\psi"] & TG \arrow[d, "\psi"] \\
HT \arrow[r, "\gamma' T"'] & HT.
\end{tikzcd}$$

Proof. In view of 5.6 and 5.7, the proof is dual to that of Theorem 5.4. Here we take ψ as the composition $\psi \cdot T\gamma$ (with ψ from 5.6). □

6. Lifting of endofunctors to modules and comodules

Given a weak monad (F, μ, η), or a weak comonad $(G\,\delta, \varepsilon)$, and any endofunctor T on the category $\mathbb{A}$, we have learned in the preceding sections when T can be lifted to an endofunctor of the compatible modules or comodules, respectively. Now, one may also ask if the lifting is again a weak monad or a weak comonad, respectively.

6.1. Entwining r-unital monads. *For weak monads (F, μ, η) and $(T, \check{\mu}, \check{\eta})$ on $\mathbb{A}$ and a natural transformation $\lambda : FT \to TF$, the following are equivalent:*

(a) *defining product and quasi-unit on TF by*

$$\overline{\mu} : TFTF \xrightarrow{T\lambda F} TTFF \xrightarrow{TT\mu} TTF \xrightarrow{\check{\mu}F} TF,$$
$$\overline{\eta} : I_{\mathbb{A}} \xrightarrow{\eta} F \xrightarrow{F\check{\eta}} FT \xrightarrow{\lambda} TF,$$

yields a weak monad $(TF, \overline{\mu}, \overline{\eta})$ on $\mathbb{A}$;

(b) *λ induces commutativity of the diagrams*

$$\begin{tikzcd} FFT \arrow[r, "F\lambda"] \arrow[d, "\mu T"'] & FTF \arrow[r, "\lambda F"] & TFF \arrow[d, "T\mu"] \\ FT \arrow[rr, "\lambda"] & & TF, \end{tikzcd} \qquad \begin{tikzcd} FT \arrow[r, "\vartheta T"] \arrow[d, "\lambda"'] \arrow[dr, "\lambda"] & FT \arrow[d, "\lambda"] \\ TF \arrow[r, "T\vartheta"'] & TF, \end{tikzcd} \tag{6.1}$$

$$\begin{tikzcd} FTT \arrow[r, "\lambda T"] \arrow[d, "F\check{\mu}"'] & TFT \arrow[r, "T\lambda"] & TTF \arrow[d, "\check{\mu}F"] \\ FT \arrow[rr, "\lambda"] & & TF, \end{tikzcd} \qquad \begin{tikzcd} FT \arrow[r, "F\check{\vartheta}"] \arrow[d, "\lambda"'] \arrow[dr, "\lambda"] & FT \arrow[d, "\lambda"] \\ TF \arrow[r, "\check{\vartheta}F"'] & TF; \end{tikzcd} \tag{6.2}$$

(c) *λ induces commutativity of the diagrams in* (6.1) *and the square in (*6.2*), and there are natural transformations*

$$\check{\mu}F : TTF \to TF \quad \textit{and} \quad \lambda \cdot F\check{\eta} : F \to TF$$

where $\check{\mu}F$ is a left and right F-module morphism and $\lambda \cdot F\check{\eta}$ is an F-module morphism.

If these conditions hold, we obtain morphisms of q-unital monads,

$$\lambda \cdot F\check{\eta} : F \to TF \quad \textit{and} \quad \lambda \cdot \eta T : T \to TF.$$

Proof. The assertions follow from the general results in Section 5 and some routine computations. ☐

6.2. Weak crossed products. Given (F, μ, η) and $T : \mathbb{A} \to \mathbb{A}$, the composition TF may have a weak monad structure without requiring such a structure on T. For example, replacing the natural transformations $\check{\mu}F$ and $\lambda \cdot F\check{\eta}$ in 6.1(c) by some natural transformations

$$\nu : TTF \to TF, \quad \xi : F \to TF,$$

similar to 6.1(a), a multiplication and a quasi-unit can be defined on TF. To make this a weak monad on $\mathbb{A}$, special conditions are to be imposed on ν and ξ which can be obtained by routine computations.

Having ν and ξ, one also has natural transformations

$$\bar{\nu}:\ TT \xrightarrow{TT\eta} TTF \xrightarrow{\nu} TF, \quad \overline{\eta}:\ I_{\mathbb{A}} \xrightarrow{\eta} F \xrightarrow{\xi} TF,$$

and it is easy to see that $\bar{\nu}$ leads to the same product on TF as ν does. Thus $\bar{\nu}$ and $\overline{\eta}$ may be used to define a weak monad structure on TF and the conditions required come out as *cocycle* and *twisted conditions*. For more details we refer, e.g., to [1], [11, Section 3].

For a weak comonad (G, δ, ε) and an endofunctor $T : \mathbb{A} \to \mathbb{A}$, we now consider liftings to the category of compatible G-comodules, $\widehat{T} : \underline{\mathbb{A}}^G \to \underline{\mathbb{A}}^G$. The case when T has a weak comonad structure is dual to 6.1:

6.3. Entwining weak comonads. *For weak comonads (F, δ, ε), $(T, \check{\delta}, \check{\varepsilon})$, and a natural transformation $\psi : TG \to GT$, the following are equivalent:*

(a) *defining coproduct and quasi-counit on TG by*

$$\widehat{\delta} : TG \xrightarrow{\check{\delta}G} TTG \xrightarrow{TT\delta} TTGG \xrightarrow{T\psi G} TGTG,$$
$$\widehat{\varepsilon} : TG \xrightarrow{\psi} GT \xrightarrow{G\check{\varepsilon}} G \xrightarrow{\varepsilon} I_{\mathbb{A}},$$

yields a weak comonad $(TG, \widehat{\delta}, \widehat{\varepsilon})$ on $\mathbb{A}$;

(b) *ψ induces commutativity of the diagrams, where $\gamma = T\varepsilon \cdot \delta$, $\check{\gamma} = T\check{\varepsilon} \cdot \check{\delta}$,*

$$\begin{tikzcd}
TG \arrow[rr, "\psi"] \arrow[d, "T\delta"'] & & GT \arrow[d, "\delta T"] \\
TGG \arrow[r, "\psi G"] & GTG \arrow[r, "G\psi"] & GGT,
\end{tikzcd}
\qquad
\begin{tikzcd}
TG \arrow[r, "T\gamma"] \arrow[d, "\psi"'] \arrow[dr, "\psi"] & TG \arrow[d, "\psi"] \\
GT \arrow[r, "\gamma T"'] & GT,
\end{tikzcd}
\tag{6.3}$$

$$\begin{tikzcd}
TG \arrow[rr, "\psi"] \arrow[d, "\check{\delta}G"'] & & GT \arrow[d, "G\check{\delta}"] \\
TTG \arrow[r, "T\psi"] & TGT \arrow[r, "\psi T"] & GTT,
\end{tikzcd}
\qquad
\begin{tikzcd}
TG \arrow[r, "\check{\gamma}G"] \arrow[d, "\psi"'] \arrow[dr, "\psi"] & TG \arrow[d, "\psi"] \\
GT \arrow[r, "G\check{\gamma}"'] & GT,
\end{tikzcd}
\tag{6.4}$$

(c) *ψ induces commutativity of the diagrams* (6.3) *and the square in* (6.4) *and we have natural transformations*

$$\check{\delta}G : TG \to TTG, \quad G\check{\varepsilon} \cdot \psi : TG \to G,$$

where $\check{\delta}G$ is a left and right G-comodule morphism and $G\check{\varepsilon} \cdot \psi$ is a left G-comodule morphism.

If these conditions hold, we obtain morphisms of q-unital comonads,

$$G\check{\varepsilon}\cdot\psi : TG \to G \quad \textit{and} \quad \varepsilon T\cdot\psi : TG \to T.$$

6.4. Weak crossed coproducts. In the situation of 6.3, the coproduct on TG can also be expressed by replacing the natural transformations $\check{\delta}G$ and $G\check{\varepsilon}\cdot\psi$ by any natural transformations

$$\nu : TG \to TTG \quad \text{and} \quad \zeta : TG \to G,$$

subject to certain conditions to obtain a weak comonad structure on TG.

Given ν and ζ as above, one may form

$$\widehat{\nu}:\ TG \xrightarrow{\nu} TTG \xrightarrow{TT\varepsilon} TT\,, \quad \widehat{\zeta}:\ TG \xrightarrow{\zeta} G \xrightarrow{\varepsilon} I_{\mathbb{A}}\,,$$

and it is easy to see that these induce a weak comonad structure on TG. This leads to the *weak crossed coproduct* as considered (for coalgebras) in [11] and [12], for example.

7. Mixed entwinings and liftings

Throughout this section let (F,μ,η) denote a weak monad and (G,δ,ε) a weak comonad on any category $\mathbb{A}$. In this section we investigate the lifting properties to compatible F-modules and compatible G-comodules, respectively.

7.1. Liftings of monads and comonads. Consider the diagrams

$$\begin{tikzcd} \underline{\mathbb{A}}_F \arrow[r, "\overline{G}"] \arrow[d, "U_F"'] & \underline{\mathbb{A}}_F \arrow[d, "U_F"] \\ \mathbb{A} \arrow[r, "G"] & \mathbb{A}, \end{tikzcd} \qquad \begin{tikzcd} \underline{\mathbb{A}}^G \arrow[r, "\widehat{F}"] \arrow[d, "U^G"'] & \underline{\mathbb{A}}^G \arrow[d, "U^G"] \\ \mathbb{A} \arrow[r, "F"] & \mathbb{A}. \end{tikzcd}$$

In both cases the lifting properties are related to a natural transformation

$$\omega : FG \to GF.$$

The lifting in the left hand case requires commutativity of the diagrams (Proposition 5.3)

$$\begin{tikzcd} FFG \arrow[r, "F\omega"] \arrow[d, "\mu G"'] & FGF \arrow[r, "\omega F"] & GFF \arrow[d, "G\mu"] \\ FG \arrow[rr, "\omega"] & & GF, \end{tikzcd} \qquad \begin{tikzcd} FG \arrow[r, "\omega"] \arrow[d, "\vartheta G"'] \arrow[dr, "\omega"] & GF \arrow[d, "G\vartheta"] \\ FG \arrow[r, "\omega"] & GF, \end{tikzcd} \tag{7.1}$$

whereas the lifting to $\underline{\mathbb{A}}^G$ needs commutativity of the diagrams (Proposition 5.7)

$$\begin{tikzcd} FG \arrow[rr, "\omega"] \arrow[d, "F\delta"'] & & GF \arrow[d, "\delta F"] \\ FGG \arrow[r, "\omega G"] & GFG \arrow[r, "G\omega"] & GGF, \end{tikzcd} \qquad \begin{tikzcd} FG \arrow[r, "\omega"] \arrow[d, "F\gamma"'] \arrow[dr, "\omega"] & GF \arrow[d, "\gamma F"] \\ FG \arrow[r, "\omega"] & GF. \end{tikzcd} \tag{7.2}$$

To make $\overline{G}$ a non-counital comonad with coproduct δ, the latter has to be an F-module morphism, in particular, $\delta F : GF \to GGF$ has to be an F-morphism and this follows by commutativity of the rectangle in (7.2) provided the square in (7.1) is commutative.

To make the lifting $\widehat{F}$ a non-unital monad with multiplication μ, the latter has to be a G-comodule morphism, in particular, $\mu G : FFG \to FG$ has to be a G-module morphism and this follows by commutativity of the rectangle in (7.1) provided the square in (7.2) is commutative.

7.2. Natural transformations. The data given in 7.1 allow for natural transformations

$$\begin{aligned} \xi : &\quad G \xrightarrow{\eta G} FG \xrightarrow{\omega} GF \xrightarrow{\varepsilon F} F\ , \\ \widehat{\kappa} : &\quad GF \xrightarrow{\eta GF} FGF \xrightarrow{\omega F} GFF \xrightarrow{G\mu} GF\ , \\ \widehat{\tau} : &\quad FG \xrightarrow{F\delta} FGG \xrightarrow{\omega G} GFG \xrightarrow{\varepsilon FG} FG, \end{aligned}$$

with the properties

$$\begin{gathered} G\mu \cdot \widehat{\kappa}F = \widehat{\kappa} \cdot G\mu, \quad \widehat{\tau}G \cdot F\delta = F\delta \cdot \widehat{\tau}, \\ \mu \cdot \xi F = \varepsilon F \cdot \widehat{\kappa}, \quad \xi G \cdot \delta = \widehat{\tau} \cdot \eta G. \end{gathered}$$

(i) *If the rectangle in (7.1) is commutative, then $\widehat{\kappa}$ is idempotent.*

(ii) *If the rectangle in (7.2) is commutative, then $\widehat{\tau}$ is idempotent.*

To make the liftings weak comonads or weak monads, respectively, we have to find pre-units or pre-counits, respectively. In what follows we consider these questions.

7.3. Lemma. *(Pre-counits for $\overline{G}$) Assume the diagrams in* (7.1) *to be commutative. Then the following are equivalent:*

(a) *for any $(A, \varphi) \in \underline{\mathbb{A}}_F$, $\varepsilon_A : G(A) \to A$ is an F-module morphism;*

(b) *$\varepsilon F : GF \to F$ is an F-morphism;*

(c) *$\vartheta = \mu \cdot F\eta$ induces commutativity of the diagram*

$$\begin{tikzcd} FG \arrow[r, "F\varepsilon"] \arrow[d, "\omega"'] & F \arrow[d, "\vartheta"] \\ GF \arrow[r, "\varepsilon F"] & F. \end{tikzcd} \tag{7.3}$$

If these conditions are satisfied, then (with $\gamma = G\varepsilon \cdot \vartheta$)

$$\mu G \cdot F\widehat{\tau} = \widehat{\tau} \cdot \mu G \quad \textit{and} \quad \widehat{\tau} = \vartheta\gamma.$$

Proof. This is shown by straightforward verification. □

7.4. Proposition. *Assume the diagrams in* (7.1), (7.2) *and* (7.3) *to be commutative. Then $(\overline{G}, \delta, \varepsilon)$ is a weak comonad on $\underline{\mathbb{A}}_F$.*

Proof. This follows from the preceding observations. □

Dual to Lemma 7.3 and 7.4 we obtain for the quasi-units for $\widehat{F}$:

7.5. Lemma. *(Pre-units for $\widehat{F}$) Assume the diagrams in* (7.2) *to be commutative. Then the following are equivalent:*

(a) *for any $(A, \upsilon) \in \underline{\mathbb{A}}^G$, $\eta_A : A \to F(A)$ is a G-comodule morphism;*

(b) *$\eta G : G \to FG$ is G-colinear;*

(c) *$\gamma = G\varepsilon \cdot \delta$ induces commutativity of the diagram*

$$\begin{tikzcd} G \arrow[r, "\eta G"] \arrow[d, "\gamma"'] & FG \arrow[d, "\omega"] \\ G \arrow[r, "G\eta"] & GF. \end{tikzcd} \tag{7.4}$$

If these conditions are satisfied, then

$$G\widehat{\kappa} \cdot \delta F = \delta F \cdot \widehat{\kappa} \quad \textit{and} \quad \widehat{\kappa} = \gamma\vartheta.$$

Summing up the above observations yields the

7.6. Proposition. *Assume the diagrams in* (7.1), (7.2) *and* (7.4) *to be commutative. Then $(\widehat{F}, \mu, \eta)$ is a weak monad on $\underline{\mathbb{A}}^G$.*

One may consider alternative choices for a pre-counit for $\overline{G}$ or a pre-unit for $\widehat{F}$.

7.7. Lemma. *Assume the diagrams in* (7.1) *to be commutative. With the notations from* 7.2*, the following are equivalent:*

(a) *for any* $(A,\varphi)\in\underline{\mathbb{A}}_F$, $\overline{\varepsilon}_A : G(A)\xrightarrow{\xi_A}F(A)\xrightarrow{\varphi}A$ *is an* F*-module morphism;*

(b) $\overline{\varepsilon}F : GF\xrightarrow{\xi F}FF\xrightarrow{\mu}F$ $(= GF\xrightarrow{\widehat{\kappa}}GF\xrightarrow{\varepsilon F}F)$ *is an* F*-morphism;*

(c) *commutativity of the diagram*

$$\begin{tikzcd} FFG \arrow[r, "F\omega"] & FGF \arrow[r, "F\varepsilon F"] & FF \arrow[d, "\mu"] \\ FG \arrow[u, "F\eta G"] \arrow[r, "\omega"] & GF \arrow[r, "\varepsilon F"] & F. \end{tikzcd} \tag{7.5}$$

If these conditions are satisfied, then

$$\widehat{\tau}=\mu G\cdot F\widehat{\tau}\cdot F\eta G.$$

Proof. The proof is obtained by some diagram constructions. □

Notice that commutativity of (7.3) implies commutativity of (7.5).

7.8. Lemma. *Assume the diagrams in* (7.2) *to be commutative. Then the following are equivalent:*

(a) *for any* $(A,\upsilon)\in\underline{\mathbb{A}}^G$, $\widehat{\eta} : A\xrightarrow{\upsilon}G(A)\xrightarrow{\xi_A}F(A)$ *is a* G*-comodule morphism;*

(b) $\widehat{\eta}G : G\xrightarrow{\eta G}FG\xrightarrow{\widehat{\tau}}FG$ $(= G\xrightarrow{\delta}GG\xrightarrow{\xi G}FG)$ *is* G*-colinear;*

(c) *commutativity of the diagram*

$$\begin{tikzcd} G \arrow[r, "\eta G"] \arrow[d, "\delta"] & FG \arrow[r, "\omega"] & GF \\ GG \arrow[r, "G\eta G"] & GFG \arrow[r, "G\omega"] & GGF. \arrow[u, "G\varepsilon F"] \end{tikzcd} \tag{7.6}$$

If these conditions are satisfied, then

$$\widehat{\kappa}=G\varepsilon F\cdot G\widehat{\kappa}\cdot\delta F.$$

Proof. The situation is dual to Lemma 7.7. □

Notice that commutativity of (7.4) implies commutativity of (7.6).

7.9. Proposition. *With the data given in* 7.1*, assume the diagrams in* (7.1), (7.2) *and* (7.5) *to be commutative.*

(1) *If* (7.6) *is commutative, then* $\overline{\varepsilon}$ *from* 7.7 *is regular for* δ*, and for* $\overline{\delta} : G \to GG$ *with*

$$\overline{\delta}F : \ GF \xrightarrow{\delta F} GGF \xrightarrow{G\widehat{\kappa}} GGF,$$

$(\overline{G}, \overline{\delta}, \overline{\varepsilon})$ *is an* r*-counital comonad on* $\underline{\mathbb{A}}_F$.

(2) *If* (7.4) *is commutative, then* $\overline{\delta}F = \delta F \cdot \widehat{\kappa}$ *and* $(\overline{G}, \overline{\delta}, \overline{\varepsilon})$ *is a weak comonad on* $\underline{\mathbb{A}}_F$.

Proof. This can be shown by suitable diagram constructions. □

7.10. Proposition. *With the data given in* 7.1*, assume the diagrams in* (7.1), (7.2)*, and* (7.6) *to be commutative.*

(1) *If* (7.5) *is commutative, then* $\widehat{\eta}$ *in* 7.8 *is regular for* μ*, and for* $\widehat{\mu} : FF \to F$ *with*

$$\widehat{\mu}G : \ FFG \xrightarrow{F\widehat{\tau}} FFG \xrightarrow{\mu G} FG,$$

$(\widehat{F}, \widehat{\mu}, \widehat{\eta})$ *is an* r*-unital monad on* $\underline{\mathbb{A}}^G$.

(2) *If* (7.3) *is commutative, then* $\widehat{\mu}G = \widehat{\tau} \cdot \mu G$ *and* $(\widehat{F}, \widehat{\mu}, \widehat{\eta})$ *is a weak monad on* $\underline{\mathbb{A}}^G$.

Proof. This is dual to Proposition 7.9. □

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Contact information

Robert Wisbauer Mathematisches Institut,
Heinrich Heine University,
40225 Düsseldorf, Germany
E-Mail: `wisbauer@math.uni-duesseldorf.de`
URL: `www.math.uni-duesseldorf.de/∼wisbauer`

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