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On one class of semiperfect semidistributive rings

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Abstract. In this paper we consider the Artinian semidistributive rings.

1. Introduction

It is well known that many important classes of rings are naturally characterized by the properties of modules over them. As examples, we mention semiperfect semidistributive rings.The first papers on the theory of semidistributive rings were appeared in the middle of XX century. The reduction theorem for SPSD-rings and decomposition theorem for semiprime right Noetherian SPSD-rings were proved in [2].

2. Reduction theorem for SPSD-rings

We write SPSDR-ring (SPSDL-ring) for a semiperfect right (left) semidistributive ring and SPSD-ring for a semiperfect semidistributive ring.

Theorem 1. *(A.Tuganbaev). A semiperfect ring A is right (left) semidistributive if and only if for any local idempotents e and f of the ring A the set eAf is a uniserial right fAf-module (uniserial left eAe-module).*

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Corollary 1. $1 = e_1 + ... + e_n$ *be a decomposition of* $1 \in A$ *into a sum of mutually orthogonal local idempotents. The ring A is right (left) semidistributive if and only if for any idempotents ei and ej of the above decomposition, the set* e_iAe_j *is a uniserial right* e_jAe_j*-module (left* e_iAe_i*-module).*

Corollary 2. *Let A be a semiperfect ring, and let* $1 = e_1 + ... + e_n$ *be a decomposition of* $1 \in A$ *into a sum of mutually orthogonal local idempotents. The ring A is right (left) semidistributive if and only if for any idempotents* e_i *and* e_j $(i = j)$ *of the above decomposition the ring* $(e_i + e_j)A(e_i + e_j)$ *is right (left) semidistributive.*

Corollary 3. *Let A be a Noetherian SPSD-ring, and let* $1 = e_1 + ... + e_n$ *be a decomposition of the identity* $1 \in A$ *into a sum of mutually orthogonal local idempotents, let* $A_{ij} = e_iAe_j$ *and let* R_i *be the Jacobson radical of a ring* A_{ii}*. Then* $R_iA_{ij} = A_{ij}R_j$ *for* $i, j = 1, ..., n$.

Example 1. Consider $A = \begin{pmatrix} R & C \\ 0 & C \end{pmatrix}$ as an R-algebra (R is the field of real numbers, C is the field of complex numbers). The Peirce decomposition of the Jacobson radical $R = R(A)$ has the form $R = \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix}$ and the R-algebra A is right serial, i.e., right semidistributive. The left indecomposable projective $Q_2 = \begin{pmatrix} C \\ C \end{pmatrix}$ has socle $\begin{pmatrix} C \\ 0 \end{pmatrix}$, which is a direct sum of two copies of the left simple module $\begin{pmatrix} R \\ 0 \end{pmatrix}$. Consequently, the R-algebra A is an SPSDR-ring but it is not an SPSDL-ring.

3. Quivers of SPSD-rings

Recall that a quiver without multiple arrows and multiple loops is called a simply laced quiver. Let A be an SPSD-ring. The quotient ring A/R^2 is right Artinian and its quiver $Q(A)$ is defined by $Q(A) = Q(A/R^2)$.

Theorem 2. *The quiver* $Q(A)$ *of an SPSD-ring A is simply laced. Conversely, for any simply laced quiver Q there exists an SPSD-ring A such that* $Q(A) = Q$.

Corollary 4. *The link graph* $LG(A)$ *of an SPSD-ring A coincides with a* $Q(A)$.

Proof. For any SPSD-ring A the following equalities hold: $LG(A) = Q(A, R) = Q(A)$.

Theorem 3. *For an Artinian ring A with $R^2 = 0$ the following conditions are equivalent: (a) A is semidistributive; (b) $Q(A)$ is simply laced and the left quiver $Q(A)$ can be obtained from $Q(A)$ by reversing all arrows.*

Definition 1. A semiperfect ring A such that A/R^2 is Artinian will be called Q-symmetric if the left quiver $Q(A)$ can be obtained from the right quiver $Q(A)$ by reversing all arrows.

Corollary 5. *Every SPSD-ring is Q-symmetric.*

Note 1. *Example 1 shows that an SPSDR-ring is not always Q- symmetric.*

Let $\mathfrak{O}$ be a discrete valuation ring with an unique maximal ideal $\mathfrak{M} = \pi\mathfrak{O} = \mathfrak{O}\pi$ Then all ideals $\mathfrak{O}$(left, right, two- sided) limited powers $\mathfrak{M}^k = \pi^k\mathfrak{O} = \mathfrak{O}\pi^k$.

Denote by $M_n(Z)$ the ring of all square $n \times n$-matrices over the ring of integers Z. Let $\mathcal{E} = M_n(Z)$. We shall call a matrix $\mathcal{E} = (a_{ij})$ an exponent matrix if $\alpha_{ij} + \alpha_{jk} \geq \alpha_{ik}$ for $i, j, k = 1, ..., n$ and $a_{ii} = 0$ for $i = 1, ..., n$. A matrix $\mathcal{E}$ is called a reduced exponent matrix if $\alpha_{ij} + \alpha_{ji} > 0$ for $i, j = 1, ..., n$. We shall use the following notation: $A = \{\mathfrak{O}, \mathcal{E}(A)\}$, where $\mathcal{E}(A) = (\alpha_{ij})$ is the exponent matrix of a ring A, i.e.,

$$A = \sum_{i,j=1}^{n} e_{ij}\pi^{\alpha_{ij}}\mathfrak{O},$$

, where the e_{ij} are the matrix units. If a tiled order is reduced, then $\alpha_{ij} + \alpha_{ji} > 0$ for $i, j = 1, ..., n, i \neq j$, i.e., $\mathcal{E}(A)$ is reduced.

Theorem 4. *The ring $A = \{\mathfrak{O}, \mathcal{E}(A)\}$ is Artinian semidistributive ring.*

Theorem 5. *Let A be a semiperfect semidistributive ring and $A_A = P_1^{n_1} \oplus \ldots \oplus P_s^{n_s}$ be a decomposition of a regular module A_A into a direct sum of indecomposable pojective A-modules. A ring A is Artinian if and only if all endomorphism rings of P_i, $(i = 1, \ldots, n)$ are Artininan.*

Proof follows from theorem 1.

References

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