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RESEARCH ARTICLE

Function algebras on rectangular bands

C. J. Maxson

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Abstract. We investigate function algebras determined by rectangular bands. The focus is on maximal semirings within these function algebras and invariants associated with certain mutations.

1. Preliminaries

For our purposes in this paper a *rectangular band* is any semigroup isomorphic to the Cartesian product $L \times R$ of arbitrary sets L and R with the binary operation, $(\ell_1, r_1)(\ell_2, r_2) = (\ell_1, r_2)$, $\ell_1, \ell_2 \in L$, $r_1, r_2 \in R$. For additional characterizations we state the following result which can be found in Howie's book ([5], p. 96).

Theorem 1.1. *If S is a semigroup the following are equivalent:*

A) *S is a rectangular band;*

B) $\forall a, b \in S, ab = ba$ *implies* $a = b$;

C) $\forall a, b \in S, aba = a$;

D) $\forall a \in S, a^2 = a$, *and* $\forall a, b, c \in S, abc = ac$.

Rectangular bands are the building blocks for bands since every band is a semilattice of rectangular bands, ([5], Theorem 3.1). Better yet, a normal band is a Clifford Semilattice (called strong semilattice by several authors) of rectangular bands, ([5], Theorem 5.14). See also Theorem 3.16

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of Howie ([5]) for a result of Petrich giving a general structure theorem for bands.

We recall that a semigroup S is a *medial semigroup* if it satisfies the identity $x(ab)y = x(ba)y$, for each x, y, a, b in S. We note that a rectangular band is a medial semigroup. In fact every normal band is medial ([11], p. 75) and, since the medial identity implies the normal identity we see that medial bands are precisely the normal bands. Our interest in medial semigroups stems from the fact that for such a semigroup $(S, +)$, the collection of semigroup endomorphisms, $\mathrm{End}(S)$, is a semiring under pointwise addition and function composition. That is $(\mathrm{End}(S), +)$ is a semigroup, $(\mathrm{End}(S), \circ)$ is a monoid with identity $id_S \equiv 1_S$ and $f \circ (g + h) = f \circ g + f \circ h$, $(g + h) \circ f = g \circ f + h \circ f$, $\forall f, g, h \in \mathrm{End}(S)$. Thus $\mathrm{End}(S)$ is a semiring in the near-semiring $(M(S), +, \circ)$ of self maps on S. We remark that the medial property does not characterize those semigroups $(S, +)$ for which $\mathrm{End}(S)$ is a semiring, ([4]).

We say that a medial semigroup, S, has the *max-end property* when $\mathrm{End}(S)$ is a maximal semiring in $M(S)$. It was shown in [9] that torsion abelian groups A have the max-end property in that $\mathrm{End}(A)$ is a maximal ring in $M(A)$. In [6] several classes of commutative semigroups were shown to have the max-end property.

One of the tools used to show the max-end property was to show that the structure is endomorphism locally cyclic. A medial semigroup is endomorphism locally cyclic, denoted by E-lc, if $\forall a, b \in S$, $\exists \alpha, \beta \in \mathrm{End}(S)$ and $\exists c \in S$ such that $\alpha(c) = a$ and $\beta(c) = b$. The proof of the next result is similar to that of the corresponding result in [6].

Proposition 1.2. *A medial band has the max-end property.*

Proof. Let $(S, +)$ be a medial band and let R be a semiring in $M(S)$ such that $\mathrm{End}(S) \subseteq R \subseteq M(S)$. We know $R \subsetneqq M(S)$ since $M(S)$ is not a semiring. Since each $a \in S$ is an idempotent, the constant map $k_a\colon\ S \to S$, $k_a(s) = a, \forall s \in S$, is an endomorphism of S. Thus for $a, b, c \in S$, $k_a(c) = a$ and $k_b(c) = b$ so S is E-lc. Thus for $\rho \in R$, $\rho(a + b) = \rho(k_a(c) + k_b(c)) = \rho(k_a + k_b)(c) = (\rho k_a + \rho k_a)(c) = \rho(a) + \rho(b)$. Hence $\rho \in S$ and $S = R$. □

From the above proof we have

Proposition 1.3. *E-lc implies max-end.*

We remark that the converse of the above implication is not true. (See [3].)

For use in the sequel we state the following known characterization of endomorphisms of rectangular bands.

Lemma 1.4 ([5], Proposition 3.4). *If φ is a homomorphism from a rectangular band $L_1 \times R_1$ into a rectangular band $L_2 \times R_2$ there exist mappings $\varphi_1\colon\ L_1 \to L_2$, $\varphi_2\colon\ R_1 \to R_2$ such that $\varphi(\ell_1, r_1) = (\varphi_1(\ell_1), \varphi_2(r_1))$ for every $(\ell_1, r_1) \in L_1 \times R_1$. Conversely, for any mappings $\varphi_1\colon\ L_1 \to L_2$, $\varphi_2\colon\ R_1 \to R_2$, the map $\varphi\colon\ L_1 \times R_1 \to L_2 \times R_2$ given by $\varphi(\ell_1, r_1) = (\varphi_1(\ell_1), \varphi_2(r_1))$ defines a homomorphism from $L_1 \times R_1$ into $L_2 \times R_2$.*

Recall that a semigroup isomorphic to the direct product of a rectangular band and a group is called a *rectangular group*. The above theorem has a generalization to rectangular groups.

Corollary 1.5 ([10], IV.4.4). *Let S_1 be the rectangular group $L_1 \times R_1 \times G_1$ and S_2 the rectangular group $L_2 \times R_2 \times G_2$. Let $\varphi_1\colon\ L_1 \to L_2$, $\varphi_2\colon\ R_1 \to R_2$ be arbitrary functions and let $\varphi_3\colon\ G_1 \to G_2$ be a group homomorphism. Then the function $\varphi(\ell, r, g) = (\varphi_1(\ell), \varphi_2(r), \varphi_3(g))$, $(\ell, r, g) \in S_1$ is a homomorphism from S_1 into S_2 and, conversely, every homomorphism of S_1 into S_2 arises in this manner.*

We further recall ([2]) that a medial semigroup, S, is simple (no two-sided ideals) if and only if S is isomorphic to a rectangular abelian group (S is a rectangular group $L \times R \times G$ and G is an abelian group).

Corollary 1.6. *A simple medial semigroup $S = L \times R \times A$ where A is a torsion abelian group has the max-end property.*

Proof. From Proposition 1.2, $L \times R$ is E-lc and from [9] A is E-lc. The result then follows from Corollary 1.5. □

2. (φ, ψ)-mutations of rectangular bands

We recall the definition of a (φ, ψ)*-mutation* of a medial semigroup. To this end, let $S = (S, +)$ be a medial semigroup, let, φ, ψ be commuting endomorphisms of S and define a new operation, $\oplus$, on S by $a \oplus b = \varphi(a) + \psi(b)$, $a, b \in S$. Using the medial property of $(S, +)$ and the commuting of φ and ψ, one finds that $(S, \oplus)$ satisfies the medial property. We say that the medial property is *invariant under* (φ, ψ)*-mutations*.

We note however that, in general, the operation $\oplus$ is not associative. If one takes φ and ψ to be idempotent endomorphisms as well, then $(S, \oplus)$ is a medial semigroup. In fact, if $(S, +, 0)$ is a monoid and φ, ψ are 0-preserving commuting endomorphisms then the idempotentcy of φ and ψ is both necessary and sufficient for $(S, \oplus)$ to be a medial semigroup. See [7] and the references given there for further information on (φ, ψ)-mutations.

We now take $(S,+)$ to be a rectangular band and take $\varphi = (\varphi_1, \varphi_2)$ and (ψ_1, ψ_2) to be commuting, idempotent endomorphisms of $S = L\times R$. Thus $\varphi_1^2 = \varphi_1$, $\varphi_2^2 = \varphi_2$, $\psi_1^2 = \psi_1$, $\psi_2^2 = \psi_2$ and $\varphi_1\psi_1 = \psi_1\varphi_1$, $\varphi_2\psi_2 = \psi_2\varphi_2$. Hence $(L \times R, \oplus)$ is a medial semigroup, $(\ell_1, r_1) \oplus (\ell_2 r_2) = \varphi(\ell_1, r_1) + \psi(\ell_2, r_2) = (\varphi_1(\ell_1), \varphi_2(\ell_2)) + (\psi_1(\ell_2), \psi_2(\ell_2)) = (\varphi_1(\ell_1), \psi_2(r_2))$. In [7] we showed that the max-end property is invariant under (φ,ψ)-mutations of finite abelian groups and certain chains. We now show that the max-end property is invariant under all (φ,ψ)-mutations of rectangular bands.

Lemma 2.1. *Let $f = (f_1, f_2) \in \mathrm{End}(S,+)$, $S = L \times R$, a rectangular band, and let $\varphi = (\varphi_1,\varphi_2)$, $\psi = (\psi_1,\psi_2)$ be commuting, idempotent endomorphisms of $(S,+)$. Then f is an endomorphism of the (φ,ψ)-mutation $(S,\oplus) \Leftrightarrow f_1$ commutes with φ_1 and f_2 commutes with ψ_2.*

Proof. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be arbitrary in $S = L \times R$. Then $f \in \mathrm{End}(S,\oplus) \Leftrightarrow f(x\oplus y) = f(x)\oplus f(y) \Leftrightarrow f(\varphi x+\psi y) = \varphi f(x)+\psi f(y) \Leftrightarrow f(\varphi_1(x_1), \psi_2(y_2)) = (\varphi_1 f_1(x_1),\ \psi_2 f_2(y_2)) \Leftrightarrow f_1\varphi_1(x_1) = \varphi_1 f_1(x_1)$ and $f_2\psi_2(y_2) = \psi_2 f_2(y_2)$. □

Theorem 2.2. *Every (φ,ψ)-mutation of a rectangular band is E-lc.*

Proof. Let $(S,+) = (L\times R,+)$ be a rectangular band and let $\varphi = (\varphi_1,\varphi_2)$, $\psi = (\psi_1,\psi_2)$ be commuting, idempotent endomorphisms of S. Let $a = (a_1, a_2)$, $b = (b_1, b_2)$ be arbitrary in S. From the above lemma, it suffices to find $f_1, g_1 \in \mathrm{Map}(L)$, $f_2, g_2 \in \mathrm{Map}(R)$ with $f_1\varphi_1 = \varphi_1 f_1$, $g_1\varphi_1 = \varphi_1 g_1$, $f_2\psi_2 = \psi_2 f_2$, $g_2\psi_2 = \psi_2 g_2$ and $c = (c_1, c_2) \in L \times R$ such that $f_1(c_1) = a_1$, $g_1(c_1) = b_1$, $f_2(c_2) = a_2$, $g_2(c_2) = b_2$. We work with L, the situation for R is similar.

Case i]: $\varphi_1(a_1) = a_1$ or $\varphi_1(b_1) = b_1$. We suppose $\varphi_1(b_1) = b_1$. Let $c_1 = a_1$, $f_1 = 1_L$ (the identity function on L) and $g_1 = c_{a_1}$, the constant function $c_{a_1}(\ell) = a_1$ for all $\ell \in L$. Now f_1 commutes with φ_1 and $f_1(c_1) = a_1$. Also $g_1(c_1) = b_1$ and for $\ell \in L$, $\varphi_1 g_1(\ell) = \varphi_1(b_1) = b_1 = g_1\varphi_1(\ell)$.

Case ii]: $\varphi_1(a_1) \neq a_1$ and $\varphi_1(b_1) \neq b_1$. In this case neither a_1 nor b_1 is in $\mathrm{Im}\ \varphi_1$. For if $\varphi_1(z) = a_1$ for some $z \in L$, then $a_1 = \varphi_1(z) = \varphi_1^2(z) = \varphi_1(a_1)$, a contradiction. We also note that for any $\ell \in L$, the fibers $\varphi_1^{-1}\varphi_1(\ell)$ are φ_1-invariant since $y \in \varphi_1^{-1}\varphi_1(\ell)$ implies $\varphi_1(y) = \varphi_1(\ell)$ and so $\varphi_1(\varphi_1(y)) = \varphi_1(\ell)$, i.e., $\varphi_1(y) \in \varphi_1^{-1}\varphi_1(\ell)$.

Case ii]a: $\varphi_1(a_1) = \varphi_1(b_1)$. Let $c_1 = a_1$ and define $g_1 \in \mathrm{Map}(L)$ by

$$g_1(x) = \begin{cases} b_1, & x = a_1; \\ \varphi_1(a_1), & x \in \varphi_1^{-1}\varphi_1(a_1), x \neq a_1; \\ x, & x \notin \varphi_1^{-1}, \varphi_1(a_1). \end{cases}$$

Then $g_1\varphi_1(a_1) = \varphi_1(a_1)$ since $a_1 \neq \varphi_1(a_1) \in \varphi_1^{-1}\varphi_1(a_1)$ and $\varphi_1 g_1(a_1) = \varphi_1(b_1) = \varphi_1(a_1)$. Moreover, for $x \in \varphi_1^{-1}\varphi_1(a_1)$, $x \neq a_1$ we get $\varphi_1 g_1(x) = \varphi_1(a_1) = g_1\varphi_1(x)$. For $x \notin \varphi_1^{-1}\varphi_1(a_1)$ one also finds $\varphi_1 g_1(x) = g_1\varphi_1(x)$ so g_1 commutes with φ_1. In this case we take $f_1 = 1_L$.
Case ii]b: $\varphi_1(a_1) \neq \varphi_1(b_1)$. Let $c_1 = a_1$ and define $g_1 \in \mathrm{Map}(L)$ by

$$g_1(x) = \begin{cases} b_1, & x = a_1; \\ \varphi_1(b_1), & x = \varphi_1(a_1); \\ b_1, & x \in \varphi_1^{-1}\varphi_1(a_1)\backslash\{a_1, \varphi_1(a_1)\}; \\ x, & x \notin \varphi_1^{-1}\varphi_1(a_1). \end{cases}$$

Suppose $x \in \varphi_1^{-1}\varphi_1(a_1)\backslash\{a_1, \varphi_1(a_1)\}$. Then $\varphi_1 g_1(x) = \varphi_1(b_1)$ and $g_1\varphi_1(x) = g_1\varphi_1(a_1) = \varphi_1(b_1)$ since $\varphi_1(x) = \varphi_1(a_1)$. In the other cases we also find $\varphi_1 g_1 = g_1\varphi_1$ so g_1 commutes with φ_1 and again we take $f_1 = 1_F$. Hence we have found $f_1, g_1 \in \mathrm{Map}\ L$, f_1, g_1 commuting with φ_1, and $c_1 \in L$ such that $f_1(c_1) = a_1$ and $g_1(c_1) = b_1$. In the same manner we find $f_2, g_2 \in \mathrm{Map}(R)$, f_2, g_2 commuting with ψ_2, and $c_2 \in R$ such that $f_2(c_2) = a_2, g_2(c_2) = b_2$. This means that $f = (f_1, f_2)$ and $g = (g_1, g_2)$ are endomorphisms of $(L \times R, \oplus)$ and $f(c_1, c_2) = (a_1, a_2)$, $g(c_1, c_2) = (b_1, b_2)$, i.e., $(L \times R, \oplus)$ is E-lc. □

From Proposition 1.3 we get our desired result.

Corollary 2.3. *The max-end property is invariant under all (φ, ψ)-mutations of a rectangular band.*

In [7] it is shown that the max-end property is invariant under all (φ, ψ)-mutations of a finite abelian group. We thus have the following result.

Corollary 2.4. *The max-end property is invariant under all (φ, ψ)-mutations of a rectangular abelian group, $L \times R \times A, A$ a finite abelian group.*

3. Maximal semirings in $M(S), S$ a rectangular band

In Section 1 we found that when S is a rectangular band, $\mathrm{End}(S)$ is a maximal semiring in $M(S)$. We now investigate how to determine other maximal semirings in $M(S)$. To this end, we recall the Galois correspondence for medial semigroups discussed in [8], here specialized to rectangular bands.

We take $S = L \times R$ and let $\boldsymbol{C} = \{C_\alpha\}$, $\alpha \in \mathcal{A}$ be a cover of S by subsemigroups, S_α, i.e., $S = \bigcup_{\alpha\in\mathcal{A}} C_\alpha$. For each cover $\boldsymbol{C} = \{C_\alpha\}$ we define $\mathcal{S}(\boldsymbol{C}) := \{f \in M(S) | f_{|C_\alpha} \in \mathrm{End}(C_\alpha), \forall C_\alpha \in \boldsymbol{C}\}$. One verifies that $\mathcal{S}(\boldsymbol{C})$ is a semiring, called the *semiring determined by* $\boldsymbol{C}$. On the other hand, for each semiring T in $M(S)$ we define $\mathcal{C}(T) := \{B | B$ is a subsemigroup of S and $f_{|_B} \in \mathrm{End}(S), \forall f \in T\}$ and note that $\mathcal{C}(T)$ is a cover of S. If Γ denotes the collection of covers of S and Λ denotes the collection of semirings in $M(S)$, then the maps $\mathcal{S}\colon\ \Gamma \to \Lambda$, $\boldsymbol{C} \mapsto \mathcal{S}(\boldsymbol{C})$, and $\mathcal{C}\colon\ \Lambda \to \Gamma$, $T \mapsto \mathcal{C}(T)$, determine a Galois correspondence between Γ and Λ. (See [8] or [1] for further details.) For $\boldsymbol{C} \in \Gamma$, $\mathcal{CS}(\boldsymbol{C}) \supseteq \boldsymbol{C}$ and $\mathcal{SCS}(\boldsymbol{C}) = \boldsymbol{C}$. We let $\overline{\boldsymbol{C}} = \mathcal{CS}(\boldsymbol{C})$ and call $\overline{\boldsymbol{C}}$ the *closure of* $\boldsymbol{C}$. Note also that $\mathcal{S}(\boldsymbol{C}) = \mathcal{S}(\overline{\boldsymbol{C}})$. The next result was given for medial semigroups in [8] and for groups/rings in [1].

Theorem 3.1. *Let $\boldsymbol{C}$ be a cover of a rectangular band S. Then $\mathcal{S}(\boldsymbol{C})$ is a maximal semiring in $M(S)$ $\Leftrightarrow$ for any cover $\boldsymbol{D}$ of S, $\boldsymbol{D} \subseteq \overline{\boldsymbol{C}} \Rightarrow \overline{\boldsymbol{D}} = \overline{\boldsymbol{C}}$.*

We mention that every maximal semiring in $M(S)$ arises as a semiring determined by a cover. For if T is a maximal semiring in $M(S)$ then $T \subseteq \mathcal{SC}(T) \subseteq M(S)$. Since $M(S)$ is not a semiring we get $T = \mathcal{SC}(T)$.

Suppose $\boldsymbol{C} = \{S\}$. Then $\mathcal{S}(\boldsymbol{C}) = \mathrm{End}(S)$ and $\overline{\boldsymbol{C}} = \{B | B$ is an $\mathrm{End}(S)$-invariant subsemigroup of $S\}$. For each $s \in S$ the constant function c_s is in $\mathrm{End}(S)$ so we have $S \subseteq B$. Thus $\overline{\boldsymbol{C}} = \boldsymbol{C}$ and so, from the above theorem $\mathrm{End}(S)$ is a maximal semiring in $M(S)$. This provides an alternate proof of Proposition 1.2 above.

We next consider the situation in which the cover $\boldsymbol{C} = \{C_\alpha\}$, $\alpha \in \mathcal{A}$, is a partition of S, hence $C_\alpha \cap C_\beta = \emptyset$, $\alpha, \beta \in \mathcal{A}$, $\alpha \neq \beta$. In the next theorem we characterize when a partition determines a maximal semiring in $M(S)$.

Theorem 3.2. *Let $\boldsymbol{C} = \{C_\alpha\}$, $\alpha \in \mathcal{A}$, be a partition of the rectangular band $(S, +)$, $S = L \times R$. The following are equivalent:*

i] *$\mathcal{S}(\boldsymbol{C})$ is not a maximal semiring in $M(S)$;*

ii] *$\boldsymbol{C} \neq \overline{\boldsymbol{C}}$, i.e., $\boldsymbol{C}$ is not a closed cover;*

iii] *$\exists C_1, C_2 \in \boldsymbol{C}$ such that $\langle C_1 \cup C_2\rangle \in \overline{\boldsymbol{C}}$ where $\langle C_1 \cup C_2\rangle$ is the rectangular band in S generated by $C_1 \cup C_2$;*

iv] *$\exists C_1, C_2 \in \boldsymbol{C}$ such that $C_1 \cup C_2 \in \overline{\boldsymbol{C}}$ or $C_1, C_1 + C_2, C_2 + C_1, C_2$ are singleton cells in $\boldsymbol{C}$.*

Proof. The equivalence of, i] and ii] is given in [8]. If $\langle C_1 \cup C_2\rangle \in \overline{\boldsymbol{C}}$ then $\boldsymbol{C} \subsetneqq \overline{\boldsymbol{C}}$. If $\boldsymbol{C} \neq \overline{\boldsymbol{C}}$, $\exists D_1 \in \overline{\boldsymbol{C}} - \boldsymbol{C}$. For $d_1 \in D_1$ we have d_1 in some

cell, C_1, of $\boldsymbol{C}$. Since $D_1 \in \overline{\boldsymbol{C}}$, $\mathcal{S}(\boldsymbol{C})d_1 \subseteq D_1$ and since $\mathcal{S}(\boldsymbol{C})d_1 = C_1$ we get $C_1 \subseteq D_1$. But $D_1 \in \boldsymbol{C}$ so $\exists d_2 \in D_2 \backslash \boldsymbol{C}_1$. Let C_2 be the cell of $\boldsymbol{C}$ containing d_2 which in turn gives $C_1 \cup C_2 \subseteq D_1$. Hence $\langle C_1 \cup C_2 \rangle \subseteq D_1$. Since $\langle C_1 \cup C_2 \rangle = C_1 \cup (C_1 + C_2) \cup (C_2 + C_1) \cup C_2$ we note that $\mathcal{S}(\boldsymbol{C})(\langle C_1 \cup C_2 \rangle) \subseteq \langle C_1 \cup C_2 \rangle$. From this and the fact that $\langle C_1 \cup C_2 \rangle \subseteq D_1$ and $D_1 \in \overline{\boldsymbol{C}}$ we get $\mathcal{S}(\boldsymbol{C})|_{\langle C_1 \cup C_2 \rangle} \subseteq \mathrm{End}(\langle C_1 \cup C_2 \rangle)$. Thus establishes $\langle C_1 \cup C_2 \rangle \in \overline{\boldsymbol{C}} \Leftrightarrow \boldsymbol{C} \neq \overline{\boldsymbol{C}}$.
iii] $\Rightarrow$ iv]. Let $C_1 = L_1 \times R_1, C_2 = L_2 \times R_2$ so we have $\langle C_1 \cup C_2 \rangle = C_1 \cup (L_1 \times R_2) \cup (L_2 \times R_1) \cup C_2 = (L_1 \cup L_2) \times (R_1 \cup R_2)$. Suppose first $L_1 \cap L_2 \neq \emptyset$, say $\ell_1 \in L_1 \cap L_2$ and take $|L_1| > 1$. For $f \in \mathcal{S}(\boldsymbol{C})$, the action of f on L_1, $f_1\colon\ L_1 \to L_1$ is independent of the action of f on $C_2, f_1'\colon\ L_2 \to L_2$, since $C_1 \cap C_2 = \emptyset$. Thus on L_1, one can have $f_1(\ell_1) \neq \ell_1$ while on L_2, $f_1'(\ell_1) = \ell_1$. But for this situation f does not determine a function on $L_1 \cup L_2$ so $\langle C_1 \cup C_2 \rangle \notin \overline{\boldsymbol{C}}$, a contradiction to the hypothesis. From this we see that, when $L_1 \cap L_2 \neq \emptyset$, $L_1 = L_2 = \{\ell\}$. Since $C_1 \cap C_2 = \emptyset$, we get $R_1 \cap R_2 = \emptyset$ or $\langle C_1 \cup C_2 \rangle = C_1 \cup C_2$ and hence $C_1 \cup C_2 \in \overline{\boldsymbol{C}}$.

If $L_1 \cap L_2 = \emptyset$ but $R_1 \cap R_2 \neq \emptyset$ then a similar argument gives $R_1 = R_2 = \{r\}$ and again $C_1 \cup C_2 = \langle C_1 \cup C_2 \rangle \in \overline{\boldsymbol{C}}$.

The remaining case is $L_1 \cap L_2 = \emptyset$ and $R_1 \cap R_2 = \emptyset$. We let $L_1 \times R_2 =: C_{12}$ and $L_2 \times R_1 =: C_{21}$. We note that C_{12} and C_{21} are in $\overline{\boldsymbol{C}}$ and using C_1 and C_{12} we find $\langle C_1 \cup C_{12} \rangle \in \overline{\boldsymbol{C}}$ and from the above, $|L_1| = 1$. Similar considerations give $|L_\alpha| = |R_1| = |R_2| = 1$. Hence $C_1, C_1 + C_2, C_2 + C_1$ and C_2 are singleton cells so must be singleton cells in $\boldsymbol{C}$.
iv] $\Rightarrow$ iii]. If $C_1 \cup C_2 \in \overline{\boldsymbol{C}}$ then $C_1 \cup C_2$ is a subsemigroup of S so $\langle C_1 \cup C_2 \rangle = C_1 \cup C_2 \in \overline{\boldsymbol{C}}$. Suppose then that C_1, $C_1 + C_2$, $C_2 + C_1, C_2$ are singleton cells in $\boldsymbol{C}$. If $L_1 = L_2$ or $R_1 = R_2$ then we get $C_1 \cup C_2 \in \overline{\boldsymbol{C}}$, so $\langle C_1 \cup C_2 \rangle = C_1 \cup C_2 \in \overline{\boldsymbol{C}}$. Otherwise $\langle C_1 \cup C_2 \rangle = C_1 \cup (C_1 + C_2) \cup (C_2 + C_1) \cup C_2$ which is in $\overline{\boldsymbol{C}}$ since these cells are all singletons. □

We next turn to the case where there are some intersections among the cells of our cover. As a first step we suppose that only two cells have a non-empty intersection. Hence we take $\boldsymbol{C} = \{C_i\}$, $i \in I$ and take $1, 2 \in I$ with $C_1 \cap C_2 \neq \emptyset$ while $C_i \cap C_j = \emptyset$, $i \neq j$, $i \in I$, $j \in I \backslash \{1, 2\}$. If $C_1 \subseteq C_2$ or $C_2 \subseteq C_1$ then we have a partition and we have the previous theorem. Hence we assume $C_1 \not\subseteq C_2$ and $C_2 \not\subseteq C_1$ so $\boldsymbol{C} \subsetneqq \overline{\boldsymbol{C}}$ since $C_1 \cap C_2 \in \overline{\boldsymbol{C}}$.

For $i_o \in I \backslash \{1, 2\}$, suppose $\exists \omega \in S \backslash C_{i_o}$ such that $\langle C_{i_o} \cup \mathcal{S}(\boldsymbol{C})\omega \rangle \in \overline{\boldsymbol{C}}$. If we let $D = \{C_i\}_{i \in I \backslash \{i_o\}} \cup \langle C_{i_o} \cup \mathcal{S}(\boldsymbol{C})\omega \rangle$ then $\mathcal{S}(D) \supsetneqq \mathcal{S}(\boldsymbol{C})$ since $\exists g \in \mathcal{S}(D)$, $g(C_{i_o}) \subseteq \mathcal{S}(\boldsymbol{C})\omega$ and $g \notin \mathcal{S}(\boldsymbol{C})$. Suppose $\langle C_1 \cup \mathcal{S}(\boldsymbol{C})\omega \rangle \in \overline{\boldsymbol{C}}$ for $\omega \notin C_1$. If $\omega \in C_i$, $i \in I \backslash \{1, 2\}$ we are in the previous case, so we take $\omega \in C_2 \backslash C_1$. We let $D = (\boldsymbol{C} \backslash \{C_1\}) \cup \langle C_1 \cup \mathcal{S}(\boldsymbol{C})\omega \rangle$ and note that $C_1 \notin \overline{D}$

so $\mathcal{S}(\boldsymbol{C})$ is not maximal. The case for $\langle C_2 \cup \mathcal{S}(\boldsymbol{C})\omega\rangle \in \overline{\boldsymbol{C}}$ is parallel. We have established the next lemma.

Lemma 3.3. *Let $\boldsymbol{C} = \{C_i\}_{i\in I}$ be a cover with $C_1 \cap C_2 \neq \emptyset$, $1, 2 \in I$ while $C_i \cap C_j = \emptyset$, $i \neq j$, $i \in I$, $j \in I\backslash\{1, 2\}$. If $\mathcal{S}(\boldsymbol{C})$ is a maximal semiring in $M(S)$ then $\forall C_i \in \boldsymbol{C}$, $\forall\omega \in S\backslash C_i$, $\langle C_i \cup \mathcal{S}(\boldsymbol{C})\omega\rangle \notin \overline{\boldsymbol{C}}$.*

In the case of a partition $\boldsymbol{C} = \{C_i\}_{i\in I}$, we note that $\forall C_i \in \boldsymbol{C}$ and each $\omega \in C_i$, $\mathcal{S}(\boldsymbol{C})\omega = C_i$. However, in the case we are now considering where $C_1 \cap C_2 \neq \emptyset$, for $\omega \in C_1 \cap C_2$, $\mathcal{S}(\boldsymbol{C})\omega \in C_1 \cap C_2$ so $\mathcal{S}(\boldsymbol{C})\omega \subsetneqq C_1$. However, we still have the existence of an $\mathcal{S}(\boldsymbol{C})$-generator in each C_i.

Lemma 3.4. *Under the conditions of Lemma 3.3, $\forall C_i \in \boldsymbol{C}$, $\exists\omega \in C_i$ such that $\mathcal{S}(\boldsymbol{C})\omega = C_i$.*

Proof. If $i \in I - \{1, 2\}$ any $\omega \in \boldsymbol{C}_i$ suffices. We give the proof for $i = 1$, the case of $i = 2$ being similar. Let $C_1 = L_1 \times R_1$, $C_2 = L_2 \times R_2$ and let $(\bar{\ell}, \bar{r})$ be arbitrary in C_1. If $L_1 \subseteq L_2$, then since $C_1 \nsubseteq C_2$, $\exists r_1 \in R_1\backslash R_2$. We fix ℓ_o arbitrary from L_1 and define

$$f\colon\ L_1 \longrightarrow L_1$$
$$f(x) = \begin{cases} \bar{\ell}, & x = \ell_o \\ x, & \text{otherwise} \end{cases}$$

and

$$g\colon\ R_1 \longrightarrow R_1$$
$$g(y) = \begin{cases} \bar{r}, & y = r_1 \\ y, & \text{otherwise.} \end{cases}$$

We use (f, g) to obtain a function $h\colon\ S \to S$. On C_1, let $h = (f, g)$. For C_i, define $f_1 = f$ on L_1 and identity on $L_2 - L_1$ and define g_1 to be the identity on R_2. We let $h = (f_1, g_1)$ on C_2 and let h be the identity function on C_i, $i \in I\backslash\{1, 2\}$. One notes that $h \in \mathcal{S}(\boldsymbol{C})$ and $h(\ell_o, r_1) = (\bar{\ell}, \bar{r})$.

When $L_1 \nsubseteq L_2$ we take $\ell_1 \in L_1\backslash L_2$ and $r_1 \in R_1 - R_2$ if such exists, otherwise fix some $r_0 \in R_1 \subseteq R_2$. As above we construct a function $h \in \mathcal{S}(\boldsymbol{C})$ such that $h(\ell_1, r_0) = (\bar{\ell}, \bar{r})$. Thus we have $\omega \in C_1$, $\mathcal{S}(\boldsymbol{C})\omega = C_1$. $\square$

Theorem 3.5. *Let $\boldsymbol{C} = \{C_i\}, i \in I$ be a cover as described in Lemma 3.3. Then $\mathcal{S}(\boldsymbol{C})$ is not a maximal semiring in $M(S) \Leftrightarrow \exists C_i \in \boldsymbol{C}$, $\omega \in S\backslash C_i$ such that $\langle C_i \cup \mathcal{S}(\boldsymbol{C})\omega\rangle \in \overline{\boldsymbol{C}}$.*

Proof. ($\Leftarrow$). Lemma 3.3.
($\Rightarrow$). We suppose $\mathcal{S}(\boldsymbol{C})$ is not a maximal semiring in $M(S)$. From Theorem 3.1, there exists a cover $\boldsymbol{D}$, $\boldsymbol{D} \subseteq \overline{\boldsymbol{C}}$ and $\overline{\boldsymbol{D}} \neq \overline{\boldsymbol{C}}$. For $\omega_i \in C_i$, $i \in I\backslash\{1,2\}$, ω_i is in some $D_i \in \boldsymbol{D}$ so $C_i \subseteq D_i$. If $C_i \subsetneqq D_i$ then $\exists\omega \in S\backslash C_i$ such that $\langle C_i \cup \mathcal{S}(\boldsymbol{C})\omega\rangle \subseteq D_i$. For each $f \in \mathcal{S}(\boldsymbol{C})$ $f(\langle C_i \cup \mathcal{S}(\boldsymbol{C})\omega\rangle) \subseteq \langle C_i \cup \mathcal{S}(\boldsymbol{C})\omega\rangle$ and since $f_{|D_i} \in \mathrm{End}(D_i)$ we get $f_{|\langle C_i \cup \mathcal{S}(\boldsymbol{C})\omega\rangle} \in \mathrm{End}\langle C_i \cup \mathcal{S}(\boldsymbol{C})\omega\rangle$. Thus $\langle C_i \cup \mathcal{S}(\boldsymbol{C})\omega\rangle \in \overline{\boldsymbol{C}}$ and we are finished. We thus take $C_i = D_i \in D$, $i \in I\backslash\{1,2\}$. Using Lemma 3.4 we see there exists $D_1 \in \boldsymbol{D}$ such that $C_1 \subseteq D_1$. If $C_1 \subsetneqq D_1$ we get $\omega \notin C_1$ such that $\langle C_1 \cup \mathcal{S}(\boldsymbol{C})\omega\rangle \subseteq D_1$. As above we get $\langle C_1 \cup \mathcal{S}(\boldsymbol{C})\omega\rangle \in \overline{\boldsymbol{C}}$ and we are finished. If this is not the case then we have C_2 contained in some $D_2 \in \boldsymbol{D}$ and since $\overline{\boldsymbol{D}} \neq \overline{\boldsymbol{C}}$, $C_2 \subsetneqq D_2$. Thus $\exists\omega \in S\backslash\boldsymbol{C}_2$, $\langle C_2 \cup \mathcal{S}(\boldsymbol{C})\omega\rangle \in \overline{\boldsymbol{C}}$ as desired. □

Example 3.6. 1) Let $S = L \times R$ with $L = R = \{1,2\}$. Let $\boldsymbol{C}$ be the cover $\boldsymbol{C} = \{C_1 = \{(1,1),(1,2)\}$, $C_2 = \{(1,1)(2,1)$, and $C_3 = \{(2,2)\}$. From Theorem 3.5, we find that $\mathcal{S}(\boldsymbol{C})$ is a maximal semiring in $M(S)$.

2) Let $S = L \times R$, $L = \{1,2,3,4\}$ and $R = \{1,2,3\}$ with cover $\boldsymbol{C} = \{C_1 = \{(1,2),(1,3),(2,2),(2,3)\}$, $C_2 = \{(1,1),(2,1),(2,2),(1,2)\}$, $C_3 = \{(3,1)(4,1)\}$, $C_4 = \{(3,2),(4,2)\}$, $C_5 = \{(3,3),(4,3)\}$. Since $\langle C_1 \cup C_2\rangle = C_1 \cup C_2 \in \overline{\boldsymbol{C}}$, we see that $\mathcal{S}(\boldsymbol{C})$ is not a maximal semiring in $M(S)$.

We close this section with the following

General Problem: Characterize, in terms of the cell structure, those covers $\boldsymbol{C}$ of a rectangular band S such that $\mathcal{S}(\boldsymbol{C})$ is a maximal semiring in $M(S)$ and extend to rectangular abelian groups $L \times R \times A$.

4. Endomorphisms of normal bands

As indicated above, every normal band is a Clifford semilattice of rectangular bands. In this section we characterize the endomorphisms of a normal band, thus determining the functions in the semiring of endomorphisms of a normal band. Since a normal band has the max-end property one might now use the characterization of the endomorphisms to see if max-end is invariant under mutations of a normal band. We leave this for a future investigation. We mention that a characterization of the endomorphisms of a Clifford semilattice of groups has been obtained by Meldrum and Samman, ([12]).

We fix some notation. Let N be a normal band with the Clifford semilattice decomposition, $N = \bigcup_{\alpha\in\Lambda} B_\alpha$ where $B_\alpha = L_\alpha \times R_\alpha$ is a rectangular band for each $\alpha \in \Lambda$. For each $\alpha, \beta \in \Lambda$ with $\alpha \geq \beta$ we let $\varphi_{\alpha,\beta}\colon\ B_\alpha \to B_\beta$ denotes a structural map of N and recall that the semigroup operation, $+$, in N, for $a \in B_\alpha$, $b \in B_\beta$, is given by $a+b = \varphi_{\alpha,\alpha\beta}(a) + \varphi_{\beta,\alpha\beta}(b)$ where the "+" on the right hand sign of the equality is the operation in the rectangular band $B_{\alpha\beta}$. Using this notation, our characterization result is as follows.

Theorem 4.1. *A function $\psi\colon\ N \to N$ is an endomorphism of $N \Leftrightarrow$*

1) *ψ determines a semilattice endomorphism $\overline{\psi}\colon\ \Lambda \to \Lambda$;*
2) *ψ acts as a homomorphism on B_α;*
3) *For each $\alpha, \beta \in \Lambda$, the following diagram commutes*

$$\begin{array}{ccc} B_\alpha & \xrightarrow{\ \psi\ } & B_{\overline{\psi}(\alpha)} \\ {\scriptstyle \varphi_{\alpha,\alpha\beta}}\big\downarrow & & \big\downarrow{\scriptstyle \varphi_{\overline{\psi}(\alpha),\overline{\psi}(\alpha\beta)}} \\ B_{\alpha\beta} & \xrightarrow{\ \psi\ } & B_{\overline{\psi}(\alpha\beta)} \end{array}.$$

Proof. Suppose $\psi\colon\ N \to N$ is a function satisfying 1)–3). Let $a, b \in N$, $a \in B_\alpha$, $b \in B_\beta$. From $\psi(a+b) = \psi(a) + \psi(b)$ we get $\psi(\varphi_{\alpha,\alpha\beta}(a) + \varphi_{\beta,\alpha\beta}(b)) = \varphi_{\overline{\psi}(\alpha),\overline{\psi}(\alpha)\overline{\psi}(\beta)}\psi(a) + \varphi_{\overline{\psi}(\beta),\overline{\psi}(\alpha)\overline{\psi}(\beta)}(\psi(b)) = \varphi_{\overline{\psi}(\alpha),\overline{\psi}(\alpha\beta)}(\psi(a)) + \varphi_{\overline{\psi}(\alpha),\overline{\psi}(\alpha\beta)}(\psi(b))$ since $\overline{\psi}$ is a semilattice endomorphism. But, then using 3), we get $\varphi_{\overline{\psi}(\alpha),\overline{\psi}(\alpha\beta)}(\psi(a)) + \varphi_{\overline{\psi}(\alpha),\overline{\psi}(\alpha\beta)}(\psi(b)) = \psi\varphi_{\alpha,\alpha\beta}(a) + \psi\varphi_{\beta,\alpha\beta}(b) = \psi(\varphi_{\alpha,\alpha\beta}(a) + \varphi_{\beta,\alpha\beta}(b))$ since $\psi_{|B_{\alpha\beta}}$ is a homomorphism. We have $\psi \in \mathrm{End}(N)$.

For the converse we let $\psi\colon\ N \to N$ be an endomorphism of N. We first show that ψ determines a function on Λ. To this end, let $x = (x_1, x_2)$, $y = (y_1, y_2)$ be elements in, say B_α. We show $\psi(x)$ and $\psi(y)$ are in the same class, B_ε. Let $\psi(x) \in B_\delta$ and $\psi(y) \in B_\varepsilon$ then $\psi((x_1, x_2) + (y_1, x_2)) = \psi(x_1, x_2)$. If $\psi(y_1, x_2) \in B_\gamma$ then we have $\delta\gamma = \delta$. Using $\psi((y_1, x_2) + (x_1, x_2)) = \psi(y_1, x_2)$ we get $\gamma\delta = \gamma$ so $\delta = \gamma$. From $\psi((y_1, x_2) + (y_1, y_2)) = \psi(y_1, y_2)$ we get $\gamma\varepsilon = \varepsilon$ and from $\psi((y_1, y_2) + (y_1, x_2)) = \psi(y_1, x_2)$ we get $\varepsilon\gamma = \gamma$. Thus we have $\varepsilon = \delta$. (Note if $B_\alpha = \{x_1\} \times R_\alpha$ one can use $x = (x_1, x_2)$ and $y = (x_1, y_2)$.) We therefore have a map $\overline{\psi}\colon\ \Lambda \to \Lambda$. For $\alpha, \beta \in \Lambda$, choose $a \in B_\alpha$, $b \in B_\beta$ and so $\psi(a+b) = \psi(a) + \psi(b)$. From this we see $\overline{\psi}(\alpha\beta) = \overline{\psi}(\alpha)\overline{\psi}(\beta)$, hence property 1) holds.

From the fact that $\psi \in \mathrm{End}(N)$ we get $\psi_{|B_\alpha}$ is a homomorphism so property 2) holds.

For property 3) we note that for any $\alpha, \beta \in \Lambda$, $a \in B_\alpha$, $b \in B_\beta$ we have $\psi(a+b) = \psi(a) + \psi(b)$ which in turn gives $\psi\varphi_{\alpha,\alpha\beta}(a) + \psi\varphi_{\beta,\alpha\beta}(b) =$

$\varphi_{\overline{\psi}(\alpha),\overline{\psi}(\alpha\beta)}(\psi(a)) + \varphi_{\overline{\psi}(\beta),\overline{\psi}(\alpha\beta)}(\psi(b))$ where each of the summands in this equality are in $B_{\overline{\psi}(\alpha\beta)}$. Representing each of these summands by an element of $B_{\overline{\psi}(\alpha\beta)}$ we get $(c, d) + (a, b) = (g, h) + (e, f)$ so $(c, b) = (g, f)$. Using $b + a$ we get $(a, b) + (c, d) = (e, f) + (g, h)$ or $(a, d) = (e, h)$. Thus $(c, d) = (g, h)$ which is property 3). □

We conclude by stating the problem mentioned above.

Problem. *Is the max-end property invariant under all (φ, ψ)-mutations of a normal band?*

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Contact information

C. J. Maxson
Department of Mathematics
Texas A&M University
College Station, TX 77843-3368, USA
E-Mail: cjmaxson@math.tamu.edu

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