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Closure operators in the categories of modules
Part I (Weakly hereditary and idempotent operators)

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Abstract. In this work the closure operators of a category of modules R-Mod are studied. Every closure operator C of R-Mod defines two functions $\mathcal{F}_1^C$ and $\mathcal{F}_2^C$, which in every module M distinguish the set of C-dense submodules $\mathcal{F}_1^C(M)$ and the set of C-closed submodules $\mathcal{F}_2^C(M)$. By means of these functions three types of closure operators are described: 1) weakly hereditary; 2) idempotent; 3) weakly hereditary and idempotent.

1. Introduction and preliminary facts

The subjects of this paper are deeply rooted in the theory of radicals and torsions in modules ([1, 2, 3, 4, 5]). Every idempotent radical (torsion) r of R-Mod defines a closure operator in the lattice of submodules $\mathbb{L}({}_RM)$ of every module $M \in R$-Mod: if $N \subseteq M$, then the closure $\bar{N}$ of N in M is defined by $\bar{N}/N = r(M/N)$. This aspect was studied by the author in the works [5, 6, 7], where the notion of *radical closure* of R-Mod was introduced as a function which in every lattice $\mathbb{L}({}_RM)$ determines a closure operator and it is compatible with the R-morphisms.

The more general notion of closure operator of a category was investigated, in particular, in the works [8, 9, 10], where the relations of closure operators with some notions and constructions in categories and in topology were shown.

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The purpose of this work is the systematic investigation of the closure operators in module categories: properties, main types, their characterization by various methods, relations with preradicals, operations, etc.

In Part I three important types of closure operators in R-Mod are analyzed: weakly hereditary, idempotent and weakly hereditary idempotent. Such closure operators C are described by the associated functions $\mathcal{F}_1^C$ and $\mathcal{F}_2^C$, which are defined by C-dense and C-closed submodules. In the theory of radicals these facts correspond to the characterization of idempotent preradicals and radicals by means of classes of torsion or torsion-free modules ([1], [5]).

Let R be an arbitrary ring with unit. We denote by R-Mod the category of unitary left R-modules. For every module $M \in R$-Mod the lattice of submodules of M is denoted by $\mathbb{L}({}_R M)$. A *preradical* of R-Mod is a subfunctor r of identity functor of R-Mod, i.e. for every $M \in R$-Mod a submodule $r(M) \subseteq M$ is defined such that $f\big(r(M)\big) \subseteq r(M')$ for any R-morphism $f: M \to M'$. The preradical r of R-Mod defines two classes of modules:

1) $\mathcal{R}(r) = \{M \in R\text{-Mod} \mid r(M) = M\}$ – the class of r-*torsion* modules;

2) $\mathcal{P}(r) = \{M \in R\text{-Mod} \mid r(M) = 0\}$ – the class of r-*torsion-free* modules.

The preradical r is called *idempotent* if $r\big(r(M)\big) = r(M)$ for every $M \in R$-Mod; r is called *radical* if $r\big(M/r(M)\big) = 0$ for every $M \in R$-Mod. Any idempotent preradical r can be re-established by the class $\mathcal{R}(r)$: $r(M) = \sum\{N_\alpha \subseteq M \mid N_\alpha \in \mathcal{R}(r)\}$; similarly, any radical r can be restored by the class $\mathcal{P}(r)$: $r(M) = \cap\,\{N_\alpha \subseteq M \mid M/N_\alpha \in \mathcal{P}(r)\}$ ([1], [5]).

We remind also that the class of all preradicals of R-Mod can be transformed in a "big lattice" $\mathbb{PR}(\wedge, \vee)$ by the rules:

$$\Big(\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha\Big)(M) = \bigcap_{\alpha\in\mathfrak{A}} r_\alpha(M), \qquad \Big(\bigvee_{\alpha\in\mathfrak{A}} r_\alpha\Big)(M) = \sum_{\alpha\in\mathfrak{A}} r_\alpha(M).$$

The principal notion of this work is the following (see [8, 9, 10]):

Definition 1.1. A closure operator of R-Mod is a function C which associates to every pair $N \subseteq M$, where $N \in \mathbb{L}({}_R M)$, a submodule of M denoted by $C_M(N)$ such that the following conditions are satisfied*:*

- (c_1) $N \subseteq C_M(N)$*;*
- (c_2) if $N \subseteq P$, where $N, P \in \mathbb{L}({}_R M)$, then $C_M(N) \subseteq C_M(P)$*;*
- (c_3) if $f: M \to M'$ is an R-morphism and $N \subseteq M$, then $f\big(C_M(N)\big) \subseteq C_{M'}\big(f(N)\big)$.

The submodule $C_M(N)$ of M will be called the *C-closure* of N in M. For $C_M(N)$ the module M is the *superior term*, and N is the *inferior term.* The condition (c_2) is the monotony in the inferior term, while the monotony in the superior term follows from (c_3):

$$(c_2') \qquad \text{if } N \subseteq P \subseteq M, \text{ then } C_P(N) \subseteq C_M(N).$$

Indeed, if $f : P \to M$ is the inclusion, then from (c_3) we have $f\big(C_P(N)\big) \subseteq C_M\big(f(N)\big)$, i.e. $C_P(N) \subseteq C_M(N)$.

We denote by $\mathbb{CO}$ the class of all closure operators of R-Mod. The partial order in $\mathbb{CO}$ is defined by:

$$C \leq D \Leftrightarrow C_M(N) \subseteq D_M(N) \text{ for every } N \subseteq M.$$

Moreover, as in the case of preradicals the class $\mathbb{CO}$ can be considered as a "big lattice" by the rules:

$$\big(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha\big)_M(N) = \bigcap_{\alpha\in\mathfrak{A}} (C_\alpha)_M(N), \qquad \big(\bigvee_{\alpha\in\mathfrak{A}} C_\alpha\big)_M(N) = \sum_{\alpha\in\mathfrak{A}} (C_\alpha)_M(N),$$

for every family $\{C_\alpha \mid \alpha \in \mathfrak{A}\} \subseteq \mathbb{CO}$ and every pair $N \subseteq M$.

Further, in the class $\mathbb{CO}$ of closure operators of R-Mod two operations are introduced ([8, 9, 10]):

1) the *product* $C \cdot D$, where $C, D \in \mathbb{CO}$, is defined by

$$(C \cdot D)_M(N) = C_M\big(D_M(N)\big) \text{ for every } N \subseteq M;$$

2) the *coproduct* $C \# D$ is defined by

$$(C \# D)_M(N) = C_{D_M(N)}(N) \text{ for every } N \subseteq M.$$

The most important types of closure operators are the following.

Definition 1.2. The closure operator C of R-Mod is called*:*

a) **weakly hereditary** if $C_M(N) = C_{C_M(N)}(N)$ for every $N \subseteq M$*;*

b) **idempotent** if $C_M(N) = C_M\big(C_M(N)\big)$ for every $N \subseteq M$.

Remark. If C is an idempotent closure operator of R-Mod, then for any $M \in R$-Mod the function $C_M(-)$ is a closure operator of the lattice $\mathbb{L}({}_RM)$.

The construction is well known by which to every closure operator C of R-Mod "the nearest" weakly hereditary or idempotent closure operator is associated. It is realized by the product and coproduct of closure operators and consists in the following ([8]).

Let $C \in \mathbb{CO}$. We define the ascending chain of closure operators C^α by:

$$C^1 = C, \quad C^{\alpha+1} = C \cdot C^\alpha \;\text{ and }\; C^\beta = \vee \,\{C^\alpha \mid \alpha < \beta\}$$

for every ordinal α and every limit ordinal β. Then $C^* = \vee\, \{C^\alpha\}$ is an idempotent closure operator such that for every idempotent closure operator $D \geq C$ we have $D \geq C^*$. The closure operator C^* is called the *idempotent hull* of C.

Dually, for $C \in \mathbb{CO}$ we can consider the descending chain C_α of closure operators defined by:

$$C_1 = C, \quad C_{\alpha+1} = C \,\#\, C^\alpha \;\text{ and }\; C_\beta = \wedge\, \{C_\alpha \mid \alpha < \beta\}.$$

Then $C_* = \wedge\, \{C_\alpha\}$ is a weakly hereditary closure operator of R-Mod such that for every weakly hereditary closure operator $D \leq C$ we have $D \leq C_*$. The closure operator C_* is called the *weakly hereditary core* of C.

The main role in the further investigations is played by the following two types of submodules defined by a closure operator C of R-Mod.

Definition 1.3. Let $C \in \mathbb{CO}$. The submodule $N \in \mathbb{L}({}_RM)$ is called*:*

a) C-**dense** in M if $C_M(N) = M$*;*

b) C-**closed** in M if $C_M(N) = N$.

For $C \in \mathbb{CO}$ and $M \in R$-Mod we denote:

$\mathcal{F}_1^C(M) = \{N \subseteq M \mid C_M(N) = M\}$ – the set of C-dense submodules of M;

$\mathcal{F}_2^C(M) = \{N \subseteq M \mid C_M(N) = N\}$ – the set of C-closed submodules of M.

It is obvious that $\mathcal{F}_1^C(M) \cap \mathcal{F}_2^C(M) = \{M\}$.

In that way any closure operator $C \in \mathbb{CO}$ defines two functions $\mathcal{F}_1^C$ and $\mathcal{F}_2^C$, which associate to every module M the sets of submodules $\mathcal{F}_1^C(M)$ and $\mathcal{F}_2^C(M)$. In continuation we will prove that if $C \in \mathbb{CO}$ is weakly hereditary, then it can be re-established by the function $\mathcal{F}_1^C$; similarly, if C is idempotent, then it is completely determined by the function $\mathcal{F}_2^C$. These facts permit to describe the named types of closure operators by the functions of indicated form.

2. Weakly hereditary closure operators

Let $C \in \mathbb{CO}$. For every module $M \in R$-Mod we consider the set of C-dense submodules:

$$\mathcal{F}_1^C(M) = \{N \subseteq M \mid C_M(N) = M\},$$

and the function $\mathcal{F}_1^C$ which in every module M separates the set of submodules $\mathcal{F}_1^C(M)$. It is obvious that the mapping $C \longmapsto \mathcal{F}_1^C$ is monotone: if $C \leq D$, then $\mathcal{F}_1^C \leq \mathcal{F}_1^D$.

Now for convenience we consider an abstracts function $\mathcal{F}$ which determine for every $M \in R$-Mod a non-empty set of submodules $\mathcal{F}(M)$ of M such that it is compatible with isomorphisms and $M \in \mathcal{F}(M)$. We will use the following conditions (properties) of $\mathcal{F}$:

1) If $N \in \mathcal{F}(M_\alpha)$, $M_\alpha \subseteq M$ $(\alpha \in \mathfrak{A})$, then $N \in \mathcal{F}(\sum\limits_{\alpha\in\mathfrak{A}} M_\alpha)$;
2) If $N \subseteq P \subseteq M$ and $N \in \mathcal{F}(P)$, then for every $K \subseteq M$ we have $N + K \in \mathcal{F}(P + K)$;
3) If $f : M \to M'$ is an R-morphism and $N \in \mathcal{F}(M)$, then $f(N) \in \mathcal{F}(f(M))$;
4) If $N \subseteq P \subseteq M$ and $N \in \mathcal{F}(M)$, then $P \in \mathcal{F}(M)$.

Remark. The implication 2) $\Rightarrow$ 4) is obvious, since if $N \subseteq P \subseteq M$ and $N \in \mathcal{F}(M)$, then by 2) $N + P \in \mathcal{F}(M + P)$, i.e. $P \in \mathcal{F}(M)$.

Proposition 2.1. *Let C be an arbitrary closure operator of R-Mod. Then the associated function $\mathcal{F}_1^C$ satisfies the conditions* 1), 2) *and* 3)*.*

Proof. 1) Let $N \in \mathcal{F}_1^C(M_\alpha)$, $M_\alpha \subseteq M$, $\alpha \in \mathfrak{A}$. Then $C_{M_\alpha}(N) = M_\alpha$ for every $\alpha \in \mathfrak{A}$ and by the monotony (c_2') we have $C_{M_\alpha}(N) \subseteq C_{\sum\limits_{\alpha\in\mathfrak{A}} M_\alpha}(N)$. Therefore $M_\alpha \subseteq C_{\sum\limits_{\alpha\in\mathfrak{A}} M_\alpha}(N)$ for every $\alpha \in \mathfrak{A}$ and $\sum\limits_{\alpha\in\mathfrak{A}} M_\alpha \subseteq C_{\sum\limits_{\alpha\in\mathfrak{A}} M_\alpha}(N)$, i.e. $\sum\limits_{\alpha\in\mathfrak{A}} M_\alpha = C_{\sum\limits_{\alpha\in\mathfrak{A}} M_\alpha}(N)$ and $N \in \mathcal{F}_1^C(\sum\limits_{\alpha\in\mathfrak{A}} M_\alpha)$.

2) Let $N \subseteq P \subseteq M$ and $N \in \mathcal{F}_1^C(P)$. Then $C_P(N) = P$ and for every $K \subseteq M$ we have $C_P(N) + K = P + K$. From the monotony of C in both terms it follows that $C_P(N) + K \subseteq C_{P+K}(N + K)$, therefore $P + K \subseteq C_{P+K}(N + K)$, i.e. $P + K = C_{P+K}(N + K)$ and $N + K \in \mathcal{F}_1^C(P + K)$.

3) Let $f : M \to M'$ be an arbitrary R-morphism and $N \in \mathcal{F}_1^C(M)$, i.e. $C_M(N) = M$. From (c_3) it follows that $f(C_M(N)) \subseteq C_{f(M)}(f(N))$ and so $f(M) \subseteq C_{f(M)}(f(N))$, i.e. $f(M) = C_{f(M)}(f(N))$ and $f(N) \in \mathcal{F}_1^C(f(M))$. □

Further we will study the inverse transition: from the abstract function $\mathcal{F}$ of R-Mod to a closure operator of $\mathbb{CO}$. For that we introduce the following notation: if $\mathcal{F}$ is an abstract function of R-Mod, let $C^{\mathcal{F}}$ be the operator defined by the rule

$$(C^{\mathcal{F}})_M(N) = \sum\{M_\alpha \subseteq M \mid N \subseteq M_\alpha, \quad N \in \mathcal{F}(M_\alpha)\} \qquad (2.1)$$

for every $N \subseteq M$. Since $N \in \mathcal{F}(N)$, the definition is correct.

It is easy to see that the mapping $\mathcal{F} \longmapsto C^{\mathcal{F}}$ is monotone: if $\mathcal{F}' \leq \mathcal{F}''$, then $C^{\mathcal{F}'} \leq C^{\mathcal{F}''}$.

Proposition 2.2. *Let $\mathcal{F}$ be an abstract function of R-Mod, which satisfies the conditions* 1), 2) *and* 3)*. Then the operator $C^{\mathcal{F}}$ defined by the rule* (2.1) *is a closure operator of R-Mod.*

Proof. (c_1) By definition $N \subseteq (C^{\mathcal{F}})_M(N)$, since $N \subseteq M_\alpha$ for every $\alpha \in \mathfrak{A}$.

(c_2) Let $N \subseteq P \subseteq M$. Then $(C^{\mathcal{F}})_M(N)$ is defined by (2.1) and

$$(C^{\mathcal{F}})_M(P) = \sum\{L_\alpha \subseteq M \mid P \subseteq L_\alpha, \quad P \in \mathcal{F}(L_\alpha)\}.$$

Since $N \in \mathcal{F}(M_\alpha)$ $(\alpha \in \mathfrak{A})$ by condition 2) of $\mathcal{F}$ we obtain $N + P \in \mathcal{F}(M_\alpha + P)$, i.e. $P \in \mathcal{F}(M_\alpha + P)$. Denoting $L_\alpha = M_\alpha + P$ we have $M_\alpha \subseteq L_\alpha$ and $P \in \mathcal{F}(L_\alpha)$. Therefore $\sum\limits_{\alpha\in\mathfrak{A}} M_\alpha \subseteq \sum\limits_{\alpha\in\mathfrak{A}} L_\alpha$, i.e. $(C^{\mathcal{F}})_M(N) \subseteq (C^{\mathcal{F}})_M(P)$.

(c_3) If $f : M \to M'$ is an R-morphism and $N \subseteq M$, then from the condition 3) of $\mathcal{F}$ we have:

$$f((C^{\mathcal{F}})_M(N)) = f(\sum_{\alpha\in\mathfrak{A}} M_\alpha) = \sum_{\alpha\in\mathfrak{A}} f(M_\alpha).$$

Since $N \in \mathcal{F}(M_\alpha)$ $(\alpha \in \mathfrak{A})$, by condition 3) of $\mathcal{F}$ we obtain $f(N) \in \mathcal{F}(f(M_\alpha))$. By definition

$$(C^{\mathcal{F}})_{M'}(f(N)) = \sum\{L_\alpha \subseteq M' \mid f(N) \subseteq L_\alpha, \quad f(N) \in \mathcal{F}(L_\alpha)\},$$

therefore $f(M_\alpha)$ coincides with some L_α, so $f(M_\alpha) \subseteq \sum\limits_{\alpha\in\mathfrak{A}} L_\alpha$ for every $\alpha \in \mathfrak{A}$. This means that $\sum\limits_{\alpha\in\mathfrak{A}} f(M_\alpha) \subseteq \sum\limits_{\alpha\in\mathfrak{A}} L_\alpha$, i.e. $f((C^{\mathcal{F}})_M(N)) \subseteq (C^{\mathcal{F}})_{M'}(f(N))$. □

Proposition 2.3. *Let $\mathcal{F}$ be an abstract function of R-Mod which satisfies the conditions* 1), 2) *and* 3). *Then the associated closure operator $C^{\mathcal{F}}$* (*Proposition* 2.2) *is weakly hereditary and the corresponding function $\mathcal{F}_1^{C^{\mathcal{F}}}$ coincides with $\mathcal{F}$* (*i.e.* $\mathcal{F} = \mathcal{F}_1^{C^{\mathcal{F}}}$).

Proof. The submodule $(C^{\mathcal{F}})_M(N)$ is defined by (2.1) and

$$(C^{\mathcal{F}})_{\sum\limits_{\alpha \in \mathfrak{A}} M_\alpha}(N) = \sum \{L_\alpha \subseteq \sum_{\alpha \in \mathfrak{A}} M_\alpha \mid N \subseteq L_\alpha, \quad N \in \mathcal{F}(L_\alpha)\}.$$

From the condition 1) of $\mathcal{F}$ and from the relations $N \in \mathcal{F}(M_\alpha)$ $(\alpha \in \mathfrak{A})$ it follows that $N \in \mathcal{F}(\sum\limits_{\alpha \in \mathfrak{A}} M_\alpha)$. Therefore $\sum\limits_{\alpha \in \mathfrak{A}} M_\alpha$ coincides with some L_α from the definition of $(C^{\mathcal{F}})_{\sum\limits_{\alpha \in \mathfrak{A}} M_\alpha}(N)$, so $\sum\limits_{\alpha \in \mathfrak{A}} M_\alpha \subseteq \sum\limits_{\alpha \in \mathfrak{A}} L_\alpha$. This means that $(C^{\mathcal{F}})_M(N) \subseteq (C^{\mathcal{F}})_{(C^{\mathcal{F}})_M(N)}(N)$ and by monotony $(C^{\mathcal{F}})_M(N) = (C^{\mathcal{F}})_{(C^{\mathcal{F}})_M(N)}(N)$, i.e. $C^{\mathcal{F}}$ is weakly hereditary.

Now we will prove that $\mathcal{F} = \mathcal{F}_1^{C^{\mathcal{F}}}$. The relation $\mathcal{F} \leq \mathcal{F}_1^{C^{\mathcal{F}}}$ is true always and follows from the definitions: if $N \in \mathcal{F}(M)$, then from (2.1) it is clear that $(C^{\mathcal{F}})_M(N) = M$, i.e. $N \in \mathcal{F}_1^{C^{\mathcal{F}}}(M)$.

The inverse relation $\mathcal{F}_1^{C^{\mathcal{F}}} \leq \mathcal{F}$ follows from the property 1) of $\mathcal{F}$: if $N \in \mathcal{F}_1^{C^{\mathcal{F}}}(M)$, then $(C^{\mathcal{F}})_M(N) = M$, i.e. $\sum\limits_{\alpha \in \mathfrak{A}} M_\alpha = M$, and from 1) we have $N \in \mathcal{F}(\sum\limits_{\alpha \in \mathfrak{A}} M_\alpha)$, i.e. $N \in \mathcal{F}(M)$. □

In continuation the consecutive use of the mappings $C \longmapsto \mathcal{F}_1^C$ and $\mathcal{F} \longmapsto C^{\mathcal{F}}$ we will consider. If $C \in \mathbb{CO}$, then by Proposition 2.1 $\mathcal{F}_1^C$ is a function with the properties 1), 2) and 3). Therefore by Proposition 2.2 the function $\mathcal{F}_1^C$ determines the closure operator $C^{\mathcal{F}_1^C}$. We denote $C_* = C^{\mathcal{F}_1^C}$.

Proposition 2.4. *For every closure operator $C \in \mathbb{CO}$ we have*:

a) $C_* \leq C$;

b) C_* *is weakly hereditary*;

c) C_* *is the greatest weakly hereditary closure operator which is contained in C.*

Proof. a) By definition

$$(C_*)_M(N) = \sum \{M_\alpha \subseteq M \mid N \subseteq M_\alpha, \quad N \in \mathcal{F}_1^C(M_\alpha)\}.$$

Since $\mathcal{F}_1^C$ satisfies the property 1) (Proposition 2.1), from the relations $N \in \mathcal{F}_1^C(M_\alpha)$ $(\alpha \in \mathfrak{A})$ it follows that $N \in \mathcal{F}_1^C(\sum_{\alpha\in\mathfrak{A}} M_\alpha)$. Therefore $C_{\sum_{\alpha\in\mathfrak{A}} M_\alpha}(N) = \sum_{\alpha\in\mathfrak{A}} M_\alpha$ and by monotony $C_{\sum_{\alpha\in\mathfrak{A}} M_\alpha}(N) \subseteq C_M(N)$, i.e. $\sum_{\alpha\in\mathfrak{A}} M_\alpha \subseteq C_M(N)$. So $(C_*)_M(N) \subseteq C_M(N)$ for every $N \subseteq M$, i.e. $C_* \leq C$.

b) Since $\mathcal{F}_1^C$ satisfies the conditions 1), 2) and 3) (Proposition 2.1), the closure operator $C_* = C^{\mathcal{F}_1^C}$ is weakly hereditary by Proposition 2.3.

c) Let D be a weakly hereditary closure operator and $D \leq C$. We must verify that $D \leq C_*$, where $C_* = C^{\mathcal{F}_1^C}$. By definition $(C_*)_M(N) = \sum_{\alpha\in\mathfrak{A}} M_\alpha$, where $N \subseteq M_\alpha$ and $N \in \mathcal{F}_1^C(M_\alpha)$. Since D is weakly hereditary and $D \leq C$, we obtain:

$$D_M(N) = D_{D_M(N)}(N) \subseteq C_{D_M(N)}(N) \subseteq D_M(N),$$

therefore $C_{D_M(N)}(N) = D_M(N)$, i.e. $N \in \mathcal{F}_1^C\big(D_M(N)\big)$. So $D_M(N)$ is one of M_α from the definition of $(C_*)_M(N)$, therefore $D_M(N) \subseteq \sum_{\alpha\in\mathfrak{A}} M_\alpha = (C_*)_M(N)$ for every $N \subseteq M$. This means that $D \leq C_*$. □

Corollary 2.5. *The closure operator $C \in \mathbb{CO}$ is weakly hereditary if and only if $C = C_*$, where $C_* = C^{\mathcal{F}_1^C}$.* □

In Section 1 we indicated the method of construction of a *weakly hereditary core* C_* of an arbitrary closure operator $C \in \mathbb{CO}$. From the previous results it follows that there is another way of construction of this closure operator: it can be obtained by the rule $C_* = C^{\mathcal{F}_1^C}$.

The main result of this section is the following

Theorem 2.6. *The mappings $C \longmapsto \mathcal{F}_1^C$ and $\mathcal{F} \longmapsto C^{\mathcal{F}}$ define a monotone bijection between the weakly hereditary closure operators C of a category R-Mod and the abstract functions $\mathcal{F}$ of this category which satisfy the conditions* 1), 2) *and* 3).

Proof. If C is a weakly hereditary closure operator of R-Mod, then $C = C^{\mathcal{F}_1^C}$ (Corollary 2.5). On the other hand, if $\mathcal{F}$ is an abstract function of R-Mod with the properties 1), 2) and 3), then $\mathcal{F} = \mathcal{F}_1^{C^{\mathcal{F}}}$ (Proposition 2.3). □

Further we will call the abstract functions $\mathcal{F}$ of R-Mod with the properties 1), 2) and 3) *the functions of type* $\mathcal{F}_1$.

3. Idempotent closure operators

The results of this section in some sense are dual to the statements of Section 2. We will show the characterization of *idempotent* closure operators C of R-Mod by the function $\mathcal{F}_2^C$ associated to C, which in every module $M \in R$-Mod separates the set of C-*closed* submodules:

$$\mathcal{F}_2^C(M) = \{N \in \mathbb{L}(_R M) \mid C_M(N) = N\}.$$

It is easy to observe that the mapping $C \longmapsto \mathcal{F}_2^C$ is antimonotone: if $C \leq D$, then $\mathcal{F}_2^C \geq \mathcal{F}_2^D$.

As in the previous case, for convenience we firstly formulate some conditions (properties) of an abstract function $\mathcal{F}$ of R-Mod (they are dual to the conditions 1) – 4) of Section 2):

1*) If $N_\alpha \in \mathcal{F}(M),\ N_\alpha \subseteq M \quad (\alpha \in \mathfrak{A})$, then $\bigcap\limits_{\alpha \in \mathfrak{A}} N_\alpha \in \mathcal{F}(M)$;

2*) If $N \subseteq P \subseteq M$ and $N \in \mathcal{F}(P)$, then for every submodule $K \subseteq M$ the relation $N \cap K \in \mathcal{F}(P \cap K)$ is true;

3*) If $g : M \to M'$ is an R-morphism and $N' \in \mathcal{F}(g(M))$, then $g^{-1}(N') \in \mathcal{F}(M)$;

4*) If $N \subseteq P \subseteq M$ and $N \in \mathcal{F}(M)$, then $N \in \mathcal{F}(P)$.

The implication 2*) $\Rightarrow$ 4*) is obvious: if $N \subseteq P \subseteq M$ and $N \in \mathcal{F}(M)$, then by 2*) we have $N \cap P \in \mathcal{F}(M \cap P)$, i.e. $N \in \mathcal{F}(P)$.

Proposition 3.1. *Let C be an arbitrary closure operator of R-Mod. Then the associated function $\mathcal{F}_2^C$ satisfies the conditions* 1*), 2*) *and* 3*).

Proof. 1*) Let $N_\alpha \in \mathcal{F}_2^C(M),\ N_\alpha \subseteq M,\ \alpha \in \mathfrak{A}$. Then $C_M(N_\alpha) = N_\alpha$ for every $\alpha \in \mathfrak{A}$ and by monotony the inclusion $\bigcap\limits_{\alpha \in \mathfrak{A}} N_\alpha \subseteq N_\alpha$ implies $C_M(\bigcap\limits_{\alpha \in \mathfrak{A}} N_\alpha) \subseteq C_M(N_\alpha) = N_\alpha$ for every $\alpha \in \mathfrak{A}$. Therefore $C_M(\bigcap\limits_{\alpha \in \mathfrak{A}} N_\alpha) \subseteq \bigcap\limits_{\alpha \in \mathfrak{A}} N_\alpha$, i.e. $C_M(\bigcap\limits_{\alpha \in \mathfrak{A}} N_\alpha) = \bigcap\limits_{\alpha \in \mathfrak{A}} N_\alpha$ and $\bigcap\limits_{\alpha \in \mathfrak{A}} N_\alpha \in \mathcal{F}_2^C(M)$.

2*) Let $N \subseteq P \subseteq M$ and $N \in \mathcal{F}_2^C(P)$, i.e. $C_P(N) = N$. Then for every submodule $K \subseteq M$ from the monotony it follows that $C_{P \cap K}(N \cap K) \subseteq C_P(N) = N$. On the other hand, the monotony implies $C_{P \cap K}(N \cap K) \subseteq C_K(N \cap K) \subseteq K$. Therefore $C_{P \cap K}(N \cap K) \subseteq N \cap K$, i.e. $C_{P \cap K}(N \cap K) = N \cap K$ and $N \cap K \in \mathcal{F}_2^C(P \cap K)$.

3*) Let $g : M \to M'$ be an R-morphism and $N' \in \mathcal{F}_2^C(g(M))$, i.e. $C_{g(M)}(N') = N'$. Using the condition (c_3) and the relation $N' =$

$g(g^{-1}(N))$, we obtain:

$$g(C_M\left(g^{-1}(N')\right)) \subseteq C_{g(M)}(g\left(g^{-1}(N')\right)) = C_{g(M)}(N') = N'.$$

Therefore $C_M\left(g^{-1}(N')\right) \subseteq g^{-1}(N')$, i.e. $C_M\left(g^{-1}(N')\right) = g^{-1}(N')$ and $g^{-1}(N') \in \mathcal{F}_2^C(M)$. $\square$

Following the scheme of the previous case, now we will show the inverse transition from an abstract function $\mathcal{F}$ of R-Mod to a closure operator of R-Mod. For that we define the operator $C_{\mathcal{F}}$ by the rule:

$$(C_{\mathcal{F}})_M(N) = \cap\,\{N_\alpha \in \mathbb{L}(_RM) \mid N \subseteq N_\alpha,\ N_\alpha \in \mathcal{F}(M)\} \tag{3.1}$$

for every $N \subseteq M$. Since $M \in \mathcal{F}(M)$, the definition is correct.

We remark that the mapping $\mathcal{F} \longmapsto C_{\mathcal{F}}$ is antimonotone: if $\mathcal{F}' \leq \mathcal{F}''$, then $C_{\mathcal{F}'} \geq C_{\mathcal{F}''}$.

Proposition 3.2. *Let $\mathcal{F}$ be an abstract function of R-Mod which satisfies the conditions* $1^*)$, $2^*)$ *and* $3^*)$. *Then the associated operator $C_{\mathcal{F}}$ defined by the rule* (3.1) *is a closure operator of R-Mod.*

Proof. (c_1) Since $N \subseteq N_\alpha$ for every $\alpha \in \mathfrak{A}$, we have $N \subseteq (C_{\mathcal{F}})_M(N)$.

(c_2) Let $N \subseteq P \subseteq M$. The submodule $(C_{\mathcal{F}})_M(N)$ is defined by (3.1) and $(C_{\mathcal{F}})_M(P) = \cap\,\{P_\alpha \subseteq M \mid P \subseteq P_\alpha,\ P_\alpha \in \mathcal{F}(M)\}$. So we have $N \subseteq P \subseteq P_\alpha$ and $P_\alpha \in \mathcal{F}(M)$, therefore P_α is some N_α from the definition of $(C_{\mathcal{F}})_M(N)$. This means that $\bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha \subseteq P_\alpha$ for every $\alpha \in \mathfrak{A}$ and so $\bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha \subseteq \bigcap\limits_{\alpha\in\mathfrak{A}} P_\alpha$, i.e. $(C_{\mathcal{F}})_M(N) \subseteq (C_{\mathcal{F}})_M(P)$.

(c_3) Let $f : M \to M'$ be an R-morphism and $N \subseteq M$. Then $(C_{\mathcal{F}})_M(N)$ is defined by (3.1) and

$$(C_{\mathcal{F}})_{M'}(f(N)) = \bigcap\,\{N'_\alpha \subseteq M' \mid f(N) \subseteq N'_\alpha,\ N'_\alpha \in \mathcal{F}(M')\}.$$

By the property $3^*)$ of $\mathcal{F}$, from $N'_\alpha \in \mathcal{F}(M')$ $(\alpha \in \mathfrak{A})$ it follows that $f^{-1}(N'_\alpha) \in \mathcal{F}(M)$, where $N'_\alpha \supseteq f(N)$, therefore $f^{-1}(N'_\alpha) \supseteq f^{-1}(f(N)) \supseteq N$. This means that $f^{-1}(N'_\alpha)$ is some N_α from the definition of $(C_{\mathcal{F}})_M(N)$, so $\bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha \subseteq f^{-1}(N'_\alpha)$ for every $\alpha \in \mathfrak{A}$. Therefore $\bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha \subseteq \bigcap\{f^{-1}(N'_\alpha) \mid f(N) \subseteq N'_\alpha,\ N'_\alpha \in \mathcal{F}(M')\}$. Using this relation we obtain:

$$f((C_{\mathcal{F}})_M(N)) = f(\bigcap_{\alpha\in\mathfrak{A}} N_\alpha) \subseteq f(\bigcap_{\alpha\in\mathfrak{A}} f^{-1}(N'_\alpha)) \subseteq \bigcap_{\alpha\in\mathfrak{A}} f(f^{-1}(N'_\alpha)) =$$

$$= \bigcap_{\alpha\in\mathfrak{A}} (N'_\alpha \cap Im\,f) \subseteq \bigcap_{\alpha\in\mathfrak{A}} N'_\alpha = (C_{\mathcal{F}})_{M'}(f(N)).$$

$\square$

Proposition 3.3. *Let $\mathcal{F}$ be an abstract function of R-Mod which satisfies the conditions* 1*), 2*) *and* 3*). *Then the associated closure operator* $C_{\mathcal{F}}$ (*Proposition* 3.2) *is idempotent and the corresponding function* $\mathcal{F}_2^{C_{\mathcal{F}}}$, *defined by*

$$\mathcal{F}_2^{C_{\mathcal{F}}}(M) = \{N \subseteq M \mid (C_{\mathcal{F}})_M(N) = N\}$$

coincides with $\mathcal{F}$ (*i.e.* $\mathcal{F} = \mathcal{F}_2^{C_{\mathcal{F}}}$).

Proof. For a function $\mathcal{F}$ with 1*), 2*) and 3*) the submodule $(C_{\mathcal{F}})_M(N)$ is defined by (3.1) and

$$(C_{\mathcal{F}})_M[(C_{\mathcal{F}})_M(N)] = \cap\, \{L_\alpha \subseteq M \mid (C_{\mathcal{F}})_M(N) \subseteq L_\alpha,\ L_\alpha \in \mathcal{F}(M)\}.$$

From the property 1*) of $\mathcal{F}$ and $N_\alpha \in \mathcal{F}(M)$ ($\alpha \in \mathfrak{A}$) it follows that $\bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha \in \mathcal{F}(M)$. Therefore $\bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha$ is some L_α, so $\bigcap\limits_{\alpha\in\mathfrak{A}} L_\alpha \subseteq \bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha$. This means that $(C_{\mathcal{F}})_M[(C_{\mathcal{F}})_M(N)] \subseteq (C_{\mathcal{F}})_M(N)$, the inverse inclusion being trivial, therefore $C_{\mathcal{F}}$ is idempotent.

Further we prove that $\mathcal{F} = \mathcal{F}_2^{C_{\mathcal{F}}}$. The relation $\mathcal{F} \leq \mathcal{F}^{C_{\mathcal{F}}}$ follows from the construction: if $N \in \mathcal{F}(M)$, then N is some N_α from the definition of $(C_{\mathcal{F}})_M(N)$, therefore $\bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha = N$, i.e. $(C_{\mathcal{F}})_M(N) = N$ and $N \in \mathcal{F}_2^{C_{\mathcal{F}}}(M)$.

The inverse relation $\mathcal{F}_2^{C_{\mathcal{F}}} \leq \mathcal{F}$ follows from the property 1*) of $\mathcal{F}$: if $N \in \mathcal{F}_2^{C_{\mathcal{F}}}(M)$, then $\cap\, \{N_\alpha \subseteq M \mid N \subseteq N_\alpha,\ N_\alpha \in \mathcal{F}(M)\} = N$ and by 1*) from $N_\alpha \in \mathcal{F}(M)$ ($\alpha \in \mathfrak{A}$) we have $\bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha \in \mathcal{F}(M)$, so $N \in \mathcal{F}(M)$. □

Now we will consider the combination of the mappings $C \longmapsto \mathcal{F}_2^C$ and $\mathcal{F} \longmapsto C_{\mathcal{F}}$ which were defined by the rules: $\mathcal{F}_2^C(M) = \{N \subseteq M \mid C_M(N) = N\}$ and $(C_{\mathcal{F}})_M(N) = \cap\, \{N_\alpha \subseteq M \mid N \subseteq N_\alpha,$ $N_\alpha \in \mathcal{F}(M)\}$. If C is an arbitrary closure operator of R-Mod, then $\mathcal{F}_2^C$ is a function with the properties 1*), 2*) and 3*) (Proposition 3.1). In its turn the function $\mathcal{F}_2^C$ defines the closure operator $C_{\mathcal{F}_2^C}$ (Proposition 3.2). We denote $C^* = C_{\mathcal{F}_2^C}$.

Proposition 3.4. *For every closure operator C of R-Mod we have:*

a) $C^* \geq C$;

b) C^* *is an* ***idempotent*** *closure operator*;

c) C^* *is the least idempotent closure operator containing* C.

Proof. a) By definition

$$(C^*)_M(N) = \cap\, \{N_\alpha \subseteq M \mid N \subseteq N_\alpha,\ N_\alpha \in \mathcal{F}_2^C(M)\}.$$

By property 1*) of $\mathcal{F}_2^C$ we have $\bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha \in \mathcal{F}_2^C(M)$, i.e. $C_M(\bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha) = \bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha$. By monotony the inclusion $N \subseteq \bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha$ implies $C_M(N) \subseteq C_M(\bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha) = \bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha$. This means that $C_M(N) \subseteq (C^*)_M(N)$ for every $N \subseteq M$, i.e. $C \leq C^*$.

b) The function $\mathcal{F}_2^C$ satisfies the properties 1*), 2*) and 3*) (Proposition 3.1), therefore by Proposition 3.3 the operator $C^* = C_{\mathcal{F}_2^C}$ is idempotent.

c) Let D be an idempotent closure operator of R-Mod and $D \geq C$. We will verify that $C^* \leq D$. By definition:

$$(C^*)_M(N) = \big(C_{\mathcal{F}_2^C}\big)_M(N) = \cap\, \{N_\alpha \subseteq M \mid N \subseteq N_\alpha,\ N_\alpha \in \mathcal{F}_2^C(M)\}.$$

Since D is idempotent and $D \geq C$ we obtain:

$$D_M(N) = D_M\big(D_M(N)\big) \ \geq\ C_M\big(D_M(N)\big) \ \geq\ D_M(N),$$

therefore $D_M(N) = C_M\big(D_M(N)\big)$, i.e. $D_M(N) \in \mathcal{F}_2^C(M)$. So $D_M(N)$ is some N_α from the definition of $(C^*)_M(N)$, therefore $\bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha \subseteq D_M(N)$. In this way $(C^*)_M(N) \subseteq D_M(N)$ for every $N \subseteq M$, i.e. $C^* \leq D$. $\square$

Corollary 3.5. *The closure operator C of R-Mod is idempotent if and only if $C = C^*$.* $\square$

In Section 1 the method of construction of *idempotent hull* C^* of an arbitrary closure operator C of R-Mod was shown. From Proposition 3.4 another way to obtain the idempotent hull of C follows, namely $C^*{=}C_{\mathcal{F}_2^C}$.

Totalizing the results of this section we obtain

Theorem 3.6. *The mappings $C \longmapsto \mathcal{F}_2^C$ and $\mathcal{F} \longmapsto C_{\mathcal{F}}$ define an anti-monotone bijection between the idempotent closure operators C of R-Mod and the abstract functions $\mathcal{F}$ of R-Mod, which satisfy the conditions* 1*), 2*) *and* 3*). $\square$

The abstract functions $\mathcal{F}$ of R-Mod with the properties 1*), 2*) and 3*) will be called in continuation *the functions of type* $\mathcal{F}_2$.

4. Weakly hereditary and idempotent closure operators

Using the previous results, now we will describe the closure operators of R-Mod which simultaneously are weakly hereditary and idempotent (in radical theory this corresponds to the characterization of idempotent radicals by the classes of torsion or torsion-free modules).

Let C be a weakly hereditary and idempotent closure operation of R-Mod. Then the operator C can be re-established both by the function $\mathcal{F}_1^C$ (Theorem 2.6) and by the function $\mathcal{F}_2^C$ (Theorem 3.6). We will show what property the abstract function $\mathcal{F}$ of R-Mod must satisfy so that the associated closure operators $C^{\mathcal{F}}$ and $C_{\mathcal{F}}$ should be weakly hereditary and idempotent. For that we consider the following condition of an abstract function $\mathcal{F}$ of R-Mod:

5) $= 5^*$) If $N \subseteq P \subseteq M,\ N \in \mathcal{F}(P)$ and $P \in \mathcal{F}(M)$, then $N \in \mathcal{F}(M)$.

This condition will be named *the property of transitivity* of $\mathcal{F}$ (it is autodual).

Proposition 4.1. *If C is an* ***idempotent*** *closure operator of R-Mod, then the associated function $\mathcal{F}_1^C$ (where $\mathcal{F}_1^C(M) = \{N \subseteq M \mid C_M(N) = M\}$) satisfies the property of transitivity* 5).

Proof. Let $N \subseteq P \subseteq M,\ N \in \mathcal{F}_1^C(P)$ and $P \in \mathcal{F}_1^C(M)$. Then $C_P(N) = P$ and $C_M(P) = M$. By monotony from $P \subseteq M$ it follows that $C_P(N) \subseteq C_M(N)$, therefore $P \subseteq C_M(N)$. Since C is monotone and idempotent, we obtain $C_M(P) \subseteq C_M(C_M(N)) = C_M(N)$, i.e. $M \subseteq C_M(N)$. So $C_M(N) = M$ and $N \in \mathcal{F}_1^C(M)$. $\square$

Proposition 4.2. *Let $\mathcal{F}$ be an abstract function of R-Mod of the type $\mathcal{F}_1$ (i.e. with the conditions* 1), 2), 3)) *which satisfies the property of transitivity* 5)*. Then the associated closure operator $C^{\mathcal{F}}$ defined by the rule*

$$(C^{\mathcal{F}})_M(N) = \sum\{M_\alpha \subseteq M \mid N \subseteq M_\alpha,\ N \in \mathcal{F}(M_\alpha)\}$$

is idempotent.

Proof. If $\mathcal{F}$ is a function of the type $\mathcal{F}_1$, then $C^{\mathcal{F}}$ is a closure operator (Proposition 2.2). By definition

$$(C^{\mathcal{F}})_M[(C^{\mathcal{F}})_M(N)] = \sum\{L_\alpha \subseteq M \mid (C^{\mathcal{F}})_M(N) \subseteq L_\alpha,\ (C^{\mathcal{F}})_M(N) \in \mathcal{F}(L_\alpha)\}.$$

From the definition of $(C^{\mathcal{F}})_M(N)$ we have $N \in \mathcal{F}(M_\alpha)$ $(\alpha \in \mathfrak{A})$ and by the property 1) of $\mathcal{F}$ we obtain $N \in \mathcal{F}(\sum\limits_{\alpha \in \mathfrak{A}} M_\alpha)$. Since we have also the relation $\sum_{\alpha \in \mathfrak{A}} M_\alpha \in \mathcal{F}(L_\alpha)$, by the transitivity of $\mathcal{F}$ we obtain $N \in \mathcal{F}(L_\alpha)$ for every $\alpha \in \mathfrak{A}$. Using once again the condition 1) of $\mathcal{F}$, we have $N \in \mathcal{F}(\sum_{\alpha \in \mathfrak{A}} L_\alpha)$. Therefore $\sum_{\alpha \in \mathfrak{A}} L_\alpha$ is some submodule M_α from the definition of $(C^{\mathcal{F}})_M(N)$, so $\sum_{\alpha \in \mathfrak{A}} L_\alpha \subseteq \sum_{\alpha \in \mathfrak{A}} M_\alpha$. This means that $(C^{\mathcal{F}})_M[(C^{\mathcal{F}})_M(N)] \subseteq (C^{\mathcal{F}})_M(N)$, the inverse inclusion being trivial, so $C^{\mathcal{F}}$ is idempotent. □

Corollary 4.3. *The mappings $C \longmapsto \mathcal{F}_1^C$ and $\mathcal{F} \longmapsto C^{\mathcal{F}}$ define a monotone bijection between the weakly hereditary and idempotent closure operators of R-Mod and the abstract functions $\mathcal{F}$ of type $\mathcal{F}_1$ (with the conditions* 1), 2), 3)) *of R-Mod with satisfy the property of transitivity* 5)*.*

Proof. By Theorem 2.6 the indicated mappings define a monotone bijection between the weakly hereditary closure operators C of R-Mod and abstract functions $\mathcal{F}$ of type $\mathcal{F}_1$. In this bijection if C is idempotent, then the function $\mathcal{F}_1^C$ is transitive (Proposition 4.1). On the other hand, if the function $\mathcal{F}$ of type $\mathcal{F}_1$ is transitive, then the weakly hereditary closure operator $C^{\mathcal{F}}$ is idempotent (Proposition 4.2). □

Thus the weakly hereditary and idempotent closure operators C of R-Mod are completely described by the abstract functions $\mathcal{F}$ of R-Mod which satisfy the conditions 1), 2), 3), 5).

Dually the characterization of weakly hereditary and idempotent closure operation C of R-Mod by abstract functions $\mathcal{F}$ of type $\mathcal{F}_2$ can be obtained.

Proposition 4.4. *If C is a* **weakly hereditary** *closure operator of R-Mod, then the associated function $\mathcal{F}_2^C$, where $\mathcal{F}_2^C(M) = \{N \subseteq M \mid C_M(N) = N\}$, satisfies the condition of transitivity* 5) = 5*)*.*

Proof. Let $N \subseteq P \subseteq M$, $N \in \mathcal{F}_2^C(P)$ and $P \in \mathcal{F}_2^C(M)$, where C is a weakly hereditary closure operator of R-Mod. Then $C_P(N) = N$ and $C_M(P) = P$. From $N \subseteq M$ by monotony it follows that $C_M(N) \subseteq C_M(P) = P$, i.e. $C_M(N) \subseteq P$. Using the monotony once again, we obtain $C_{C_M(N)}(N) \subseteq C_P(N) = N$. Since C is weakly hereditary, we have $C_{C_M(N)}(N) = C_M(N)$, therefore $C_M(N) \subseteq N$, i.e. $C_M(N) = N$ and $N \in \mathcal{F}_2^C(M)$. This proves that $\mathcal{F}_2^C$ is transitive. □

Proposition 4.5. *If $\mathcal{F}$ is an abstract function of R-Mod of the type $\mathcal{F}_2$ (i.e. with the conditions* 1^*), 2^*), 3^*)) *which satisfies the transitivity property* 5^*), *then the corresponding closure operator $C_{\mathcal{F}}$, defined by the rule*

$$(C_{\mathcal{F}})_M(N) = \cap \, \{N_\alpha \subseteq M \mid N \subseteq N_\alpha, \; N_\alpha \in \mathcal{F}(M)\},$$

is weakly hereditary.

Proof. By definition

$$(C_{\mathcal{F}})_{(C_{\mathcal{F}})_M(N)}(N) = \cap \, \{L_\alpha \subseteq M \mid N \subseteq L_\alpha \subseteq (C_{\mathcal{F}})_M(N),$$
$$L_\alpha \subseteq \mathcal{F}\big((C_{\mathcal{F}})_M(N)\big)\}.$$

From the definition of $(C_{\mathcal{F}})_M(N)$ we have $N_\alpha \subseteq \mathcal{F}(M)$ $(\alpha \in \mathfrak{A})$ and by condition 1^*) of $\mathcal{F}$ it follows that $\bigcap\limits_{\alpha \in \mathfrak{A}} N_\alpha \in \mathcal{F}(M)$, i.e. $(C_{\mathcal{F}})_M(N) \in \mathcal{F}(M)$.

On the other hand, from the relations $L_\alpha \in \mathcal{F}\big((C_{\mathcal{F}})_M(N)\big)$ $(\alpha \in \mathfrak{A})$ by condition 1^*) of $\mathcal{F}$ we have $\bigcap\limits_{\alpha \in \mathfrak{A}} L_\alpha \in \mathcal{F}\big((C_{\mathcal{F}})_M(N)\big)$. Using the transitivity in the situation $\bigcap\limits_{\alpha \in \mathfrak{A}} L_\alpha \subseteq (C_{\mathcal{F}})_M(N) \subseteq M$, we obtain $\bigcap\limits_{\alpha \in \mathfrak{A}} L_\alpha \in \mathcal{F}(M)$. Therefore the submodule $\bigcap\limits_{\alpha \in \mathfrak{A}} L_\alpha$ is some N_α from the definition of $(C_{\mathcal{F}})_M(N)$, so $\bigcap\limits_{\alpha \in \mathfrak{A}} N_\alpha \subseteq \bigcap\limits_{\alpha \in \mathfrak{A}} L_\alpha$. This means that $(C_{\mathcal{F}})_M(N) \subseteq (C_{\mathcal{F}})_{(C_{\mathcal{F}})_M(N)}(N)$. The inverse inclusion follows from $M \supseteq (C_{\mathcal{F}})_M(N)$. This proves that $C_{\mathcal{F}}$ is weakly hereditary. □

From Propositions 4.4 and 4.5, using Theorem 3.6, we obtain

Corollary 4.6. *The mappings $C \longmapsto \mathcal{F}_2^C$ and $\mathcal{F} \longmapsto C_{\mathcal{F}}$ define an antimonotone bijection between the weakly hereditary and idempotent closure operators C of R-Mod and the abstract functions $\mathcal{F}$ of R-Mod which satisfy the conditions* 1^*), 2^*), 3^*), 5^*) (*i.e. the transitive functions $\mathcal{F}$ of type $\mathcal{F}_2$*). □

Combining Corollaries 4.3 and 4.6, is obvious the

Corollary 4.7. *The mappings*

$$\mathcal{F} \longmapsto C^{\mathcal{F}} \longmapsto {\mathcal{F}_2}^{C^{\mathcal{F}}}, \quad \mathcal{F} \longmapsto C_{\mathcal{F}} \longmapsto {\mathcal{F}_1}^{C_{\mathcal{F}}}$$

define an antimonotone bijection between the transitive abstract functions of type $\mathcal{F}_1$ and the transitive abstract functions of type $\mathcal{F}_2$. □

Let C be a weakly hereditary and idempotent closure operator of R-Mod. For any module $M \in R$-Mod we can indicate a direct way to obtain the sets of submodules $\mathcal{F}_1^C(M)$ and $\mathcal{F}_2^C(M)$ one by another ([6], Proposition 2.3):

$$\mathcal{F}_1^C(M) = \{N \subseteq M \mid P \notin \mathcal{F}_2^C(M) \text{ for every } P \text{ such that } N \subseteq P \subsetneqq M\},$$

$$\mathcal{F}_2^C(M) = \{N \subseteq M \mid N \notin \mathcal{F}_1^C(P) \text{ for every } P \text{ such that } N \subsetneqq P \subseteq M\}.$$

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