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On maximal and minimal linear matching property

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Abstract. The matching basis in field extentions is introduced by S. Eliahou and C. Lecouvey in [2]. In this paper we define the minimal and maximal linear matching property for field extensions and prove that if K is not algebraically closed, then K has minimal linear matching property. In this paper we will prove that algebraic number fields have maximal linear matching property. We also give a shorter proof of a result established in [6] on the fundamental theorem of algebra.

1. Introduction

Throughout this paper we will consider a field extension $K \subset L$ where K is commutative and central in L. Let G be an additive group and $A, B \subset G$ be nonempty finite subsets of G. A *matching* from A to B is a map $\phi : A \to B$ which is bijective and satisfies the condition

$$a + \phi(a) \notin A$$

for all $a \in A$. This notion was introduced in [3] by Fan and Losonczy, who used matchings in $\mathbb{Z}^n$ as a tool for studying an old problem of Wakeford concerning canonical forms for symmetric tensors [7]. Eliahou

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and Lecouvey extended this notion to subspaces in a field extension, here we will introduce a notion from [2].

Let $K \subset L$ be a field extension and A, B be n-dimensional K-subspaces of L. Let $\mathcal{A} = \{a_1, \ldots, a_n\}$, $\mathcal{B} = \{b_1, \ldots, b_n\}$ be basis of A and B respectively. It is said that $\mathcal{A}$ is *matched* to $\mathcal{B}$ if

$$a_i b \in A \quad \Rightarrow \quad b \in \langle b_1, \ldots, \hat{b_i}, \ldots, b_n \rangle$$

for all $b \in B$ and $i = 1, \ldots, n$, where $\langle b_1, \ldots, \hat{b_i}, \ldots, b_n \rangle$ is the hyperplane of B spanned by the set $\mathcal{B} \setminus \{b_i\}$. Also it is said that A is *matched* to B if every basis of A can be matched to a basis of B.

It is said that L has the *linear matching property* from K if, for every $n \geq 1$ and every n-dimensional K-subspaces A and B of L with $1 \notin B$, the subspace A is matched to B. By this we mean linear matching property for K-subspaces.

As we mentioned, the above notion was introduce by Eliahou and Lecouvey in [2], where they proved that if $K \subset L$ is a field extension and $[L : K]$ is prime, then L has linear matching property (see Theorem 5.3 in [2]). We extend this property to the family of field extensions and introduce the notions of minimal and maximal linear matching properties.

2. Definitions and the main results

Definition 2.1. Let K be a field. We say K has *minimal linear matching property* if there exists a finite field extension L of K, such that L has linear matching property from K.

Definition 2.2. Let K be a field. We say K has *maximal linear matching property* if for any positive integer n, there exists a field extension L_n of K, such that $[L_n : K] = n$ and L_n has linear matching property from K.

We shall prove the following results in section 5.

Theorem 2.3. *Let K be a field which is not algebraically closed, then K has the minimal linear matching property.*

Theorem 2.4. *Algebraic number fields have the maximal linear matching property.*

Theorem 2.5. *Suppose that K is a field and has the maximal linear matching property, then K is infinite.*

To prove our main results, we will use Theorem 3.1 which can be regarded as an improvement of the foundamental theorem of algebra.

In [6], Shipman gives an algebraic proof of the foundamental theorem of algebra in special cases, but here we present a different proof which is independent Shipman's proof.

3. An improvment of the fundamental theorem of algebra

Theorem 3.1. *Let K be a field such that every polynomial of prime degree in $K[x]$ has a root in K, then K is algebraically closed.*

Proof. First, we claim there exists a prime p such that for any non-linear irreducible polynomial $f(x) \in K[x]$, p divides the degree of $f(x)$. Suppose that this claim is false, and $p_1, \ldots, p_n$ are prime divisors of the degree of $f(x)$, then there exists $g_i \in K[x]$ such that $p_i \not| \deg g_i(x)$ and $g_i(x)$ is an irreducible polynomials in $K[x]$, where $1 \leq i \leq n$.

Now set $F(x) := f^{k_0}(x) g_1^{k_1}(x) \cdots g_n^{k_n}(x)$ where $k_0, k_1, \ldots, k_n$ are non-negative integers. It is clear that $\gcd(\deg f(x), \deg g_1(x), \ldots, \deg g_n(x)) = 1$ and $\deg F = k_0 \deg f + k_1 \deg g_1 + \cdots + k_n \deg g_n$. By Dirichlet's Theorem on primes, since the k_i's are non-negative integers, we can choose $k_0, \ldots, k_n$ such that $\deg F$ becomes a prime number. So $F(x)$ has a root in K and this is a contradiction. Therefore there exists a prime p such that p divide the degree of every irreducible polynomials in $K[x]$. Now if L is a field extension of K of degree p and $\alpha \in L \setminus K$, then $L = K(\alpha)$ and if $f(x) \in K[x]$ is the minimal polynomial of α, then $\deg f(x) = p$ and $f(x)$ has a root in K and this is a contradiction, hence K has no field extension of degree p. Let L be a Galois extension of K with $[L : K] = p^r \cdot m$ where $r, m \in \mathbb{N}$, $(m, p) = 1$. By Galois fundamental theorem and Cauchy theorem, there is an intermediate field L', $K \subset L' \subset L$ such that $[L : L'] = p^r$, then $[L' : K] = m$. If $m > 1$ we can choose $\alpha \in L' \setminus K$, and assume $f(x)$ is the minimal polynomial of α over K, then $\deg f(x) | m$, also $f(x)$ is irreducible, then $p | \deg f(x)$, so $p | m$, a contradiction. Hence $m = 1$ and $[L : K] = p^r$, again by using Galois fundamental theorem and Cauchy theorem there exists an intermediate field L', $K \subseteq L' \subset L$ such that $[L : L'] = p^{r-1}$, then $[L' : K] = p$, but since we proved that K has no field extension of degree p, this is a contradiction. Thus K has no Galois extension and it is algebraically closed. $\square$

Corollary 3.2 *Let K be a field such that every polynomial of prime degree in $K[x]$ is reducible on K. Then K is algebraically closed.*

4. Preliminary results about field extensions and linear matching property

We use the following result from [4].

Theorem 4.1. *Let L be a finite field of characteristic $p > 0$ where $\mathbb{Z}_p$ is embedded in L and $[L : \mathbb{Z}_p] = n$. Then for any divisor m of n, L has a subfield with p^m elements.*

We also use the following result from [5] which is about field extensions with no proper intermediate subfield.

Theorem 4.2. *If K is an algebraic number field, then for every positive integer n there exist infinitely many field extensions of K with degree n having no proper subfields over K.*

The following theorem was proved in [2], see also [1].

Theorem 4.3. *Let $K \subset L$ be a field extension. Then L has linear matching property if and only if $K \subset L$ has no proper intermediate subfield with finite degree over K.*

Now we are ready to prove the main results.

5. Proof of main results

Proof of Theorem 2.3

Proof. By Corollary 3.2 there exists an irreducible polynomial $f(x)$ of prime degree in $K[x]$. Now if L is the splitting field of $f(x)$ over K, then $[L : K]$ is prime and by Theorem 4.3 L has the linear matching property from K, so K has the minimal linear matching property. $\square$

Proof of Theorem 2.4

Proof. Let K be an algebraic number field. Then by theorem 4.2 for any positive integer n, there exists an extension L_n of K with $[L_n : K] = n$ and this field extension has no proper intermediate subfield, then by Theorem 4.3, L_n has the linear matching property from K, so K has the maximal linear matching property. $\square$

Proof of Theorem 2.5

Proof. Let K be a finite field with $|K| = p^n$ and p a prime and n a positive integer. Now let q and m be positive integers with $n < q < m$

and $q|m$. If L is an extension of K of degree m, then $[L:\mathbb{Z}_p] = mn$ and by Theorem 4.1, $\mathbb{Z}_p \subseteq L$ has an intermediate subfield K' of degree p^q. Now since finite fields with the same cardinality are isomorphic, K' is a finite proper intermediate subfield in the extension $K \subset L$ with finite degree over K, then by Theorem 4.3, L does not have linear matching property from K, hence K does not have maximal linear matching property. $\square$

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