

Labelling matrices and index matrices of a graph structure

T. Dinesh and T. V. Ramakrishnan

Communicated by D. Simson

ABSTRACT. The concept of graph structure was introduced by E. Sampathkumar in ‘Generalised Graph Structures’, Bull. Kerala Math. Assoc., Vol 3, No.2, Dec 2006, 65-123. Based on the works of Brouwer, Doob and Stewart, R.H. Jeurissen has (‘The Incidence Matrix and Labelings of a Graph’, J. Combin. Theory, Ser. B30 (1981), 290-301) proved that the collection of all admissible index vectors and the collection of all labellings for 0 form free F -modules (F is a commutative ring). We have obtained similar results on graph structures in a previous paper. In the present paper, we introduce labelling matrices and index matrices of graph structures and prove that the collection of all admissible index matrices and the collection of all labelling matrices for 0 form free F -modules. We also find their ranks in various cases of bipartition and char F (equal to 2 and not equal to 2).

Introduction

E. Sampathkumar introduced the concept of graph structure in [9]. It is in particular, a generalisation of the notions like graphs [5], signed graphs [2], [11], [12] and edge-coloured graphs [6] with the colourings. He defined a graph structure G as $G = (V, R_1, R_2, \dots, R_k)$ where V is a non-empty set and R_1, R_2, \dots, R_k are relations on V which are mutually disjoint such that each $R_i, i = 1, 2, \dots, k$ is symmetric and irreflexive.

2010 MSC: 05C07, 05C78.

Key words and phrases: Graph structure, R_i -labelling, R_i -index vector, admissible R_i -index vector, labelling matrix, index matrix, admissible index matrix..

R.H. Jeurissen [8], based on the works of Brouwer [1], Doob [4] and Stewart [10], has proved some results using incidence matrices of graphs. He has defined index vectors and labelings and also admissible index vectors. He has proved that if F is a commutative ring and G a graph, the collection of all admissible index vectors and the collection of all labellings for 0 are free F -modules. He has also found out the ranks of these modules in various cases, namely, G is bipartite, non-bipartite, $\text{char } F = 2$, $\text{char } F \neq 2$ etc.

We have introduced similar concepts in graph structures in [3]. Instead of index vectors and labellings, we have introduced R_i -index vectors and R_i -labellings there and proved some results. Here we are introducing labelling matrices and index matrices of a graph structure. We are also proving that the collection of all admissible index matrices and the collection of all labelling matrices for 0 form free F -modules. We are also finding their ranks in various cases, namely, completely bipartite, not R_i -bipartite for some i s, $\text{char } F = 2$, $\text{char } F \neq 2$ etc.

1. Preliminaries

We first go through some concepts introduced in [9].

Definition 1 ([9]). In a graph structure $G = (V, R_1, R_2, \dots, R_k)$, if $(u, v) \in R_i$, (u, v) is an R_i -edge.

Definition 2 ([9]). An R_i -path between two vertices u and v is an alternating sequence of vertices and edges consisting only of R_i -edges.

Definition 3 ([9]). A set S of vertices in a graph structure $G = (V, R_1, R_2, \dots, R_k)$ is R_i -connected for any given i if any two vertices in S are connected by an R_i -path.

Now we recall the concept of R_i -distance [3].

Definition 4 ([3]). The minimum number of R_i -edges from a vertex u to a vertex v in any R_i -path of a graph structure $G = (V, R_1, R_2, \dots, R_k)$ is called the R_i -distance from u to v . It is the number of R_i -edges from a vertex u to a vertex v in an R_i -tree.

The incidence matrix of a graph structure is defined as follows in [9].

Definition 5 ([9]). The incidence matrix B of a graph structure $G = (V, R_1, R_2, \dots, R_k)$ is a $k \times p$ matrix $b = (b_{ij})$ where b_{ij} is the number of R_i -edges incident to the vertex v_j .

We have defined the R_i -incidence matrix of a graph structure in [3] similar to the incidence matrix of a graph as follows.

Definition 6 ([3]). Let $G = (V, R_1, R_2, \dots, R_k)$ be a graph structure. The R_i -incidence matrix of G is defined as $I_{R_i} = (b_{ij})$, where $b_{rs} = 1$ if v_r is incident with an R_i -edge e_s and 0 otherwise.

First we select a spanning R_i -tree and label the R_i -edges inside and outside the spanning R_i -tree as in [3]. We recall the procedure.

Let G be a finite R_i -connected graph structure on p vertices with q_i number of R_i -edges, $i = 1, 2, \dots, k$. Choose a spanning R_i -tree T_i as follows.

Let v_0 be the root. Number the vertices as v_1, v_2, \dots successively. First those at R_i -distance 1 from v_0 , then those at R_i -distance 2 and so on (Those at same R_i -distance are numbered arbitrarily). Label the R_i -edge (v_r, v_s) with $r < s$ in the spanning R_i -tree T_i as e_i^s . Number the R_i -edges outside the spanning R_i -tree T_i as $e_i^{p_i}, e_i^{p_i+1}, \dots, e_i^{q_i}$.

Then the T_i -part of the R_i -incidence matrix will be of the form

$$\begin{matrix}
 v_0 \\
 v_1 \\
 v_2 \\
 \cdot \\
 \cdot \\
 \cdot \\
 \cdot \\
 \cdot \\
 \cdot \\
 \cdot \\
 \cdot \\
 v_{p-1}
 \end{matrix}
 \begin{bmatrix}
 1 & \cdot & \cdot & \cdot & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & 0 & 0 & 0 & 0 & 1 & \cdot & \cdot & \cdot & 1 & 0 & 0 & 0 & 0 & 0 \\
 0 & \cdot & \cdot & \cdot & 0 & 1 & 0 & 0 & 0 & 0 & 1 & \cdot & \cdot & \cdot & 1 \\
 0 & \cdot & \cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdot & \cdot & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
 \end{bmatrix}$$

Consider a vertex v_s and let $v_s, v_{k_1}, v_{k_2}, \dots, v_{k_r}, v_0$ be the unique R_i -path in T_i from v_s to v_0 . Denote it by c_i^s . Then $c_i^s = c_i^{k_1} - c_i^{k_2} + c_i^{k_3} - \dots + (-1)^s c_i^t = [(-1)^t 00\dots 0]^t$.

For each T_i , there exists an upper triangular $p - 1 \times p - 1$ matrix D_i with elements $0, \pm 1$ and an all 1-diagonal such that

$$I_{T_i} D_i = \begin{bmatrix}
 1 & \pm 1 & \pm 1 & \cdot & \cdot & \cdot & \pm 1 \\
 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 1 & \cdot & \cdot & \cdot & \cdot & 0 \\
 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
 0 & \cdot & \cdot & \cdot & \cdot & 1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 1
 \end{bmatrix}$$

The top row has +1 or -1 in j th position depending on whether v_j is at odd or even R_i -distance from v_0 .

Consider a column of B_i , the part of I_{R_i} corresponding to the R_i -edges outside T_i . Suppose its 1's are in j th and k th rows with $j < k$. Subtracting j th and k th columns of $I_{T_i}D_i$ from it (if $j = 0$, only k), we get

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \text{ if } R_i\text{-distance in } T_i \text{ from } v_j \text{ to } v_0 \text{ is odd and that from } v_k \text{ to } v_0 \text{ is even or if } R_i\text{-distance in } T_i \text{ from } v_j \text{ to } v_0 \text{ is even and that from } v_k \text{ to } v_0 \text{ is odd.}$$

$$\begin{bmatrix} 2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ if } v_j \text{ and } v_k \text{ are at even } R_i\text{-distance from } v_0$$

$$\begin{bmatrix} -2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ if } v_j \text{ and } v_k \text{ are at odd } R_i\text{-distance from } v_0.$$

Let E_i be $(0, -1)$ matrix with one or two -1s in each column.

$$(I_{T_i}D_i|B_i) \begin{bmatrix} I & E_i \\ 0 & I \end{bmatrix} \text{ gives}$$

$$\left[\begin{array}{cccc|cccccccc} 1 & \pm 1 & \pm 1 & \dots & \pm 1 & 0 & \dots & 0 & 2 & \dots & 2 & -2 & \dots & -2 \\ 1 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{array} \right]$$

after renumbering R_i -edges outside T_i .

The column operations on the R_i -incidence matrices will give the incidence matrix of the graph structure in the form

$$(I_G D|B) \begin{bmatrix} I & E \\ \Theta & I \end{bmatrix} = \begin{bmatrix} I_1 & \Theta & \dots & \dots & \Theta \\ \Theta & I_2 & \Theta & \dots & \Theta \\ \dots & \Theta & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \Theta \\ \Theta & \Theta & \dots & \dots & \Theta & I_k \end{bmatrix}$$

where

$$I_G = \begin{bmatrix} I_{T_1}|B_1 & \Theta & \cdot & \cdot & \cdot & \Theta \\ \Theta & I_{T_2}|B_2 & \Theta & \cdot & \cdot & \Theta \\ \cdot & \Theta & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \Theta \\ \Theta & \Theta & \cdot & \cdot & \Theta & I_{T_k}|B_k \end{bmatrix},$$

$$D = \begin{bmatrix} D_1 & \Theta & \cdot & \cdot & \cdot & \Theta \\ \Theta & D_2 & \Theta & \cdot & \cdot & \Theta \\ \cdot & \Theta & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \Theta \\ \cdot & \cdot & \cdot & \cdot & \cdot & \Theta \\ \Theta & \Theta & \cdot & \cdot & \Theta & D_k \end{bmatrix}, \quad E = \begin{bmatrix} E_1 & \Theta & \cdot & \cdot & \cdot & \Theta \\ \Theta & E_2 & \Theta & \cdot & \cdot & \Theta \\ \cdot & \Theta & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \Theta \\ \Theta & \Theta & \cdot & \cdot & \Theta & E_k \end{bmatrix}$$

and each I_i has the form of

$$(I_{T_i}D_i|B_i) \begin{bmatrix} I & E_i \\ 0 & I \end{bmatrix}.$$

2. Labelling matrix and index matrix

We first recall the concepts of R_i -labelling and R_i -index vector introduced in [3].

Definition 7 ([3]). Let F be an abelian group or a ring and $G = (V, R_1, R_2, \dots, R_k)$ be a graph structure with vertices v_0, v_1, \dots, v_{p-1} and q_i number of R_i -edges. A mapping from V to F is an R_i -index vector and a mapping from R_i to F is an R_i -labelling.

Definition 8 ([3]). An R_i -labelling x_i is an R_i -labelling for the R_i -index vector r_i if for each j , $r_i(v_j) = \sum_{e_r \in E_i^j} x_i(e_r)$, where E_i^j is the set of all R_i -edges incident with v_j .

Definition 9 ([3]). R_i -index vectors for which an R_i -labelling exists are called admissible R_i -index vectors.

Now we introduce the concepts of labelling matrix and index matrix of a graph structure.

Definition 10. Let F be an abelian group or a ring. Let r_i be an R_i -index vector and x_i be an R_i -labelling for $i = 1, 2, \dots, k$. Then

$$\text{a) } r = \begin{bmatrix} r_1 & 0 & \dots & 0 \\ 0 & r_2 & 0 & \dots & 0 \\ \cdot & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \dots & 0 & r_k \end{bmatrix} \text{ is defined to be an index matrix for } G$$

$$\text{b) } x = \begin{bmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & 0 & \dots & 0 \\ \cdot & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \dots & 0 & x_k \end{bmatrix} \text{ is defined to be a labelling matrix for } G.$$

Definition 11. The map

$$x : \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_k \end{bmatrix} \rightarrow F^k$$

is a labelling for $r : V^k \rightarrow F^k$ if $\sum_{m \in E_s} x_i(m) = r_i(x_s)$ for $s = 0, 1, 2, \dots, p-1$; $i = 1, 2, \dots, k$.

Now we prove a necessary and sufficient condition for x to be a labelling matrix for r . For that first we recall the following lemma of [3].

Lemma 1 ([3]). *Let $G = (V, R_1, R_2, \dots, R_k)$ be a graph structure. x_i is an R_i -labelling for R_i -index vector r_i iff $I_{R_i} x_i = r_i$.*

Now we move on to prove the condition for x to be a labelling matrix for r .

Lemma 2. *Let $G = (V, R_1, R_2, \dots, R_k)$ be a graph structure. x is a labelling for r iff $I_G x = r$ where*

$$I_G = \begin{bmatrix} I_{R_1} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & I_{R_2} & 0 & \cdot & \cdot & 0 \\ \cdot & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & 0 & I_{R_k} \end{bmatrix}.$$

Proof. Let x be a labelling for r . Then by Lemma 2,

$$\begin{aligned}
 I_G x &= \begin{bmatrix} I_{R_1} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & I_{R_2} & 0 & \cdot & \cdot & 0 \\ \cdot & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & 0 & I_{R_k} \end{bmatrix} \begin{bmatrix} x_1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & x_2 & 0 & \cdot & \cdot & 0 \\ \cdot & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & 0 & x_k \end{bmatrix} \\
 &= \begin{bmatrix} I_{R_1} x_1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & I_{R_2} x_2 & 0 & \cdot & \cdot & 0 \\ \cdot & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & 0 & I_{R_k} x_k \end{bmatrix} = \begin{bmatrix} r_1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & r_2 & 0 & \cdot & \cdot & 0 \\ \cdot & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & 0 & r_k \end{bmatrix} = r.
 \end{aligned}$$

Conversely, let $I_G x = r$. Then $I_{R_i} x_i = r_i$ for $i = 1, 2, \dots, k$. So x_i is a labelling for $r_i, i = 1, 2, \dots, k$. i.e., x is a labelling for r . \square

Definition 12. Let $G = (V, R_1, R_2, \dots, R_k)$ be a graph structure. If r_i is an admissible R_i -index vector for $i = 1, 2, \dots, k$, then

$$r = \begin{bmatrix} r_1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & r_2 & 0 & \cdot & \cdot & 0 \\ \cdot & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & 0 & r_k \end{bmatrix}$$

is called an admissible index matrix for G .

Now we recall the definition of complete bipartition given in [9].

Definition 13 ([9]). A graph structure G is completely bipartite if for each $i, 1 \leq i \leq k$, G is R_i -bipartite (with same sets of bipartition).

Now we go through some of the results proved in [3].

Theorem 1 ([3]). *If F is an abelian group and v_1, v_2, \dots, v_p are vertices of an R_i -bipartite graph structure G , then r_i is an admissible R_i -index vector iff $\sum_{j=1}^s r_i(v_j) = \sum_{j=s+1}^p r_i(v_j)$ where $S = \{v_1, v_2, \dots, v_s\}$ and $U = \{v_{s+1}, v_{s+2}, \dots, v_p\}$ are the sets of R_i -bipartition.*

Theorem 2 ([3]). *If F is an abelian group and v_1, v_2, \dots, v_p are the vertices of a non- R_i -bipartite graph structure G , then r_i is an admissible R_i -index vector iff $\sum_{j=1}^p r_i(v_j) = 2f_i$, for some $f_i \in F$.*

We now prove certain preliminary results.

Theorem 3. *Let $G = (V, R_1, R_2, \dots, R_k)$ be a complete bipartite graph structure. If F is an abelian group and v_1, v_2, \dots, v_p are vertices of G , then r is an admissible index matrix iff $\sum_{j=1}^{s_i} r_i(v_j) = \sum_{j=s_i+1}^p r_i(v_j)$, for $i = 1, 2, \dots, k$ where $S = \{v_1, v_2, \dots, v_{s_i}\}$ and $U = \{v_{s_i+1}, v_{s_i+2}, \dots, v_p\}$ are sets of bipartition.*

Proof. Let r be admissible. Then r_i is an admissible R_i -index vector for $i = 1, 2, \dots, k$. G is R_i -bipartite for $i = 1, 2, \dots, k$. Therefore by Theorem 1,

$$\sum_{j=1}^{s_i} r_i(v_j) = \sum_{j=s_i+1}^p r_i(v_j) = \sum_{j=s_i+1}^p r_i(v_j)$$

for $i = 1, 2, \dots, k$.

Conversely, let

$$\sum_{j=1}^{s_i} r_i(v_j) = \sum_{j=s_i+1}^p r_i(v_j) = \sum_{j=s_i+1}^p r_i(v_j)$$

for $i = 1, 2, \dots, k$.

Then by Theorem 1, r_i is an R_i -admissible index vector for $i = 1, 2, \dots, k$. So r is admissible. \square

Theorem 4. *Let $G = (V, R_1, R_2, \dots, R_k)$ be a graph structure. If F is an abelian group and v_1, v_2, \dots, v_p are the vertices of a graph structure G which is not R_i -bipartite for $i = 1, 2, \dots, k$, then r is admissible iff $\sum_{j=1}^p r_i(v_j) = 2f_i$ for some $f_i \in F, i = 1, 2, \dots, k$.*

Proof. Let r be admissible. Then r_i is an admissible R_i -index vector, $i = 1, 2, \dots, k$. G is not R_i -bipartite for $i = 1, 2, \dots, k$. Therefore by Theorem 2, $\sum_{j=1}^p r_i(v_j) = 2f_i$ for some $f_i \in F, i = 1, 2, \dots, k$.

Conversely, let $\sum_{j=1}^p r_i(v_j) = 2f_i$ for some $f_i \in F, i = 1, 2, \dots, k$. Then by Theorem 2, r_i is an admissible R_i -index vector for $i = 1, 2, \dots, k$. Therefore r is admissible. \square

Now we move on to prove that the collection of all admissible index matrices for a graph structure form a free F -module and find its rank.

Theorem 5. *Let $G = (V, R_1, R_2, \dots, R_k)$ be a graph structure. If F is an integral domain, the admissible index matrices for G form a free F -module. Its rank is*

- (i) $k(p - 1)$ if G is completely bipartite or $\text{char } F = 2$.
- (ii) kp if G is not R_i -bipartite for $i = 1, 2, \dots, k$ and $\text{char } F \neq 2$.
- (iii) kp if G is not R_i -bipartite except for $i = i_1, i_2, \dots, i_r$, $\text{char } F \neq 2$ and 2 is invertible.
- (iv) $kp - r$ if G is not R_i -bipartite except for $i = i_1, i_2, \dots, i_r$, $\text{char } F \neq 2$ and 2 is not invertible.

Proof. Index matrices of G belong to $F^{kp \times k}$. Let $A =$ set of all admissible index matrices. It is a subset of $F^{kp \times k}$.

Let $r, s \in A$. Then r_i, s_i are admissible R_i -index vectors for $i = 1, 2, \dots, k$. Therefore there exists R_i -labellings x_{r_i}, x_{s_i} , for r_i and s_i for $i = 1, 2, \dots, k$, and we have

$$\begin{aligned}
 r - s &= \begin{bmatrix} r_1 & 0 & 0 & \dots & 0 \\ 0 & r_2 & 0 & \dots & 0 \\ 0 & 0 & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & 0 \\ 0 & 0 & \dots & 0 & r_k \end{bmatrix} - \begin{bmatrix} s_1 & 0 & 0 & \dots & 0 \\ 0 & s_2 & 0 & \dots & 0 \\ 0 & 0 & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & 0 \\ 0 & 0 & \dots & 0 & s_k \end{bmatrix} \\
 &= \begin{bmatrix} r_1 - s_1 & 0 & 0 & \dots & 0 \\ 0 & r_2 - s_2 & 0 & \dots & 0 \\ 0 & 0 & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & 0 \\ 0 & 0 & \dots & 0 & r_k - s_k \end{bmatrix}.
 \end{aligned}$$

$$I_G(x_r - x_s) = \begin{bmatrix} I_{R_1} & 0 & 0 & \dots & 0 \\ 0 & I_{R_2} & 0 & \dots & 0 \\ 0 & 0 & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & 0 \\ 0 & 0 & \dots & 0 & I_{R_k} \end{bmatrix}$$

$$\left\{ \begin{bmatrix} x_{r_1} & 0 & 0 & \cdot & \cdot & 0 \\ 0 & x_{r_2} & 0 & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 0 & x_{r_k} \end{bmatrix} - \begin{bmatrix} x_{s_1} & 0 & 0 & \cdot & \cdot & 0 \\ 0 & x_{s_2} & 0 & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 0 & x_{s_k} \end{bmatrix} \right\} \\ = \begin{bmatrix} r_1 - s_1 & 0 & 0 & \cdot & \cdot & 0 \\ 0 & r_2 - s_2 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & 0 & r_k - s_k \end{bmatrix},$$

that is, $x_r - x_s$ is a labelling for $r - s$ since $x_{r_i} - x_{s_i}$ is a labelling for $r_i - s_i$ for $i = 1, 2, \dots, k$. So $r - s \in A$.

Let $f \in F, r \in A$. Then r_i is R_i -admissible for $i = 1, 2, \dots, k$. There exists an R_i -labelling x_i for $r_i, i = 1, 2, \dots, k$. fx_i is a labelling for $fr_i, i = 1, 2, \dots, k$. So fx is a labelling for fr and hence $fr \in A$. Therefore A is an F -module and it is a submodule of $F^{kp \times k}$.

Case 1: G is completely bipartite

r is admissible iff $\sum_{j=1}^s r_i(v_j) = \sum_{j=s+1}^p r_i(v_j)$ for $i = 1, 2, \dots, k$ by Theorem 3.

$$r_i(v_p) = r_i(v_1) + \dots + r_i(v_s) - r_i(v_{s+1}) - \dots - r_i(v_{p-1}).$$

$$r = \begin{bmatrix} r_1(v_1) & 0 & \cdot & \cdot & \cdot & 0 \\ r_1(v_2) & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ r_1(v_p) & 0 & \cdot & \cdot & \cdot & \cdot \\ 0 & r_2(v_1) & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & r_2(v_p) & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & r_k(v_1) & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 0 & r_k(v_p) \end{bmatrix} = \begin{bmatrix} r_1(v_1) & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ r_1(v_1) & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \end{bmatrix}$$

$$\begin{array}{c}
\begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 \\ r_1(v_2) & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ r_1(v_2) & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \end{bmatrix} & + \dots + & \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ r_1(v_{p-1}) & \cdot & \cdot & \cdot & \cdot & \cdot \\ -r_1(v_{p-1}) & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \end{bmatrix} \\
+ \dots + & & + \dots + \\
\begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 0 & -r_k(v_1) \end{bmatrix} & + \dots + & \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 0 & -r_k(v_{p-1}) \end{bmatrix}
\end{array}$$

Denote the above $k(p-1)$ matrices by

$$r_{11}, r_{12}, \dots, r_{1(p-1)}, \dots, r_{k1}, \dots, r_{k(p-1)}.$$

Let $f_{11}, \dots, f_{k(p-1)}$ be elements of F and let $f_{11}r_{11} + \dots + f_{k(p-1)}r_{k(p-1)} = 0$. Then $f_{11}r_{11} = \dots = f_{k(p-1)}r_{k(p-1)} = 0$. Therefore $f_{11} = f_{12} = \dots = f_{k(p-1)} = 0$ and so $r_{11}, r_{12}, \dots, r_{k(p-1)}$ are linearly independent. Therefore A is a free F -module of rank $k(p-1)$.

Case 2: char $F = 2$

The case when G is completely bipartite has already been discussed. So it is enough if we consider G to be not completely bipartite.

Two subcases arise.

Subcase 1: G is not R_i -bipartite for $i = 1, 2, \dots, k$.

By theorem 4, r is admissible iff $\sum_{j=1}^p r_i(v_j) = 2f_i$ for some $f_i \in F, i = 1, 2, \dots, k$. Since $\text{char } F = 2, \sum_{j=1}^p r_i(v_j) = 0, i = 1, 2, \dots, k$. Therefore $r_i(v_p) = -r_i(v_1) - r_i(v_2) - \dots - r_i(v_{p-1}), i = 1, 2, \dots, k$. So we can write

$$\begin{aligned}
 r = & \begin{bmatrix} r_1(v_1) & 0 & \dots & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \dots & \dots & \vdots \\ 0 & \vdots & \dots & \dots & \vdots \\ -r_1(v_1) & \dots & \dots & \dots & \vdots \\ 0 & \dots & \dots & \dots & \vdots \\ \vdots & \dots & \dots & \dots & \vdots \\ \vdots & \dots & \dots & \dots & \vdots \\ \vdots & \dots & \dots & \dots & \vdots \\ 0 & 0 & \dots & \dots & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \dots & \dots & 0 \\ r_1(v_2) & 0 & \dots & \dots & 0 \\ 0 & \vdots & \dots & \dots & \vdots \\ 0 & \vdots & \dots & \dots & \vdots \\ -r_1(v_2) & \dots & \dots & \dots & \vdots \\ 0 & \dots & \dots & \dots & \vdots \\ \vdots & \dots & \dots & \dots & \vdots \\ \vdots & \dots & \dots & \dots & \vdots \\ \vdots & \dots & \dots & \dots & \vdots \\ 0 & 0 & \dots & \dots & 0 \end{bmatrix} + \dots \\
 & + \begin{bmatrix} 0 & 0 & \dots & \dots & 0 \\ \vdots & 0 & \dots & \dots & 0 \\ 0 & \vdots & \dots & \dots & \vdots \\ r_1(v_{p-1}) & \dots & \dots & \dots & \vdots \\ -r_1(v_{p-1}) & \dots & \dots & \dots & \vdots \\ 0 & \dots & \dots & \dots & \vdots \\ \vdots & \dots & \dots & \dots & \vdots \\ \vdots & \dots & \dots & \dots & \vdots \\ \vdots & \dots & \dots & \dots & \vdots \\ 0 & 0 & \dots & \dots & 0 \end{bmatrix} + \dots + \begin{bmatrix} 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots \\ 0 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots \\ 0 & 0 & \dots & \dots & -r_k(v_1) \end{bmatrix} + \dots \\
 & + \begin{bmatrix} 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots \\ 0 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots \\ 0 & 0 & \dots & \dots & -r_k(v_{p-1}) \end{bmatrix}
 \end{aligned}$$

Let these elements be denoted by $r_{11}, r_{12}, \dots, r_{k1}, \dots, r_{k(p-1)}$ and let $f_{11}, f_{12}, \dots, f_{1(p-1)}, \dots, f_{k1}, \dots, f_{k(p-1)} \in F$.

and the remaining have the form

$$\begin{bmatrix} 0 & 0 & \cdot & 0 & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & r_i(v_j) & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & -r_i(v_j) & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 0 & \cdot & 0 \end{bmatrix}$$

Clearly they are linearly independent. Therefore A is a free F -module of rank $k(p-1)$.

Case 3: G is not R_i -bipartite for $i = 1, 2, \dots, k$ and $\text{char } F \neq 2$

Subcase 1: G is not R_i -bipartite for $i = 1, 2, \dots, k$ and $\text{char } F \neq 2$

a) 2 is invertible

$$\sum_{j=1}^p r_i(v_j) = f_i = (2 \cdot 2^{-1} f_i) = 2(2^{-1} f_i)$$

for some $f_i \in F, i = 1, 2, \dots, k$, i.e., $\sum_{j=1}^p r_i(v_j) = 2f'_i, f'_i = 2^{-1}f_i \in F$ for $i = 1, 2, \dots, k$. Therefore r_i is R_i -admissible for $i = 1, 2, \dots, k$ by Theorem 2. So r is admissible. Hence rank of A is rank of $F^{kp \times k} = kp$.

b) 2 is not invertible

Consider

$s_{11}, s_{12}, \dots, s_{kp}$ are linearly independent and hence A is a free F -module with rank equal to the rank of $F^{kp \times k} = kp$.

Subcase 2: G is R_i -bipartite for $i = i_1, i_2, \dots, i_r, 1 < r < k$ and not R_i -bipartite for $i = i_{r+1}, i_{r+2}, \dots, i_k$ where $i_1, i_2, \dots, i_k \in \{1, 2, \dots, k\}$

a) 2 is invertible

Then $\sum_{j=1}^p r_i(v_j) = f_i = 2f'_i$ where $f'_i = 2^{-1}f_i \in F$.

So r_i is an admissible R_i -index vector for $i = 1, 2, \dots, k$ and hence r is admissible by Theorem 2. Therefore rank of A is equal to rank of $F^{kp \times k} = kp$.

b) 2 is not invertible

As in subcase 1 b, define s_{11}, \dots, s_{kp} . Then $s_{ij} \in A, i = 1, 2, \dots, k; j = 1, 2, \dots, p$. But for $i = i_1, i_2, \dots, i_r, s_{ip} = s_{i1} + s_{i2} + \dots + s_{is} - s_{i(s+1)} - \dots - s_{i(p-1)}$. Therefore rank of A is equal to rank of $F^{kp \times k} - r = kp - r$. \square

Now we prove that the collection of all labelling matrices for 0 form a free F -module and find its rank. For that, first we recall a theorem from [3].

Theorem 6 ([3]). *Let $G = (V, R_1, R_2, \dots, R_k)$ be a graph structure. If F is an integral domain, the R_i -labelling of G for the R_i -index vector 0 form a free F -module. Its rank is $q_i - p + 1$ if G is R_i -bipartite or $\text{char } F = 2$ and $q_i - p$ if G is not R_i -bipartite and $\text{char } F \neq 2$.*

Theorem 7. *Let $G = (V, R_1, R_2, \dots, R_k)$ be a graph structure. If F is an integral domain, the labellings of G for the index matrix 0 form a free F -module. Its rank is*

- (i) $q_1 + q_2 + \dots + q_k - k(p - 1)$ if G is completely bipartite or $\text{char } F = 2$,
- (ii) $q_1 + q_2 + \dots + q_k - kp$ if G is not R_i -bipartite for $i = 1, 2, \dots, k$ and $\text{char } F \neq 2$,
- (iii) $q_1 + q_2 + \dots + q_k - kp + r$ if G is R_i -bipartite for $i = i_1, i_2, \dots, i_r$ and $\text{char } F \neq 2$.

Proof. Define $\phi : F^{(q_1+q_2+\dots+q_k) \times k} \rightarrow \{I_G x\}$ as $\phi(x) = I_G x$. Let $x, y \in F^{(q_1+q_2+\dots+q_k) \times k}$.

$$\phi(x + y) = I_G(x + y) = I_G x + I_G y = \phi(x) + \phi(y),$$

$$\text{For } f \in F, x \in F^{(q_1+q_2+\dots+q_k) \times k}, \phi(fx) = I_G(fx) = f(I_G x) = f\phi(x).$$

Therefore ϕ is a homomorphism. It is onto also. Let

$K = \{x \in F^{(q_1+q_2+\dots+q_k) \times k} : I_G x = 0\}$. It is the kernel of ϕ . Clearly it is

a submodule of $F^{(q_1+q_2+\dots+q_k)\times k}$. Hence K is an F -module.

r is admissible iff $I_G x = r$ by Lemma 2. ie., iff $(I_G C)(C^{-1}x) = r$. So 0 is admissible iff $(I_G C)(C^{-1}x) = 0$. ie., iff $(I_G C)y = 0$ where $y = C^{-1}x$.

Therefore a general solution for y pre-multiplied by C is a labelling for 0.

A general solution for y for $r = 0$ has the form

$$S = \begin{bmatrix} \Theta & A_1 & 0 & \cdot & \cdot & \cdot & 0 \\ \Theta & 0 & A_2 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \Theta & \cdot & \cdot & \cdot & \cdot & \cdot & A_k \end{bmatrix}$$

where Θ is a zero matrix of suitable order,

$$A_1 = \begin{bmatrix} \alpha_p & \cdot & \cdot & \cdot & \alpha_{s_1} & \cdot & \cdot & \cdot & \alpha_{t_1} & \cdot & \cdot & \cdot & \alpha_{q_1} \end{bmatrix},$$

$$\dots, A_k = \begin{bmatrix} \alpha_p & \cdot & \cdot & \cdot & \alpha_{s_k} & \cdot & \cdot & \cdot & \alpha_{t_k} & \cdot & \cdot & \cdot & \alpha_{q_k} \end{bmatrix}$$

Also

$$2(\alpha_{t_{i+1}} + \alpha_{t_{i+2}} + \dots + \alpha_{s_{i+1}} - \alpha_{s_i} - \dots - \alpha_{q_i}) = 0 \quad (**)$$

for $i = 1, 2, \dots, k$. So the number of zeroes in the top row of each $I_{R_i} C_i$ is $t_i - p + 1$.

Any labelling has the form CS . Therefore if x is a labelling,

$$x = C \left\{ \begin{bmatrix} \Theta & A_{1p} & 0 & \cdot & \cdot & \cdot & \cdot & \Theta \\ \Theta & \Theta & \cdot & \cdot & \cdot & \cdot & \cdot & \Theta \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \Theta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \Theta \end{bmatrix} + \begin{bmatrix} \Theta & A_{1(p+1)} & \Theta & \cdot & \cdot & \cdot & \cdot & \Theta \\ \Theta & \Theta & \cdot & \cdot & \cdot & \cdot & \cdot & \Theta \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \Theta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \Theta \end{bmatrix} + \dots \right.$$

$$+ \begin{bmatrix} \Theta & A_{1q_1} & \Theta & \cdot & \cdot & \cdot & \cdot & \Theta \\ \Theta & \Theta & \cdot & \cdot & \cdot & \cdot & \cdot & \Theta \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \Theta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \Theta \end{bmatrix} + \dots + \begin{bmatrix} \Theta & \Theta & \cdot & \cdot & \cdot & \cdot & \cdot & \Theta \\ \Theta & \Theta & \cdot & \cdot & \cdot & \cdot & \cdot & \Theta \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \Theta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & A_{kp} \end{bmatrix}$$

$$+ \dots + \left[\begin{array}{cccccccc} \Theta & \Theta & . & . & . & . & . & \Theta \\ \Theta & \Theta & . & . & . & . & . & \Theta \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ \Theta & . & . & . & . & . & . & A_{kq_k} \end{array} \right] \}$$

where

$$A_{1p} = \left[\begin{array}{cccccc} \alpha_p & 0 & . & . & . & 0 \end{array} \right],$$

$$A_{1(p+1)} = \left[\begin{array}{cccccc} 0 & \alpha_{p+1} & 0 & . & . & 0 \end{array} \right], \dots, A_{1q_1} = \left[\begin{array}{cccccc} 0 & . & . & . & 0 & \alpha_{q_1} \end{array} \right],$$

$$\dots,$$

$$A_{kp} = \left[\begin{array}{cccccc} \alpha_p & 0 & . & . & 0 & 0 \end{array} \right], \dots, A_{kq_k} = \left[\begin{array}{cccccc} 0 & . & . & . & 0 & \alpha_{q_k} \end{array} \right].$$

Case 1: char $F = 2$

(**) is automatically satisfied for each i . Therefore the elements of the set of labellings for 0 are linearly independent. So the rank of the free F -module of labellings for 0 is $q_1 - p + 1 + q_2 - p + 1 + \dots + q_k - p + 1 = (q_1 + q_2 + \dots + q_k) - kp + k = q_1 + q_2 + \dots + q_k - (p - 1)k$ by Theorem 6.

Case 2: G is completely bipartite

Since G is R_i -bipartite for each i , the top elements of $I_{R_i}C_i$ corresponding to the R_i -edges outside the spanning R_i -tree will be 0 for $i = 1, 2, \dots, k$. So (**) is irrelevant and hence the elements of the set of labellings for 0 are linearly independent. Therefore the rank of the free F -module of labellings for 0 is $q_1 - p + 1 + q_2 - p + 1 + \dots + q_k - p + 1 = (q_1 + q_2 + \dots + q_k) - kp + k = q_1 + q_2 + \dots + q_k - (p - 1)k$ by Theorem 6.

Case 3: G is not completely bipartite and char $F \neq 2$

Subcase 1: G is not R_i -bipartite for $i = 1, 2, \dots, k$

In this case, due to (**), one of the matrices can be expressed as a linear combination of others for each i . So the rank of the free F -module of labellings for 0 is $q_1 - p + q_2 - p + \dots + q_k - p = (q_1 + q_2 + \dots + q_k) - kp$ by theorem 6.

Subcase 2: G is R_i -bipartite for $i = i_1, i_2, \dots, i_r, 1 < r < k$ and not R_i -bipartite for $i = i_{r+1}, \dots, i_k; i_1, i_2, \dots, i_k \in \{1, 2, \dots, k\}$

In this case, due to (**), one of the matrices can be represented as a linear combination of others for each i except for $i = i_1, i_2, \dots, i_r$. so the rank of the free F -module of labellings for 0 is $(q_1 + q_2 + \dots + q_k) - kp + r$ by Theorem 6. \square

References

- [1] A.E. Brouwer, *A note on magic graphs*, Report **ZN 47/72**(Internal communication), Mathematisch Centrum, Amsterdam, 1972.
- [2] D. Cartwright and F. Harary, *Structural balance: a generalization of Heider's theory*, Psychological Review, **63**, 1956, pp. 277-293.
- [3] T. Dinesh and T.V. Ramakrishnan, *R_i -labellings and R_i -index vectors of a graph structure*, Adv. Appl. Discrete Math., **7(1)**, 2011, pp. 63-82.
- [4] M. Doob, *Generalizations of magic graphs*, J. Combin. Theory Ser. **B17**, 1974, pp.205-217.
- [5] L. Euler, *Solutio problematis ad geometriam situs pertinentis*, Comment. Academiae Sci. I. Petropolitanae, **8**, 1736, pp. 128-140.
- [6] S. Florini and Robin James Wilson, *Edge-colourings of graphs*, Research Notes in Mathematics, London: Pitman, **16**, 1977.
- [7] F. Harary, *Graph Theory*, Narosa Pub. House, 1995.
- [8] R.H. Jeurissen, *The incidence matrix and labelings of a graph*, J. Combin. Theory Ser. **B30**, 1981, pp.290-301.
- [9] E. Sampathkumar, *Generalised graph structures*, Bull. Kerala Math. Assoc., **3(2)**, 2006, pp.65-123.
- [10] B.M. Stewart, *Magic graphs*, Canad. J. Math. **18**, 1966, pp.1031-1059.
- [11] T. Zaslavsky, *Characterizations of signed graphs*, J. Graph Theory, **5**, 1981, pp.401-406.
- [12] T. Zaslavsky, *Signed graphs*, Discrete Appl. Math., **4(1)**, 1982, pp. 47-74.

CONTACT INFORMATION

T. Dinesh

Department of Mathematics,
Nehru Arts and Science College,
Padannakkad P.O., Kasaragod District - 671
314, Kerala, India
E-Mail: dineshthek@gmail.com

T. V. Ramakrishnan

Department of Mathematics,
S.E.S. College, Sreekandapuram, Kannur Dis-
trict - 670 631, Kerala, India
E-Mail: ramakrishnantvkn@gmail.com

Received by the editors: 25.07.2011
and in final form 29.05.2012.