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# Inverse semigroups generated by group congruences. The Möbius functions

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Abstract. The computation of the Möbius function of a Möbius category that arises from a combinatorial inverse semigroup has a distinctive feature. This computation is done on the field of finite posets. In the case of two combinatorial inverse semigroups, order isomorphisms between corresponding finite posets reduce the computation to one of the semigroups. Starting with a combinatorial inverse monoid and using a group congruence we construct a combinatorial inverse semigroup such that the Möbius function becomes an invariant to this construction. For illustration, we consider the multiplicative analogue of the bicyclic semigroup and the free monogenic inverse monoid.

## 1. Introduction

A systematic theory of Möbius functions of partially ordered sets (posets) was developed first by Rota [16]. In the theory of semigroups, Steinberg [22], [23] explored several applications of Rota's theory of Möbius functions to the representation theory. In their papers, Content,Lemay, Leroux [1] and Haigh [4] set up the Möbius function theory for categories. Recently, Leinster [11], [12] Lawvere and Menni [9], Noguchi [14], Fiore, Lück and Sauer [3] recalled the Möbius inversion for categories providing a new abstract framework.

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In [18]-[20] the author established links between inverse semigroups and Möbius categories. These papers described conditions such that Leech's division category of an inverse monoid becomes a Möbius category: *the reduced division category of an inverse monoid S relative to an idempotent transversal F of the $\mathscr{D}$-classes of S with* $1 \in F$ *is a Möbius category if and only if the inverse monoid is combinatorial and locally finite.* (If $S$ contains zero, the above statement is true if "division category" is replaced by "Clifford category" and "locally finite" by "0-locally finite".) So, many examples of Möbius categories arise from the theory of inverse semigroups.The peculiarity of these Möbius categories is the fact that the computation of their Möbius functions can be reduced to the computation of Möbius functions of finite posets.

The concept of (*abstract*) *division category* was introduced by Leech [10]. A right cancellative category $C$ (i.e. any morphism in $C$ is an epimorphism) with pushouts and with a quasi-initial object $I$ (i.e. $Hom(I, A) \neq \emptyset$ for any object $A \in ObC$) is called a *division category*. Following Lawson [7] (see also [6]), a *Clifford category* is a right cancellative category such that any two morphisms with a common domain which can be completed to a commutative square have a pushout.

Now, a small category $C$ is a *Möbius category* ([1]) if

1) any morphism $f$ in $C$ has only a finite number of factorizations of the form $f = gh$;
2) an incidence function $\xi$ of $C$ (i.e. a complex-valued function defined on the set $MorC$ of all morphisms of $C$) has a convolution inverse (the convolution being defined by $(\xi * \eta)(\alpha) = \sum_{\beta\gamma=\alpha} \xi(\beta)\eta(\gamma)$) if and only if $\xi(\alpha) \neq 0$ for any identity morphism $\alpha$.

The *convolution identity* is the incidence function $\delta : \delta(\alpha) = 1$ if $\alpha$ is an identity morphism and 0 otherwise. The *Möbius function* $\mu$ of a Möbius category $C$ is the convolution inverse of the zeta function $\zeta : \zeta(\alpha) = 1$ for any morphism $\alpha$ in $C$. The *Möbius inversion formula* is nothing but the statement: $\eta = \xi * \zeta \Leftrightarrow \xi = \eta * \mu$.

A nice characterization of Möbius categories is given in [9] involving intervals. The definition of interval of a category was introduced by Lawvere (see note in [9, p.226]). We call this intervals Lawvere intervals (in [12] the term "fine intervals" is used). If $C$ is a small category and $\alpha$ a morphism of $C$, then the *Lawvere interval* $I(\alpha)$ of $\alpha$ is defined by:

1) $ObI(\alpha) =$ the set of all factorizations of $\alpha$;
2) $Hom_{I(\alpha)}(\beta_1\beta_0, \gamma_1\gamma_0) = \{\lambda \in MorC | \lambda\beta_0 = \gamma_0 \text{ and } \gamma_1\lambda = \beta_1\}$;(see Diagram 1)

$$\begin{tikzcd} & {} \arrow[dd, "\lambda"'] \arrow[dr, "\beta_1"] & \\ X \arrow[ur, "\beta_0"] \arrow[dr, "\gamma_0"'] & & Y \\ & {} \arrow[ur, "\gamma_1"'] & \end{tikzcd}$$

Diagram 1

3) The composition in $I(\alpha)$ is the same as in $C$.

A small category $C$ is a Möbius category if and only if all Lawvere intervals of C are finite and one-way (i.e. $Hom(X,Y) \neq \emptyset$, $Hom(Y,X) \neq \emptyset$ $\Rightarrow X = Y$ and $|Hom(X,X)| = 1$ for any object $X$ in $C$).

In this paper we use the fact that the Möbius function of a Möbius category is completely determined by the Möbius functions of Lawvere intervals ([9],[21]). (A similar result is given by Leinster [12, Proposition 3.7] for coarse Möbius functions and coarse intervals.) In the case of Möbius categories that arise from combinatorial inverse semigroups, the Lawvere intervals are finite posets ([21]). So, in our paper order isomorphisms between corresponding Lawvere intervals of two Möbius categories lead us to conclude that the Möbius functions "are the same". Starting with a combinatorial inverse semigroup, we construct (using a group congruence) a combinatorial inverse semigroup such that the Möbius function becomes an invariant of this construction. The construction is inspired by one in a Dombi/Gilbert paper ([2]). The construction is in fact a Rees quotient of a semidirect product of a semilattice by an inverse semigroup. We consider the multiplicative analogue of the bicycle semigroup and the free monogenic inverse semigroup to illustrate our results.

We say that an inverse semigroup $S$ is *locally finite* (*0-locally finite*) if the poset of idempotents $E(S)$ ($E^*(S)$) is locally finite (i.e. any interval is finite). Otherwise, the notions and notations of inverse semigroup theory used in this paper are quite standard ([5],[15]).

## 2. The $\varrho$-semigroups with zero. The Möbius functions

Throughout this section, $S$ denotes a locally finite combinatorial inverse monoid without zero. Then the reduced division category $C_F(S)$ relative to an idempotent transversal $F$ of the $\mathscr{D}$- classes of S with $1 \in F$ is a Möbius category. The category $C_F(S)$ is defined by:

– $ObC_F(S) = F$;

- $Hom_{C_F(S)}(e,f) = \{(s,e)|s \in S; s^{-1}s \leq e \text{ and } ss^{-1} = f\}$
- The composition of two morphisms $(s,e) : e \to f$ and $(t,f) : f \to g$ is given by $(t,f) \cdot (s,e) = (ts,e)$.

We denote by $\mu_S$ the Möbius function of the Möbius category $C_F(S)$ and we call it the initial Möbius function.

Let $\varrho$ be a congruence relation on $S$ such that $G_\varrho = S/\varrho$ is a group. We denote by $\varrho(s), \varrho(t), \ldots$ or simply by $\overline{x}, \overline{y}, \ldots$ the congruence classes (i.e. the elements of $G_\varrho$).

**Definition 2.1.** *We call the semigroup*

$$S_\varrho = (G_\varrho \times S) \cup \{0\}$$

*with the multiplication " $\cdot$ " defined by:*

$$(\overline{x}, s) \cdot (\overline{y}, t) = \begin{cases} (\overline{x}, st) & if \quad \overline{x}\varrho(s) = \overline{y} \\ 0 & otherwise \end{cases}$$

$$0 \cdot (\overline{x}, s) = (\overline{x}, s) \cdot 0 = 0 \cdot 0 = 0,$$

*the $\varrho$-semigroup (with zero) of $S$.*

We will establish the basic properties of the $\varrho$-semigroup of $S$ to conclude that it is a 0-locally finite combinatorial inverse semigroup.

**Theorem 2.1.** *We have the following:*

(1) *A non-zero element $(\overline{x}, s)$ of $S_\varrho$ is an idempotent if and only if $s \in E(S)$.*

(2) *The semigroup $S_\varrho$ is inverse, with $(\overline{x}, s)^{-1} = (\overline{x}\varrho(s), s^{-1})$.*

(3) *If $(\overline{x}, s)$ is a non-zero element of $S_\varrho$ then*

$$(\overline{x}, s) \cdot (\overline{x}, s)^{-1} = (\overline{x}, ss^{-1}) \quad and \quad (\overline{x}, s)^{-1} \cdot (\overline{x}, s) = (\overline{x}\varrho(s), s^{-1}s).$$

(4) *If $(\overline{x}, s)$,$(\overline{y}, t)$ are non-zero idempotents then*

$$(\overline{x}, s) \leq (\overline{y}, t) \quad if \ and \ only \ if \quad \overline{x} = \overline{y} \quad and \quad s \leq t.$$

(5) *The poset of non-zero idempotents $E^*(S_\varrho), \leq)$ is locally finite.*

(6) *$(\overline{x}, s)\mathscr{L}(\overline{y}, t) \Leftrightarrow \overline{x}\varrho s = \overline{y}\varrho t$ and $s\mathscr{L}t$; $(\overline{x}, s)\mathscr{R}(\overline{y}, t) \Leftrightarrow \overline{x} = \overline{y}$ and $s\mathscr{R}t$ (where $\mathscr{L}$ and $\mathscr{R}$ are the first two Green's relations).*

(7) *The inverse semigroup $S_\varrho$ is combinatorial.*

*Proof.* (1) Since $G_\varrho = S/\varrho$ is a group, $E(S)$ is contained in a single $\varrho$-class which is the identity element of $G_\varrho$. It follows $(\overline{x}, s) \in E^*(S_\varrho)$ if and only if $s \in E(S)$.

(2) $(\overline{x}, s) \cdot (\overline{x}\varrho(s), s^{-1}) \cdot (\overline{x}, s) = (\overline{x}, ss^{-1}) \cdot (\overline{x}, s) = (\overline{x}, ss^{-1}s) = (\overline{x}, s)$; $(\overline{x}\varrho(s), s^{-1}) \cdot (\overline{x}, s) \cdot (\overline{x}\varrho(s), s^{-1}) = (\overline{x}\varrho(s), s^{-1}s) \cdot (\overline{x}\varrho(s), s^{-1}) = (\overline{x}\varrho(s), s^{-1})$; and the idempotent of $S_\varrho$ commute.

(3) The result follows by using (2).

(4) $(\overline{x}, s) \leq (\overline{y}, t) \Leftrightarrow (\overline{x}, s) = (\overline{x}, s) \cdot (\overline{x}, s)^{-1} \cdot (\overline{y}, t) \Leftrightarrow (\overline{x}, s) = (\overline{x}, ss^{-1}) \cdot (\overline{y}, t) \Leftrightarrow \overline{x} = \overline{y}$ and $s \leq t$.

(5) follows from (1) and (4) taking into account that $(E(S), \leq)$ is locally finite.

(6) follows from (3).

(7) Since $S$ is combinatorial and

$$(\overline{x}, s)\mathscr{H}(\overline{y}, t) \Leftrightarrow \overline{x}\varrho(s) = \overline{y}\varrho(t),\ \overline{x} = \overline{y} \ \ \textit{and} \ \ s\mathscr{H}t \ \ \textit{in} \ \ S,$$

it follows that $\mathscr{H}$ is the equality relation on $S_\varrho$. So, $S_\varrho$ is combinatorial. □

To find the reduced Clifford category of $S_\varrho$ (i.e. the Möbius category of $S_\varrho$) we have to determine a non-zero idempotent transversal $F_\varrho$ of the $\mathscr{D}$-classes of $S_\varrho$. If $F$ is an idempotent transversal of the $\mathscr{D}$-classes of $S$ (with $1 \in F$) we have:

**Theorem 2.2.** *The set*

$$F_\varrho = G_\varrho \times F$$

*is an idempotent transversal of the non-zero $\mathscr{D}$-classes of $S_\varrho$.*

*Proof.* First we show that for any two non-zero idempotents $(\overline{x}, e)$ and $(\overline{y}, f)$ of $S_\varrho$,

$$(\overline{x}, e)\mathscr{D}(\overline{y}, f) \quad \Leftrightarrow \quad \overline{x} = \overline{y}\varrho(s),\ s^{-1}s = e \ \ \textit{and} \ \ ss^{-1} = f \ \ \textit{for some} \ \ s \in S.$$

We have:

$$\begin{aligned}
&(\overline{x}, e)\mathscr{D}(\overline{y}, f)\\
&\ \Leftrightarrow (\overline{x}, e) = (\overline{z}, s)^{-1} \cdot (\overline{z}, s) \ \textit{and} \ (\overline{y}, f) = (\overline{z}, s) \cdot (\overline{z}, s)^{-1} \ \textit{for some} \ (\overline{z}, s) \in S_\varrho\\
&\ \Leftrightarrow \ (\overline{x}, e) = (\overline{z}\varrho(s), s^{-1}s) \ \textit{and} \ (\overline{y}, f) = (\overline{z}, ss^{-1}) \ \textit{for some} \ (\overline{z}, s) \in S_\varrho\\
&\ \Leftrightarrow \ \overline{x} = \overline{y}\varrho(s),\ s^{-1}s = e \ \textit{and} \ ss^{-1} = f \ \textit{for some} \ s \in S.
\end{aligned}$$

Let $(\overline{y}, f)$ be a non-zero idempotent of $S_\varrho$ and let $e \in F$ such that $e\mathscr{D}f$. Then there exists $s \in S$ such that $s^{-1}s = e$ and $ss^{-1} = f$. Denote $\overline{y}\varrho(s) = \overline{x}$. So, $(\overline{y}, f)$ and $(\overline{x}, e)$ are $\mathscr{D}$ related in $S_\varrho$ such that $(\overline{x}, e) \in F_\varrho$.

If $(\overline{x}, e), (\overline{y}, f) \in F_\varrho$ are two $\mathscr{D}$-related idempotents, then there exists $s \in S$ such that $\overline{x} = \overline{y}\varrho(s)$, $s^{-1}s = e$ and $ss^{-1} = f$ for some $s \in S$. It follows $e = f$ (because $e, f \in F$) and $s\mathscr{H}e$. Since $S$ is combinatorial we have $s = e$ and therefore $\overline{x} = \overline{y}$. Therefore, the assertion of the theorem is proved. □

Now, the reduced Clifford category of $S_\varrho$ (the same construction as for a reduced division category) denoted by $C_{F_\varrho}(S_\varrho)$ is the following:

- $ObC_{F_\varrho}(S_\varrho) = F_\varrho$;
- $Hom_{C_{F_\varrho}(S_\varrho)}((\overline{x}, e), (\overline{y}, f)) = \{(\overline{y}, s, \overline{x}, e) | s \in S; \overline{y}\varrho(s) = \overline{x}, s^{-1}s \leq e$ and $ss^{-1} = f\}$;
- The composition of two morphisms $(\overline{y}, s, \overline{x}, e) : (\overline{x}, e) \to (\overline{y}, f)$ and $(\overline{z}, t, \overline{y}, f) : (\overline{y}, f) \to (\overline{z}, g)$ is given by $(\overline{z}, t, \overline{y}, f) \cdot (\overline{y}, s, \overline{x}, e) = (\overline{z}, ts, \overline{x}, e)$.

The following theorem suggests a way to know more about the Möbius function of the Möbius category $C_{F_\varrho}(S_\varrho)$. Let $(\overline{z}, r, \overline{x}, e) : (\overline{x}, e) \to (\overline{z}, g)$ be a morphism in $C_{F_\varrho}(S_\varrho)$. The Lawvere intervals $I(\overline{z}, r, \overline{x}, e)$ in $C_{F_\varrho}(S_\varrho)$ and $I(r, e)$ in $C_F(S)$ are partially ordered sets (since both $C_{F_\varrho}(S_\varrho)$ and $C_F(S)$ are right cancellative). We have:

**Theorem 2.3.** *The map* $\theta : I(\overline{z}, r, \overline{x}, e) \to I(r, e)$ *defined by*

$$\theta((\overline{z}, t, \overline{y}, f) \cdot (\overline{y}, s, \overline{x}, e)) = (t, f) \cdot (s, e)$$

*is an order isomorphism.*

*Proof.* Since $(\overline{z}, r, \overline{x}, e) : (\overline{x}, e) \to (\overline{z}, g)$ is a morphism in $C_{F_\varrho}(S_\varrho)$, we have:

$$\overline{z}\varrho(r) = \overline{x},\ r^{-1}r \leq e \text{ and } rr^{-1} = g.$$

To show that $\theta$ is surjective, let $(t, f) \cdot (s, e) \in I(r, e)$, that is $ts = r,\ s^{-1}s \leq e,\ ss^{-1} = f,\ t^{-1}t \leq f$ and $tt^{-1} = rr^{-1} = g$. Since

$$(\overline{x}[\varrho(s)]^{-1}, s, \overline{x}, e) \in Hom_{C_{F_\varrho}(S_\varrho)}((\overline{x}, e), (\overline{x}[\varrho(s)]^{-1}, f)),$$

$$(\overline{z}, t, \overline{x}[\varrho(s)]^{-1}, f) \in Hom_{C_{F_\varrho}(S_\varrho)}((\overline{x}[\varrho(s)]^{-1}, f), (\overline{z}, g)),$$

and

$$(\overline{z}, t, \overline{x}[\varrho(s)]^{-1}, f) \cdot (\overline{x}[\varrho(s)]^{-1}, s, \overline{x}, e) = (\overline{z}, r, \overline{x}, e),$$

it follows

$$\theta(\overline{z}, t, \overline{x}[\varrho(s)]^{-1}, f) \cdot (\overline{x}[\varrho(s)]^{-1}, s, \overline{x}, e) = (t, f) \cdot (s, e).$$

Now, for all $[(\overline{z},t,\overline{y},f)\cdot(\overline{y},s,\overline{x},e)]$ and $[(\overline{z},t',\overline{y'},f')\cdot(\overline{y'},s',\overline{x},e)]$ in $I(\overline{z},r,\overline{x},e)$, Diagram 2 is commutative in $I(r,e)$ (that is in $C_F(S)$) if and only if Diagram 3 is commutative in $I(\overline{z},r,\overline{x},e)$ (that is in $C_{F_\varrho}(S_\varrho)$); where $\overline{y}=\overline{x}[\varrho(s)]^{-1}$ and $\overline{y'}=\overline{x}[\varrho(s')]^{-1}$.

$I(r,e)$:

$$\begin{tikzcd}
 & f \arrow[dr, "{(t,f)}"] \arrow[dd, "{(p,f)}"] & \\
e \arrow[ur, "{(s,e)}"] \arrow[dr, "{(s',e)}"'] & & g \\
 & f' \arrow[ur, "{(t',f')}"'] &
\end{tikzcd}$$

Diagram 2

$I(\bar{z},r,\bar{x},e)$:

$$\begin{tikzcd}
 & (\bar{y},f) \arrow[dr, "{(\bar{z},t,\bar{y},f)}"] \arrow[dd, "{(\bar{y}',p,\bar{y},f)}"] & \\
(\bar{x},e) \arrow[ur, "{(\bar{y},s,\bar{x},e)}"] \arrow[dr, "{(\bar{y}',s',\bar{x},e)}"'] & & (\bar{z},g) \\
 & (\bar{y}',f') \arrow[ur, "{(\bar{z},t',\bar{y}',f')}"'] &
\end{tikzcd}$$

Diagram 3

Therefore

$$\theta[(\overline{z},t,\overline{y},f)\cdot(\overline{y},s,\overline{x},e)]\leq\theta[(\overline{z},t',\overline{y'},f')\cdot(\overline{y'},s',\overline{x},e)]$$

if and only if

$$[(\overline{z},t,\overline{y},f)\cdot(\overline{y},s,\overline{x},e)]\leq[(\overline{z},t',\overline{y'},f')\cdot(\overline{y'},s',\overline{x},e)].$$

Hence the map $\theta: I(\overline{z},r,\overline{x},e)\to I(r,e)$ is an order isomorphism. $\square$

Since the two posets $I(\overline{z},r,\overline{x},e)$ and $I(r,e)$ are order isomorphic (one of the orders can be obtained from the other just by renaming of elements), it follows:

**Theorem 2.4.** *Let $\mu_S$ be the initial Möbius function (i.e. the Möbius function of the Möbius category $C_F(S)$). For any morphism $(\overline{z},r,\overline{x},e)$ :*

*$(\overline{x}, e) \to (\overline{z}, g)$, in $C_{F_\varrho}(S_\varrho)$, we have:*

$$\mu_\varrho(\overline{z}, r, \overline{x}, e) = \mu_S(r, e),$$

*where $\mu_\varrho$ is the Möbius function of the Möbius category $C_{F_\varrho}(S_\varrho)$.*

## 3. Examples

### 3.1. The classical Möbius function

Let $S = Z^+ \times Z^+$ ($Z^+$ being the set of positive integers) be the multiplicative analogue of the bicyclic semigroup:

$$(a, b) \cdot (c, d) = \left( \frac{a[b, c]}{b}, \frac{[b, c]d}{c} \right),$$

where $[b, c]$ denotes the least common multiple of $b$ and $c$. Then the reduced division category $C_F(S)$, relative to the idempotent transversal $F = \{(1, 1)\}$, is a monoid (a category with the set of objects a singleton), namely the $\mathscr{R}$-class of $S$ containing the identity $(1, 1)$. This monoid is isomorphic to the monoid of positive integers $Z^+$ with the usual multiplication. It follows that the incidence functions of $C_F(S)$ are arithmetical functions and the convolution of incidence functions is the Dirichlet convolution of arithmetical functions. The Möbius function of the reduced division category $C_F(S)$ is then nothing but the classical Möbius function $\mu$:

$$\mu(a) = \begin{cases} 1 & \text{if} \quad a = 1 \\ (-1)^k & \text{if} \quad a \textit{ is a product of } k \textit{ distinct primes} \\ 0 & \text{if} \quad a \textit{ has one or more repeated prime factors.} \end{cases}$$

Now, the minimum group congrunece $\sigma$ on $S$ is given by:

$$(a, b)\sigma(c, d) \Leftrightarrow \frac{b}{a} = \frac{d}{c}.$$

So, the maximum group homomorphic image of $S$ is isomorphic to the multiplicative group of positive rational numbers $Q^+$, that is $S/\sigma \cong Q^+$. The multiplication on the $\sigma$-semigroup of S, $S_\sigma = (Q^+ \times S) \cup \{0\}$, is defined as follows

$$(\alpha, a, b) \cdot (\beta, c, d) = \begin{cases} (\alpha, \frac{a[b,c]}{b}, \frac{[b,c]d}{c}) & \textit{if} \quad \alpha \frac{b}{a} = \beta \\ 0 & \textit{otherwise} \end{cases}$$

and

$$0 \cdot (\alpha, a, b) = (\alpha, a, b) \cdot 0 = 0 \cdot 0 = 0.$$

The reduced Clifford category $C_{F_\sigma}(S_\sigma)$ is (isomorphic to) the category defined by:

- $ObC_{F_\sigma}(S_\sigma) = Q^+$;
- $Hom_{C_{F_\sigma}(S_\sigma)}(\alpha, \beta) = \{(\alpha, a) \in Q^+ \times Z^+ | \frac{\alpha}{a} = \beta\}$;
- The composition of two morphisms $(\alpha, a) : \alpha \to \beta$ and $(\beta, b) : \beta \to (\gamma)$ is given by $(\beta, b) \cdot (\alpha, a) = (\alpha, ab)$.

It is straightforward to see that this reduced Clifford category is a poset $(Q^+, \preceq)$ with a partial order $\preceq$ defined by:

$$\alpha \preceq \beta \Leftrightarrow \frac{\alpha}{\beta} \in Z^+$$

(any poset $(X, \leq)$ can be viewed as a category in a natural way: there's a unique morphism from $x$ to $y$ if and only if $x \leq y$).

Now, if $\mu_\sigma$ denotes the Möbius function of the poset $(Q^+, \preceq)$ and $(\alpha, \beta)$ denote a closed interval of this poset, then by Theorem 2.4 it follows:

$$\mu_\sigma(\alpha, \beta) = \begin{cases} 1 & \text{if} \quad \alpha = \beta \\ (-1)^k & \text{if} \quad \textit{the integer } \frac{\alpha}{\beta} \textit{ is a product of } k \textit{ distinct primes} \\ 0 & \text{if} \quad \textit{the integer } \frac{\alpha}{\beta} \textit{ has one or more repeated prime factors.} \end{cases}$$

## 3.2. The Möbius function in a case of one-dimensional tiling semigroup

Tiling semigroups are very similar to the free inverse semigroups. The tiling semigroups of one-dimensional tilings have been studied both from language-theoretic viewpoint ([8],[13]) and in terms of algebraic structures ([5],[2]). Dombi and Gilbert in [2] characterized the structure of one-dimensional tiling semigroups in the periodic case. A special description was given if the period has length $m$ and involves each tile exactly once. In what follows we will use this description.

The set

$$P_m = (Z_m \times Z_- \times Z_+ \times Z) \cup \{0\}$$

(where $Z_m$ is the cyclic group of addition modulo $m$; $Z_-$ is the set of non-positive integers; $Z_+$ is the set of non-negative integers and Z - the set of all integers) is the underlying set of the one-dimensional tiling semigroup in the periodic case: the period has length $m$ and involves each tile exactly once. The multiplication of two non-zero elements is given by:

$$(\overline{x}, i, a, u)\cdot(\overline{y}, j, b, v) = \begin{cases} (\overline{x}, min(i, j+u), max(a, b+u), u+v) & if \quad \overline{x}+\overline{u}=\overline{y} \\ 0 & otherwise \end{cases}$$

(the residue class of the integer $x$ is denoted by $\overline{x}$).
It is straightforward to see that this is a $\varrho$-semigroup of the monogenic free inverse monoid $FIS_1$ (see the free inverse semigroup copy $C_1$ of Petrich ([15, p. 394]). So, the set

$$F_m = \{(\overline{x}, i, 0, 0) \in E^*(P_m)\}$$

is an idempotent transversal of the non-zero $\mathscr{D}$-classes of $P_m$, and the reduced Clifford category $C_{F_m}(P_m)$ is given by:

- $ObC_{F_m}(P_m) = Z_m \times Z_-$;
- $Hom_{C_{F_m}(P_m)}((\overline{x}, i), (\overline{y}, j)) = \{(a, \overline{x}, i, j) | a \in Z_+, a \leq i-j, \overline{a}+\overline{x} = \overline{y}\}$;
- $(b, \overline{y}, j, k) \cdot (a, \overline{x}, i, j) = (a+b, \overline{x}, i, k)$ is the composition of two morphisms $(a, \overline{x}, i, j) : (\overline{x}, i) \to (\overline{y}, j)$ and $(b, \overline{y}, j, k) : (\overline{y}, j) \to (\overline{z}, k)$.

In [19] the author determined the Möbius function of the Möbius category of the free monogenic inverse monoid $FIS_1$, but the computation was made on Scheiblich's [17] isomorphic copy of the free monogenic inverse monoid. This fact does not change the situation, and therefore by Theorem 2.4 we conclude (see [19, Proposition 3.3] with the transition from one copy to another) that the Möbius function $\mu_m$ of the tilling semigroup $P_m$ (that is of the Möbius category $C_{F_m}(P_m)$) is defined by:

$$\mu_m(a, \overline{x}, i, j) = \begin{cases} 1 & \text{if} \quad a=0 \; and \; j=i \qquad or \quad a=1 \; and \; j=i-2 \\ -1 & \text{if} \quad a=0 \; and \; j=i-1 \quad or \quad a=1 \; and \; j=i-1 \\ 0 & \qquad otherwise \end{cases}$$

for any morphism $(a, \overline{x}, i, j)$ of $C_{F_m}(P_m)$.

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