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RESEARCH ARTICLE

On locally nilpotent derivations of Fermat rings

Paulo Roberto Brumatti and Marcelo Oliveira Veloso

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Abstract. Let $B_n^m = \frac{\mathbb{C}[X_1,\ldots,X_n]}{(X_1^m+\cdots+X_n^m)}$ (*Fermat ring*), where $m \geq 2$ and $n \geq 3$. In a recent paper D. Fiston and S. Maubach show that for $m \geq n^2 - 2n$ the unique locally nilpotent derivation of B_n^m is the zero derivation. In this note we prove that the ring B_n^2 has non-zero irreducible locally nilpotent derivations, which are explicitly presented, and that its ML-invariant is $\mathbb{C}$.

Introduction

Let $\mathbb{C}[X_1, \ldots, X_n]$ be the polynomial ring in n variables over complex numbers $\mathbb{C}$. Define

$$B_n^m = \frac{\mathbb{C}[X_1, \ldots, X_n]}{(X_1^m + \cdots + X_n^m)},$$

where $m \geq 2$ and $n \geq 3$. This ring is known as *Fermat ring*.

In a recent paper [3] D. Fiston and S. Maubach show that for $m \geq n^2 - 2n$ the unique locally nilpotent derivation of B_n^m is the zero derivation. Consequently the following question naturally arises: is the unique locally nilpotent derivation of the Fermat ring B_n^m for $m \geq 2$ and $n \geq 3$ the zero derivation?

In this work we show that the answer to this question is negative for $m = 2$ and $n \geq 3$. In other words, there exist nontrivial locally nilpotent derivations over B_n^2 (see examples 1 and 2). Furthemore, we show that

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these derivations are irreducible (see Theorem 2). In the general case, we prove that for certain classes of derivations of B_n^m the unique locally nilpotent derivation is the zero derivation (see Proposition 2).

The material is organized as follows. Section 1 provides the basic definitions, notations and results that are needed in this paper. In section 2 we present some results on the locally nilpotent derivations of the ring of Fermat. In section 3 we show examples of linear derivations in $LND(B_n^2)$ and some results on these derivations.

1. Generalities

In the following the word *"ring"* means commutative ring with a unit element and characteristic zero. Furthermore, we denote the group of units of a ring A by A^* and the polynomial ring $A[X_1, \ldots, X_n]$ by $A^{[n]}$. A *"domain"* is an integral domain. If A is a subring of B ($A \leq B$) and B is a domain, then $Frac\,(B)$ is its field of fractions and $trdeg_A(B)$ is the transcendence degree of $Frac\,(B)$ over $Frac\,(A)$.

Let R be a ring. An additive mapping $D : R \to R$ is said to be a *derivation* of R if it satisfies the Leibniz rule: $D(ab) = aD(b) + D(a)b$, for all $a, b \in R$. If $A \leq R$ is a subring and D is a derivation of R satisfying $D(A) = 0$, we call D an A-derivation. We denote the set of all derivations of R by $Der(R)$ and the set of all A-derivations of R by $Der_A(R)$. A derivation D is *irreducible* if it satisfies: given $b \in R$, $D(R) \subseteq bR$ if and only if $b \in R^*$.

A derivation D is *locally nilpotent* if for each $r \in R$ there is an integer $n \geq 0$ such that $D^n(r) = 0$. Let us denote by $LND(R)$ the set of all locally nilpotent derivations of R. If A is a subring of B, we will make use of the following notations

$$LND_A(B) = \{D \in LND(B) \mid D \in Der_A(B)\}$$

$$KLND(B) = \{A; A = ker\, D, D \in LND(B)\}.$$

Given $D \in LND(B)$ define $\nu_D(b) = min\{n \in \mathbb{N} \mid D^{n+1} = 0\}$, for $0 \neq b \in B$. In addition, define $\nu_D(0) = -\infty$. The *degree function* ν_D induced by a derivation D is a degree function on B (see [2]).

In this note $x, y, z, \ldots$ will represent residue classes of variables X, Y, $Z, \ldots$ module an ideal.

Note that since $\mathbb{C}$ is algebraically closed given $G = \sum_{i=1}^n a_i X_i^m$ with $a_i \in \mathbb{C}^*$ there exists a $\mathbb{C}$-automorphism φ of $\mathbb{C}[X_1, \ldots, X_n]$ such that $\varphi(X_i) = b_i X_i$, $b_i \in \mathbb{C}^*$ and $\varphi(X_1^m + \cdots + X_n^m) = G$. In this case φ

induces a $\mathbb{C}$-isomorphism of the $Der_{\mathbb{C}}(B_n^m)$ in $Der_{\mathbb{C}}(\frac{\mathbb{C}[X_1,\ldots,X_n]}{(G)})$. Thus all the results obtained in this paper about the module $Der_{\mathbb{C}}(B_n^m)$ can be extended to the module $Der_{\mathbb{C}}(\frac{\mathbb{C}[X_1,\ldots,X_n]}{(G)})$. In this paper, derivation of Fermat ring means $\mathbb{C}$-derivation and therefore we will use the notation $Der(B_n^m)$ to denote $Der_{\mathbb{C}}(B_n^m)$.

The following facts are well known (see [1] or [4]).

Lemma 1. *Let B be an integral domain and $D_1, D_2 \in LND(B)$ such that $ker\, D_1 = A = ker\, D_2$. If there exists $s \in B$ such that $0 \neq D_1(s) \in A$, then $0 \neq D_2(s) \in A$ and $D_2(s)D_1 = D_1(s)D_2$.* □

Lemma 2. *Let B be a domain satisfying ascending chain condition for principal ideals, let $A \in KLND(B)$ and consider the set*

$$S = \{D \in LND_A(B) \mid D \text{ is an irreducible derivation}\}.$$

Then $S \neq \emptyset$ and $LND_A(B) = \{aD \mid a \in A \text{ and } D \in S\}$. □

Proposition 1. *Let B be a domain and $D \in LND(B)$ a nonzero derivation. Suppose that $A = ker\, D$, then:*

a) A is a factorially closed subring of B. In particular $B^* = A^*$.

b) If K is any field contained in B then D is a K-derivation.

c) If $s \in B$ satisfy $Ds = 1$ then $B = A[s] = A^{[1]}$.

d) Let $S = A \setminus \{0\}$, then $S^{-1}B = (Frac\, A)^{[1]}$ and $trdeg_A B = 1$.

e) If $A' \in KLND(B)$ and $A' \subseteq A$ then $A' = A$ □

2. The set $LND(B_n^m)$

In this section we obtain some results that state that certain classes of derivations of $\mathbb{C}[X_1,\ldots,X_n]$ do not induce derivations of B_n^m or are not locally nilpotent if they do.

Let K be a field and let $S = \frac{K^{[n]}}{I}$ be a finitely generated K-algebra. Consider the $K^{[n]}$-submodule $\mathcal{D}_I = \{D \in Der_K(K^{[n]}) \mid D(I) \subseteq I\}$ of the module $Der_K(K^{[n]})$. It is well known that the $K^{[n]}$-homomorfism $\varphi : \mathcal{D}_I \to Der_K(S)$ given by $\varphi(D)(g + I) = D(g) + I$ induces a $K^{[n]}$-isomorfism of $\frac{\mathcal{D}_I}{IDer_K(K^{[n]})}$ in $Der_K(S)$. From this fact we obtain the following result.

Proposition 2. *Let d be a derivation of the B_n^m. If $d(x_1) = a \in \mathbb{C}$ and for each i, $1 < i \leq n$, $d(x_i) \in \mathbb{C}[x_1,\ldots,x_{i-1}]$, then d is the zero derivation.*

Proof. Let F be the Fermat polynomial $X_1^m + \cdots + X_n^m$. We know that there exists $D \in Der(\mathbb{C}^{[n]})$ such that $D(F) \in F\mathbb{C}^{[n]}$ and that $d(x_i) = D(X_i) + F\mathbb{C}^{[n]}$, $\forall i$. Thus we have $D(X_1) - a \in F\mathbb{C}^{[n]}$, and for each $i > 1$ there exists $G_i = G_i(X_1, \ldots, X_{i-1}) \in \mathbb{C}[X_1, \ldots, X_{i-1}]$ such that $D(X_i) - G_i \in F\mathbb{C}^{[n]}$. Since $D(F) = m\sum_{i=1}^{n} X_i^{m-1} D(X_i) \in F\mathbb{C}^{[n]}$ and $D(F) = m\sum_{i=1}^{n} X_i^{m-1}(D(X_i) - G_i) + m\sum_{i=1}^{n} X_i^{m-1} G_i$, where $G_1 = a$, we obtain $\sum_{i=1}^{n} X_i^{m-1} G_i \in F\mathbb{C}^{[n]}$ and then obviously $G_i = 0$ for all i. Thus d is the zero derivation. □

Corollary 1. *Let d be a locally nilpotent derivation of the Fermat ring B_n^m. If $d(x_i) = \alpha_i x_1^{m_1} \cdots x_n^{m_n}$, where $\alpha_i \in \mathbb{C}$ for all i, then d is the zero derivation.*

Proof. Let ν_d be a *degree function* induced by a derivation d. Since the polynomial F is symmetric we can suppose, without loss of generality, that

$$\nu_d(x_1) \leq \nu_d(x_2) \leq \cdots \leq \nu_d(x_k) \leq \cdots \leq \nu_d(x_n).$$

Suppose that for some $k \in \{1, \ldots, n\}$ we have $0 \neq d(x_k)$. Thus

$$\nu_d(x_k) - 1 = m_1\nu_d(x_1) + m_2\nu_d(x_2) + \cdots + m_k\nu_d(x_k) + \cdots + m_n\nu_d(x_n).$$

This implies that $m_n = m_{n-1} = \cdots = m_k = 0$. Thus, as d satisfies the conditions of the Proposition 2, we can conclude that d is the zero derivation. □

3. Linear derivations

This section is dedicated to the study of the locally nilpotent linear derivation of the Fermat ring.

Definition 1. A derivation d of the ring B_n^m is called **linear** if

$$d(x_i) = \sum_{j=1}^{n} a_{ij} x_j \text{ for } i = 1, \ldots, n, \text{ where } a_{ij} \in \mathbb{C}.$$

The matrix $[a_{ij}]$ is called the **associated matrix** of the derivation d. □

Lemma 3. *Let d be a linear derivation of B_n^m and $[a_{ij}]$ its associated matrix. Then d is locally nilpotent if and only if $[a_{ij}]$ is nilpotent.*

Proof. The following equality can be verified by induction over s.

$$\begin{bmatrix} d^s(x_1) \\ \vdots \\ d^s(x_n) \end{bmatrix} = [a_{ij}]^s \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}. \tag{1}$$

We know that d is locally nilpotent if and only if there exists $r \in \mathbb{N}$ such that $d^r(x_i) = 0$ for all i. As $\{x_1, \dots, x_n\}$ is linearly independent over $\mathbb{C}$ by the equality 1, we can conclude the result. □

Proposition 3. *If $d \in LND(B_n^m)$ is linear and $m > 2$, then $d = 0$.*

Proof. Let $A = [a_{ij}]$ be the associated matrix of d. Thus, for all i, $d(x_i) = \sum_{j=1}^{n} a_{ij}x_j$. Since $x_1^m + \cdots + x_n^m = 0$ we infer that $x_1^{m-1}d(x_1) + \cdots + x_n^{m-1}d(x_n) = 0$. Then

$$0 = x_1^{m-1}(\sum_{j=1}^{n} a_{1j}x_j) + x_2^{m-1}(\sum_{j=1}^{n} a_{2j}x_j) + \cdots + x_n^{m-1}(\sum_{j=1}^{n} a_{nj}x_j)$$

and as $x_1^m = -x_2^m - \cdots - x_n^m$ we deduce that

$$0 = (a_{22} - a_{11})x_2^m + \cdots + (a_{nn} - a_{11})x_n^m + \textstyle\sum_{j\neq 1}^{n} a_{1j}x_jx_1^{m-1} + \sum_{j\neq 2}^{n} a_{2j}x_jx_2^{m-1} + \cdots + \sum_{j\neq n}^{n} a_{nj}x_jx_n^{m-1}. \quad (*)$$

Observe that if $m > 2$, then the set

$$\{x_2^{m-1}, \dots, x_n^{m-1}\} \cup \{x_jx_i^{m-1};\ 1 \leq i < j \leq n,\} \cup \{x_jx_i^{m-1};\ 1 \leq j < i \leq n\}$$

is linearly independent over $\mathbb{C}$. Thus, we can conclude that

$$a_{11} = a_{22} = \cdots = a_{nn} = a \text{ and } a_{ij} = 0 \text{ if } i \neq j.$$

Since $d(x_1) = ax_1$ and d is locally nilpotent, we infer that $a = 0$. Thus, the matrix $A = [a_{ij}]$ is null and $d = 0$. □

The next result characterizes the linear derivations of the $LND(B_n^2)$.

Theorem 1. *If $d \in Der(B_n^2)$ is linear, then $d \in LND(B_n^2)$ if and only if its associated matrix is nilpotent and anti-symmetric.*

Proof. Let $d \in Der(B_n^2)$ be a linear derivation and $A = [a_{ij}]$ be the associated matrix of d. Using the same arguments used in Proposition 3 we obtain

$$0 = (a_{22} - a_{11})x_2^2 + \cdots + (a_{nn} - a_{11})x_n^2 + \sum_{i<j}(a_{ij} + a_{ji})x_i x_j$$

Since the set $\{x_2^2, \ldots, x_n^2\} \cup \{x_i x_j;\ 1 \leq i < j \leq n\}$ is linearly independent over $\mathbb{C}$, we know that

$$a_{11} = a_{22} = \cdots = a_{nn} = a \text{ and } a_{ij} = -a_{ji} \text{ if } i < j,$$

but if A is nilpotent then its trace na is null and thus A is also anti-symmetric.

Now we can conclude by Lemma 3 that d is locally nilpotent if and only if A is nilpotent and anti-symmetric. □

The next lemma helps us to find nilpotent and anti-symmetric matrices.

First, we introduce some notation. Given a natural number $n > 1$, $\mathbb{M}_n$ denotes the ring of matrices $n \times n$ with entries in $\mathbb{C}$, $I_n \in \mathbb{M}_n$ is the identity matrix and S_n is the group of permutations of $\{1, \ldots, n\}$. Let σ be an element of S_n, $F_\sigma = \{i \in \mathbb{N};\ 1 \leq i \leq n \quad \text{and} \quad \sigma(i) = i\}$ and $(-1)^\sigma = 1$ if σ is even and -1 if σ is odd.

Let $A = (a_{ij}) \in \mathbb{M}_n$. An elementary result involving A and its characteristic polynomial is given by the following lemma:

Lemma 4. *Let A be a matrix in $\mathbb{M}_n$ and let*

$$f(X) = det(XI_n - A) = X^n + b_{n-1}X^{n-1} + \cdots + b_1 X + b_0$$

be the characteristic polynomial of A.

a) If $a_{ii} = 0$ for every i, $1 \leq i \leq n$, then for all j, $0 \leq j \leq n-1$, $b_j = \sum_{\sigma \in F_j}(-1)^\sigma(-1)^{n-j}(\prod_{i \neq \sigma(i)} a_{i\sigma(i)})$, where $F_j = \{\sigma \in S_n; \sharp(F_\sigma) = j\}$. In particular $b_{n-1} = 0$.

b) If A is anti-symmetric, then $b_{n-2} = \sum_{i<j} a_{ij}^2$.

Proof. **a)** Just observe that if $C = X.I_n - A = (c_{ij})$ and $\sigma \in S_n$, then

$$(-1)^\sigma c_{1\sigma(1)} \cdots c_{n\sigma(n)} = (-1)^\sigma(-1)^{n-\sharp(F_\sigma)}(\prod_{i \neq \sigma(i)} a_{i\sigma(i)}).X^{\sharp(F_\sigma)}.$$

We know that $b_{n-1} = -trace(A)$ and then $b_{n-1} = 0$.

b) If $\sigma \in S_n$ then $\sharp(F_\sigma) = n-2$ if and only if σ is a transposition, i.e., $\sigma = (ij)$, $i \neq j$. Hence the result is proved as (ij) is odd and $a_{ij} = -a_{ji}$. □

Remark 1. Let $\mathbb{R}$ be the field of the real numbers. From Theorem 1 and Lemma 4 we conclude that the zero derivation is the unique derivation of ring $B = \frac{\mathbb{R}[X_1,\ldots,X_n]}{(X_1^2+\cdots+X_n^2)}$ that is locally nilpotent and linear.

In the following we present explicit examples of locally nilpotent derivations of B_n^2 that are linear.

Example 1. Let n be an odd number and $i = \sqrt{-1} \in \mathbb{C}$. Let D_I be a linear derivation of $\mathbb{C}^{[n]}$ defined by the anti-symmetric matrix $n \times n$

$$I = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 & -1 \\ 0 & 0 & \ldots & 0 & 0 & -i \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & -1 \\ 0 & 0 & \ldots & 0 & 0 & -i \\ 1 & i & \ldots & 1 & i & 0 \end{bmatrix}.$$

It is easy to verify that

$$D_I(X_n) = X_1 + iX_2 + \cdots + X_{n-2} + iX_{n-1},$$

and for $k < n$

$$D_I(X_k) = \begin{cases} -X_n, & \text{if} \quad k \text{ is odd.} \\ -iX_n, & \text{if} \quad k \text{ is even.} \end{cases}$$

But $D_I(X_1^2 + \cdots + X_n^2) = 2\sum_{i=1}^{n-1} X_i D_I(X_i) + 2X_n D_I(X_n)$ and then

$$D_I(X_1^2 + \cdots + X_n^2) = -2X_n D_I(X_n) + 2X_n D_I(X_n) = 0.$$

Thus, D_I induces a linear derivation, d_I, of B_n^2 given by

$$d_I(x_n) = x_1 + ix_2 + \cdots + x_{n-2} + ix_{n-1},$$

and for $k < n$

$$d_I(x_k) = \begin{cases} -x_n, & \text{if} \quad k \text{ is odd.} \\ -ix_n, & \text{if} \quad k \text{ is even.} \end{cases}$$

Now is easy to check that $I^3 = 0$. Thus, d_I is a locally nilpotent linear derivation of B_n^2 by Theorem 1.

Example 2. Let n be an even number and let ε be a primitive $(n-1)$-th root of unity . Let D_P be a linear derivation of $\mathbb{C}^{[n]}$ defined by the anti-symmetric matrix $n \times n$

$$P = \begin{bmatrix} 0 & 0 & \dots & 0 & \dots & 0 & -1 \\ 0 & 0 & 0 & \dots & 0 & 0 & -\varepsilon \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & -\varepsilon^k \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & -\varepsilon^{n-2} \\ 1 & \varepsilon & \dots & \varepsilon^k & \dots & \varepsilon^{n-2} & 0 \end{bmatrix}$$

It is easy to verify that

$$D_P(X_k) = -\varepsilon^{k-1} X_n, \text{ for } k < n$$

and

$$D_P(X_n) = X_1 + \varepsilon X_2 + \cdots + \varepsilon^{k-1} X_k + \cdots + \varepsilon^{n-2} X_{n-1}.$$

As in example 1 it is easy to check that $D_P(X_1^2 + \cdots + X_n^2) = 0$. Thus, D_P induces a linear derivation, d_P, of B_n^2 given by

$$d_P(x_k) = -\varepsilon^{k-1} x_n, \text{ for } k < n$$

and

$$d_P(x_n) = x_1 + \varepsilon x_2 + \cdots + \varepsilon^{k-1} x_k + \cdots + \varepsilon^{n-2} x_{n-1}.$$

Since $1 + \varepsilon + \varepsilon^2 + \cdots + \varepsilon^{n-2} = 0$ and $\{1, \varepsilon, \ldots, \varepsilon^{n-2}\} = \{1, \varepsilon^2, \ldots, \varepsilon^{2(n-2)}\}$ it is easy to check that $P^3 = 0$. Thus, d_P is a locally nilpotent linear derivation of B_n^2 by Theorem 1.

The next step is to show that the derivations d_I and d_P are irreducible. But for this we need the following elementary result.

Lemma 5. *Let h be an element of the B_n^m. Then for each $k \in \{1, \ldots, n\}$ there exists a unique $G \in \mathbb{C}[X_1, \ldots, X_n]$ satisfying*

$$h = G(x_1, \ldots, x_n) \text{ and } deg_{X_k}(G) < m.$$

Proof. By the Euclidean algorithm for the ring $\mathbb{C}[X_1, \ldots, X_n]$ it is sufficient to observe that for all k the polinomial $F = X_1^m + \cdots + X_n^m$ is monic in X_k. □

In the Fermat ring B_n^2 for each $k \in \{1, \ldots, n\}$ define the subring B_k of the ring B_n^2 by $\mathbb{C}[x_1, \ldots, \widehat{x_k}, \ldots, x_n]$ where $\widehat{x_k}$ signifies that the element x_k was omitted in the ring B_n^2 .

Lemma 6. *Let $h \in B_n \subset B_n^2$. Then:*

1) *$d_P(h) \in x_n B_n$ if n is even, d_P defined in example 2;*
2) *$d_I(h) \in x_n B_n$ if n is odd, d_I defined in example 1.*

Proof. Suppose that n is even and let $h \in B_n$. Then

$$h = \sum_{i=(i_1,\ldots,i_{n-1})} a_i x_1^{i_1} \cdots x_{n-1}^{i_{n-1}}, \text{ hence}$$

$$\begin{aligned} d_P(h) &= \frac{\partial h}{\partial x_1} d_P(x_1) + \cdots + \frac{\partial h}{\partial x_k} d_P(x_k) + \cdots + \frac{\partial h}{\partial x_{n-1}} d_P(x_{n-1}) \\ &= \frac{\partial h}{\partial x_1}(-x_n) + \cdots + \frac{\partial h}{\partial x_k}(-\varepsilon^{k-1} x_n) + \cdots + \frac{\partial h}{\partial x_{n-1}}(-\varepsilon^{n-2} x_n) \end{aligned}$$

then $d_P(h) \in x_n B_n$. The proof of the case n odd is analogous. □

Lemma 7. *Let $h \in B_n^2$. Then*

1) *$d_P(h) = 0$ if and only if $d_P(h) = 0$ and $h \in B_n$, if n is even;*
2) *$d_I(h) = 0$ if and only if $d_I(h) = 0$ and $h \in B_n$, if n is odd.*

Proof. Suppose that n is even and let $h \in B_n^2$. By Lemma 5 there exists a unique $h_0, h_1 \in B_n$ such that $h = h_1 x_n + h_0$. Assume $h_1 \neq 0$. Now note that

$$0 = d_P(h) = d_P(h_1)x_n + h_1 d_P(x_n) + d_P(h_0). \tag{2}$$

From Lemma 6 we have $d_P(h_1), d_P(h_0) \in x_n B_n$. Thus, $d_P(h_1) = b x_n$ for some $b \in B_n$. Hence $d_P(h_1)x_n = (b x_n)x_n = b x_n^2 = b(-x_1^2 - \cdots - x_{n-1}^2)$ $\in B_n$. As $d_P(x_n) = x_1 + \varepsilon x_2 + \cdots + \varepsilon^{i-1} x_i + \cdots + \varepsilon^{n-2} x_{n-1}$ we have $h_1 d_P(x_n) \in B_n$. Thus $d_P(h_1)x_n + h_1 d_P(x_n) \in B_n$ and by Lemma 6 $d_P(h_0) = c x_n$ for some $c \in B_n$, then by Lemma 5 and (2) we infer that $0 = d_P(h_1)x_n + h_1 d_P(x_n) = d_P(h_1 x_n)$. As $ker\, d_P$ is factorially closed $x_n \in ker\, d_P$, so $d_P(x_n) = 0$. But since $d_P(x_n) \neq 0$, this is a contradiction. Hence $h_1 = 0$. The proof of the case n odd is analogous. □

Lemma 8. *Let $n \geq 3$ be a natural number. Then*

1) *$ker\, d_P = \mathbb{C}[x_1 - \varepsilon^{(n-2)}x_2, \ldots, x_1 - \varepsilon^{(n-k)}x_k, \ldots, x_1 - \varepsilon x_{n-1}]$, if n is even.*

2) *$ker\, d_I = \mathbb{C}[x_1 + ix_2, x_1 - x_3, \ldots, x_1 - x_{k-2}, x_1 - ix_{k-1}]$, if n is odd.*

Proof. Suppose that n is even and let A be the subring

$$\mathbb{C}[x_1 - \varepsilon^{(n-2)}x_2, \ldots, x_1 - \varepsilon^{(n-k)}x_k, \ldots, x_1 - \varepsilon x_{n-1}]$$

of B_2^n. As

$$d_P(x_1 - \varepsilon^{(n-k)}x_k) = d_P x_1 - \varepsilon^{(n-k)} d_P(x_k) = -x_n - \varepsilon^{(n-k)}(-\varepsilon^{(k-1)}x_n) = 0,$$

for every $k < n$, we deduce that $A \subseteq ker\, d_P$. Given

$$y_2 = x_1 - \varepsilon^{(n-2)}x_2, \;\ldots,\; y_k = x_1 - \varepsilon^{(n-k)}x_k, \;\ldots, y_{n-1} = x_1 - \varepsilon x_{n-1}$$

observe that

$$A[x_1] = \mathbb{C}[x_1, y_2, \ldots, y_{n-1}] = \mathbb{C}[x_1, \ldots, x_{n-1}] = B_n,$$

thus the set $\{x_1, y_2, \cdots, y_{n-1}\}$ is algebraically independent over $\mathbb{C}$.

By Lemma 7 for each $h \in ker\, d_P$, we have $d_P(h) = 0$ and $h \in B_n$, then we may write $h = \sum_{i=0}^{n} a_i x_1^i$ where $a_i \in A \subseteq ker\, d_P$ for all $i \in \{0, \ldots, n\}$. Assume $n > 0$ and remember that $d_P(x_1) = -x_n$. So

$$0 = d_P(h) = -[a_1 + 2a_2 x_1 + \cdots + n a_n x_1^{n-1}]x_n.$$

By the uniqueness of Lemma 5 we have $a_1 + 2a_2x_1 + \cdots + na_nx_1^{n-1} = 0$ and hence $a_i = 0$ for $i = 1, \ldots, n$. Therefore $h = a_0 \in A \subseteq ker\, d_P$. The proof of the case n odd is analogous.. □

Theorem 2. *Let $n \geq 3$ be a natural number.*

1) *If n is even, then $d_P \in LND(B_n^2)$, where d_P was defined in the example 2, is irreducible and*

$$LND_A(B_n^2) = \{ad_P \mid a \in A\},$$

where $A = \mathbb{C}[x_1 - \varepsilon^{(n-2)}x_2, \ldots, x_1 - \varepsilon^{(n-k)}x_k, \ldots, x_1 - \varepsilon x_{n-1}]$.

2) *If n is odd, then $d_I \in LND(B_n^2)$, where d_I was defined in the example 1, is irreducible and*

$$LND_S(B_n^2) = \{sd_I \mid s \in S\},$$

where $S = \mathbb{C}[x_1 + ix_2, x_1 - x_3, \ldots, x_1 - x_{n-2}, x_1 - ix_{n-1}]$.

Proof. Suppose that n is even and $d \in LND_A(B_n^2)\setminus\{0\}$. By Proposition 1 we have $ker\, d = A$. Observe that

$$d_P^2(x_n) = d_P(\sum_{k=1}^{n-1} \varepsilon^{k-1} x_k) = \sum_{k=1}^{n-1} \varepsilon^{k-1} d_P(x_k) = x_n(\sum_{k=1}^{n-1} \varepsilon^{2(k-1)}) = 0$$

thus $d_p(x_n) \in A$. Then, by Lemma 1, $d(x_n) \in A$ and

$$d_P(x_n)d = d(x_n)d_P. \tag{3}$$

By definition $d_P(x_1) = -x_n$, so

$$d_P(x_n)d(x_1) = -d(x_n)x_n. \tag{4}$$

We know that $d(x_1) = g_1x_n + g_0$ with $g_0, g_1 \in B_n$. Then, (4) implies that $d_P(x_n)g_1x_n + d_P(x_n)g_0 = -d(x_n)x_n$. Since $d_P(x_n) \in A \subseteq B_n$, by the uniqueness of Lemma 5 we obtain $d(x_n) = -d_P(x_n)g_1$. As $d(x_n) \in A$ we know that $d_P(d(x_n)) = 0$. Thus $0 = d_P(d(x_n)) = d_P(-d_P(x_n)g_1)$ and then $d_P(g_1) = 0$, i.e., $g_1 \in A$. Since $d(x_n) = -d_P(x_n)g_1$, (3) implies that

$$d_P(x_n)d = d(x_n)d_P = -d_P(x_n)g_1d_P.$$

Therefore $d = -g_1d_P$, where $-g_1 \in A$. The Lemma 2 implies that $d_P = hd_0$ for some $h \in A$ and some irreducible $d_0 \in LND(B_n^2)$. As we saw $d_0 = h_0d_P$ for some $h_0 \in A$. So $d_P = hd_0 = h(h_0d_P) = (hh_0)d_P$. Thus $h \in A^* = \mathbb{C}$ and hence d_P is irreducible. The proof of the case n odd is analogous. □

Let B be a $\mathbb{C}$-domain and $\theta \in Aut_{\mathbb{C}}(B)$. It is well known that if $D \in LND(B)$, then $\theta D\theta^{-1} \in LND(B)$ and $ker\ \theta D\theta^{-1} = \theta(ker\, D)$.

Let S_n be the symmetric group and $\sigma \in S_n$. The permutation σ induces a $\mathbb{C}$-automorphism of $\mathbb{C}^{[n]} = \mathbb{C}[X_1, \ldots, X_n]$ which is also called σ and defined by relations $\sigma(X_i) = X_{\sigma(i)}$ for every i. Now since

$$\sigma(X_1^2 + \cdots + X_n^2) = X_1^2 + \cdots + X_n^2$$

then σ induces a $\mathbb{C}$-automorphism of B_n^2 which is also called σ and defined by relations $\sigma(x_i) = x_{\sigma(i)}$ for every i. Suppose that n is even. Given $j < n$ we denote the transposition $(j\ n) \in S_n$ by τ_j and the derivation $\tau_j d_P {\tau_j}^{-1}$ by d_{P_j}. Hence we have $d_{P_j} \in LND(B_n^2)$ and

$$ker\, d_{P_j} = \tau_j(\mathbb{C}[x_1 - \varepsilon^{(n-2)}x_2, \ldots, x_1 - \varepsilon^{(n-k)}x_k, \ldots, x_1 - \varepsilon x_{n-1}]).$$

Observe that

$$\tau_j(x_1 - \varepsilon^{(n-k)}x_k) = \begin{cases} x_n - \varepsilon^{(n-k)}x_k, & \text{if } j = 1 \\ x_1 - \varepsilon^{(n-k)}x_n, & \text{if } j = k \\ x_1 - \varepsilon^{(n-k)}x_k, & \text{otherwise.} \end{cases}$$

This implies that $ker\, d_{P_j} \subset B_j$.

Now suppose that n is odd. For each $1 \leq j \leq n$ denote the derivation $\tau_j d_I {\tau_j}^{-1}$ by d_{I_j} . Thus we have

$$ker\, d_{I_j} = \tau_j(\mathbb{C}[x_1 + ix_2, x_1 - x_3, \dots, x_1 - x_{n-2}, x_1 - ix_{n-1}]).$$

if k is odd

$$\tau_j(x_1 - x_k) = \begin{cases} x_n - x_k, & \text{if } j = 1 \\ x_1 - x_n, & \text{if } j = k \\ x_1 - x_k, & \text{otherwise.} \end{cases}$$

If k is even

$$\tau_j(x_1 - ix_k) = \begin{cases} x_n - ix_k, & \text{if } j = 1 \\ x_1 - ix_n, & \text{if } j = k \\ x_1 - ix_k, & \text{otherwise.} \end{cases}$$

Is follows that $ker\, d_{I_j} \subset B_j$.

The concept of ML-invariant of the a ring R was introduced by L. Makar-Limanov. This invariant has proved very useful in studying the group of automorphisms of a ring (see [5]) .

Definition 2. Let B be a ring. The intersection of the kernels of all locally nilpotent derivation of B is called the ML-invariant of B.

The next result shows that the ML-invariant of B_n^2 is $\mathbb{C}$. Note that for $m \geq n^2 - 2n$ the ML-invariant of B_n^m is B_n^m.

Theorem 3. *The ML-invariant of the ring B_n^2 is $\mathbb{C}$.*

Proof. We define $d_j = d_{I_j}$ if n is odd, and $d_j = d_{P_j}$ if n is even. In both cases, by previous observations, we have $ker\, d_j \subset B_j$ and

$$\cap_{j=1}^n ker\, d_j \subset \cap_{j=1}^n B_j = \mathbb{C}.$$

Since $\mathbb{C} \subset ker\, d_j$, for all $j \in \{1, \dots, n\}$, then the result follows. $\square$

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Contact information

P. Brumatti
IMECC-Unicamp, Rua Sérgio Buarque de Holanda 651, Cx. P. 6065
13083-859, Campinas-SP, Brazil
E-Mail: `brumatti@ime.unicamp.br`

M. Veloso
Defim-UFSJ, Rodovia MG 443 Km 7
36420-000, Ouro Branco-MG, Brazil
E-Mail: `veloso@ufsj.edu.br`

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