Algebra and Discrete Mathematics
Volume 16 **(2013)**. Number 2. pp. 287 – 292

RESEARCH ARTICLE

Relative symmetric polynomials and money change problem

M. Shahryari

Communicated by V. A. Artamonov

Abstract. This article is devoted to the number of non-negative solutions of the linear Diophantine equation

$$a_1t_1 + a_2t_2 + \cdots + a_nt_n = d,$$

where $a_1, \ldots, a_n$, and d are positive integers. We obtain a relation between the number of solutions of this equation and characters of the symmetric group, using *relative symmetric polynomials*. As an application, we give a necessary and sufficient condition for the space of the relative symmetric polynomials to be non-zero.

Suppose $a_1, \ldots, a_n$, and d are positive integers, and consider the following linear Diophantine equation:

$$a_1t_1 + a_2t_2 + \cdots + a_nt_n = d.$$

Let $Q_d(a_1, \ldots, a_n)$ be the number of non-negative integer solutions of this equation. Computing the exact values of the function Q_d is the well-known *money change problem*. It is easy to see that a generating function for $Q_d(a_1, \ldots, a_n)$ is

$$\prod_{i=1}^{n} \frac{1}{1 - t^{a_i}}.$$

This article is devoted for an interesting relation between Q_d and irreducible complex characters of the symmetric group S_m, where $m = a_1 + \cdots + a_n$. In fact, we will show that Q_d is a permutation character of

2010 MSC: Primary 05A17, Secondary 05E05 and 15A69.

Key words and phrases: Money change problem; Partitions of integers; Relative symmetric polynomials; Symmetric groups; Complex characters.

S_m, and then we will find its irreducible constituents. Our main tool, in the investigation of Q_d, is the notion of *relative symmetric polynomials*, which is introduced by the author in [3]. Once, we find the irreducible constituents of Q_d, we can also obtain a necessary and sufficient condition for vanishing of the space of relative symmetric polynomials. A similar result was obtained in [2], for vanishing of *symmetry classes of tensors*, using the same method.

We need a survey of results about relative symmetric polynomials in this article. For a detailed exposition, one can see [3].

Let G be a subgroup of the full symmetric group S_m of degree m and suppose χ is an irreducible complex character of G. Let $H_d[x_1,\ldots,x_m]$ be the complex space of homogenous polynomials of degree d with the independent commuting variables $x_1,\ldots,x_m$. Suppose $\Gamma^+_{m,d}$ is the set of all m-tuples of non-negative integers, $\alpha=(\alpha_1,\ldots,\alpha_m)$, such that $\sum_i \alpha_i = d$. For any $\alpha\in\Gamma^+_{m,d}$, define X^α to be the monomial $x_1^{\alpha_1}\cdots x_m^{\alpha_m}$. So the set $\{X^\alpha : \alpha\in\Gamma^+_{m,d}\}$ is a basis of $H_d[x_1,\ldots,x_m]$. We define also an inner product on $H_d[x_1,\ldots,x_m]$ by

$$< X^\alpha, X^\beta >= \delta_{\alpha,\beta}.$$

The group G acts on $H_d[x_1,\ldots,x_m]$ via

$$q^\sigma(x_1,\ldots,x_m) = q(x_{\sigma^{-1}(1)},\ldots,x_{\sigma^{-1}(m)}).$$

It also acts on $\Gamma^+_{m,d}$ by

$$\sigma\alpha = (\alpha_{\sigma(1)},\ldots,\alpha_{\sigma(m)}).$$

Let Δ be a set of representatives of orbits of $\Gamma^+_{m,d}$ under the action of G.

Now consider the idempotent

$$T(G,\chi) = \frac{\chi(1)}{|G|}\sum_{\sigma\in G}\chi(\sigma)\sigma$$

in the group algebra $\mathbb{C}G$. Define the space of *relative symmetric polynomials of degree* d with respect to G and χ to be

$$H_d(G,\chi) = T(G,\chi)(H_d[x_1,\ldots,x_m]).$$

Let $q\in H_d[x_1,\ldots,x_m]$. Then we set

$$q^* = T(G,\chi)(q)$$

and we call it a *symmetrized polynomial* with respect to G and χ. Clearly

$$H_d(G,\chi) =< X^{\alpha,*} : \alpha \in \Gamma^+_{m,d} >,$$

where $< set \;\; of \;\; vectors >$ denotes the subspace generated by a given set of vectors.

Recall that the inner product of two characters of an arbitrary group K is defined as follows,

$$[\phi,\psi]_K = \frac{1}{|K|}\sum_{\sigma\in K}\phi(\sigma)\psi(\sigma^{-1}).$$

In the special case where K is a subgroup of G and ϕ and ψ are characters of G, the notation $[\phi,\psi]_K$ will denote the inner product of the restrictions of ϕ and ψ to K.

It is proved it [3] that for any α, we have

$$||X^{\alpha,*}||^2 = \chi(1)\frac{[\chi,1]_{G_\alpha}}{[G:G_\alpha]},$$

where G_α is the stabilizer subgroup of α under the action of G. Hence, $X^{\alpha,*}\neq 0$, if and only if $[\chi,1]_{G_\alpha}\neq 0$. According to this result, let Ω be the set of all $\alpha\in\Gamma^+_{m,d}$, with $[\chi,1]_{G_\alpha}\neq 0$ and suppose $\bar{\Delta}=\Delta\cap\Omega$.

We proved in [3], the following formula for the dimension of $H_d(G,\chi)$

$$\dim H_d(G,\chi) = \chi(1)\sum_{\alpha\in\bar{\Delta}}[\chi,1]_{G_\alpha}.$$

Note that, $\bar{\Delta}$ depends on χ, but Δ depends only on G. Since, $[\chi,1]_{G_\alpha}=0$, for all $\alpha\in\Delta-\bar{\Delta}$, we can re-write the above formula, as

$$\dim H_d(G,\chi) = \chi(1)\sum_{\alpha\in\Delta}[\chi,1]_{G_\alpha}.$$

There is also another interesting formula for the dimension of $H_d(G,\chi)$. This is the formula which employs the function Q_d and so it connects the money change problem to relative symmetric polynomials. Let $\sigma\in G$ be any element with the cycle structure $[a_1,\ldots,a_n]$, (i.e. σ is equal to a product of n disjoint cycles of lengths $a_1,\ldots,a_n$, respectively). Define $Q_d(\sigma)$ to be the number of non-negative integer solutions of the equation

$$a_1t_1+a_2t_2+\cdots+a_nt_n=d,$$

so, we have $Q_d(\sigma) = Q_d(a_1, \dots, a_n)$. If we consider the free vector space $\mathbb{C}[\Gamma^+_{m,d}]$ as a $\mathbb{C}G$-module, then for all $\sigma \in G$, we have

$$Tr\ \sigma = Q_d(\sigma),$$

and hence, Q_d is a permutation character of G. It is proved in [3], that we have also

$$\dim H_d(G, \chi) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) Q_d(\sigma).$$

Note that, we can write this result as $\dim H_d(G, \chi) = \chi(1)[\chi, Q_d]_G$. Now, comparing two formulae for the dimension of $H_d(G, \chi)$ and using the *reciprocity relation* for induced characters, we obtain

$$Q_d(a_1, \dots, a_n) = \sum_{\alpha \in \Delta} (1_{G_\alpha})^G(\sigma),$$

where $\sigma \in S_m$ is any permutation of the cycle structure $[a_1, \dots, a_n]$, G is any subgroup of S_m containing σ and $m = a_1 + \cdots + a_n$. It is clear that, if α and β are in the same orbit of $\Gamma^+_{m,d}$, then $(1_{G_\alpha})^G = (1_{G_\beta})^G$, so we have also

$$Q_d(a_1, \dots, a_n) = \frac{1}{|G|} \sum_{\alpha \in \Gamma^+_{m,d}} |G_\alpha| (1_{G_\alpha})^G(\sigma).$$

As our main result in this section, we have,

Theorem A.

$$Q_d = \frac{1}{|G|} \sum_{\alpha \in \Gamma^+_{m,d}} |G_\alpha| (1_{G_\alpha})^G.$$

For a fixed equation

$$a_1 t_1 + a_2 t_2 + \cdots + a_n t_n = d,$$

suppose σ is a permutation of the cycle structure $[a_1, \dots, a_n]$ and let G be the cyclic group generated by σ. Clearly this G is the smallest subgroup of S_m which we can use to obtain the number of non-negative solutions of the above equation. By Theorem A, the required number is

$$Q_d(\sigma) = \frac{1}{|G|} \sum_{\alpha \in \Gamma^+_{m,d}} |G_\alpha| (1_{G_\alpha})^G(\sigma).$$

On the other hand, if we want to have a formula for all values of Q_d, we must let $G = S_m$, which contains any possible cycle structure. So in the remaining part of this article, we will assume that, $G = S_m$, then using representation theory of symmetric groups, we will find, irreducible constituents of Q_d.

We need some standard notions from representation theory of symmetric groups. Ordinary representations of S_m are in one to one correspondence with *partitions* of m. Let

$$\pi = (\pi_1, \ldots, \pi_s)$$

be any partition of m. The irreducible character of S_m, corresponding to a partition π is denoted by χ^π. There is also a subgroup of S_m, associated to π, which is called the *Young subgroup* and it is defined as,

$$S_\pi = S_{\{1,\ldots,\pi_1\}} \times S_{\{\pi_1+1,\ldots,\pi_1+\pi_2\}} \times \cdots .$$

Therefore, we have $S_\pi \cong S_{\pi_1} \times \cdots \times S_{\pi_s}$.

Let $\pi = (\pi_1, \ldots, \pi_s)$ and $\mu = (\mu_1, \ldots, \mu_l)$ be two partitions of m. We say that μ majorizes π, iff for any $1 \leq i \leq min\{s, l\}$, the inequality

$$\pi_1 + \cdots + \pi_i \leq \mu_1 + \cdots + \mu_i$$

holds. In this case we write $\lambda \trianglelefteq \mu$. This is clearly a partial ordering on the set of all partitions of m. A generalized μ-tableau of type π is a function

$$T : \{(i, j) : 1 \leq i \leq h(\mu), 1 \leq j \leq \mu_i\} \to \{1, 2, \ldots, m\}$$

such that for any $1 \leq i \leq m$, we have $|t^{-1}(i)| = \pi_i$. This generalized tableau is called semi-standard if for each i, $j_1 < j_2$ implies $T(i, j_1) \leq T(i, j_2)$ and for any j, $i_1 < i_2$ implies $T(i_1, j) < T(i_2, j)$. In other words, T is semi-standard, iff every row of T is non-descending and every column of T is ascending. The number of all such semi-standard tableaux is denoted by $K_{\mu\pi}$ and it is called the *Kostka number*. It is well known that $K_{\mu\pi} \neq 0$ iff μ majorizes π, see [1], for example. We have also,

$$\begin{aligned} K_{\mu\pi} &= [(1_{S_\pi})^{S_m}, \chi^\mu]_{S_m} \\ &= [1, \chi_\mu]_{S^\pi}. \end{aligned}$$

For any $\alpha \in \Gamma^+_{m,d}$, the multiplicity partition is denoted by $M(\alpha)$, so, to obtain $M(\alpha)$, we must arrange the multiplicities of numbers $0, 1, \ldots, d$ in α in the descending order. It is clear that $(S_m)_\alpha \cong S_{M(\alpha)}$, the Young subgroup. If $M(\alpha) = (k_1, \ldots, k_s)$, then we have

$$|(S_m)_\alpha| = k_1!k_2! \ldots k_s!.$$

In what follows, we use the notation $M(\alpha)!$ for the number $k_1!k_2!\dots k_s!$. On the other hand, we have

$$\begin{aligned}(1_{G_\alpha})^G &= (1_{S_{M(\alpha)}})^{S_m} \\ &= \sum_{M(\alpha)\trianglelefteq\pi} K_{\pi,M(\alpha)}\chi^\pi.\end{aligned}$$

Now, using Theorem A, we obtain,

Theorem B.

$$Q_d = \frac{1}{m!}\sum_{\alpha\in\Gamma^+_{m,d}}\sum_{M(\alpha)\trianglelefteq\pi} M(\alpha)!K_{\pi,M(\alpha)}\chi^\pi.$$

As a result, we can compute the dimension of $H_d(S_m,\chi^\pi)$, in a new fashion. We have,

$$\begin{aligned}\dim H_d(S_m,\chi^\pi) &= \chi^\pi(1)[\chi^\pi, Q_d]_{S_m} \\ &= \frac{\chi^\pi(1)}{m!}\sum_{\alpha\in\Gamma^+_{m,d}, M(\alpha)\trianglelefteq\pi} M(\alpha)!K_{\pi,M(\alpha)}.\end{aligned}$$

Note that, this generalizes the similar formulae in the final part of the second section of [3]. Now, as a final result, we have also, a necessary and sufficient condition for $H_d(S_m,\chi^\pi)$ to be non-zero.

Theorem C. *We have $H_d(S_m,\chi^\pi)\neq 0$, if and only if there exists $\alpha\in\Gamma^+_{m,d}$, such that $M(\alpha)\trianglelefteq\pi$.*

References

[1] B. Sagan, *The Symmetric Group: Representation, Combinatorial Algorithms and Symmetric Functions*, Wadsworth and Brook/ Cole math. series, 1991.

[2] M. Shahryari, M. A. Shahabi, On a permutation character of S_m, Linear and Multilinear Algebra, **44** (1998), 45 – 52.

[3] M. Shahryari, Relative symmetric polynomials, Linear Algebra and its Applications, **433** (2010), 1410 – 1421.

CONTACT INFORMATION

M. Shahryari Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran
E-Mail: mshahryari@tabrizu.ac.ir

Received by the editors: 08.04.2012
and in final form 28.04.2012.