Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume 16 **(2013)**. Number 2. pp. 171 – 187

Reducibility and irreducibility of monomial matrices over commutative rings

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Communicated by V. V. Kirichenko

Dedicated to the memory of Professor Petro M. Gudivok

Abstract. Let R be a local ring with nonzero Jacobson radical. We study monomial matrices over R of the form

$$\begin{pmatrix} 0 & \dots & 0 & t^{s_n} \\ t^{s_1} & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & t^{s_{n-1}} & 0 \end{pmatrix},$$

and give a criterion for such matrices to be reducible when $n \leq 6$, $s_1 \dots, s_n \in \{0, 1\}$ and the radical is a principal ideal with generator t. We also show that the criterion does not hold for $n = 7$.

Introduction

The problem of classifying, up to similarity, all the matrices over a commutative ring (which is not a field) is usually very difficult; in most cases it is "unsolvable" (wild), as in the case of the rings of residue classes [1]. Special cases of matrices of small orders were considered by many authors (see, e.g., [2]–[5]). In such situation, an important place is occupied by irreducible matrices over rings. Our paper is devoted to this subject.

2010 MSC: 15B33, 15A30.

Key words and phrases: irreducible matrix, similarity, local ring, Jacobson radical.

Throughout the paper R denotes a commutative ring with identity, which is not a field, and R^* the group of its invertible elements.

We say that an $n \times n$ matrix M over R is *reducible over* R, or simply *reducible*, if it is similar (over R) to a matrix

$$N = \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix},$$

where A_i is an $n_i \times n_i$ matrix over R; $i = 1, 2$, n_1, $n_2 > 0$ (i. e. there exists an invertible matrix X over R such that $X^{-1}MX = N$). Otherwise we say that M is *irreducible over* R, or simply *irreducible*.

We consider the question: when is an $n \times n$ matrix over R of the form

$$M(t, s_1, \dots, s_n) = \begin{pmatrix} 0 & \dots & 0 & t^{s_n} \\ t^{s_1} & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & t^{s_{n-1}} & 0 \end{pmatrix} \tag{1}$$

with $t \in R$ irreducible?

The answer to this question is only known in some cases. Obviously, $M(t, s_1, \dots, s_n)$ is reducible if $t^{s_1} = \dots = t^{s_n}$ with $n > 1$ (in particular, $t \in \{0, 1\}$ or $s_1 = \dots = s_n$). If R is local and its radical is a principle ideal with generator t, then $M(t, 0, \dots, 0, 1)$ is irreducible since its characteristic polynomial $x^n + (-1)^{n+1}t$ is irreducible; and $M(t, 0, 1, \dots, 1)$ is irreducible (probed by Gudivok and Tylyshchak [6]).

1. Reducible matrices $M(t, s_1, \dots, s_n)$

$\mathbb{Z}[\lambda]$ denotes the ring of polynomials of the variable λ over the ring $\mathbb{Z}$ of integer numbers. Its field of fractions is denoted by F. By a basis of a vector space we mean an ordered basis. As usual, n denotes a natural number.

Proposition 1. *Let $s_1, \dots, s_n$ be natural numbers such that $s = \sum_{i=1}^n s_i$ and n are not coprime. Then for any common divisors $d > 1$ of s and n, the matrix*

$$M = M(\lambda, s_1, \dots, s_n)$$

over $\mathbb{Z}[\lambda]$ is similar (over $\mathbb{Z}[\lambda]$) to a matrix of the form

$$N = \begin{pmatrix} A & D \\ 0 & B \end{pmatrix},$$

where A is an $\frac{n}{d} \times \frac{n}{d}$ matrix.

Proof. Let $s = dk$, $n = dm$.

To illustrate the idea of the proof, we consider the following special case:

n	s	$s_1, \ldots, s_n$	d	m	k
15	9	0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1	3	5	3

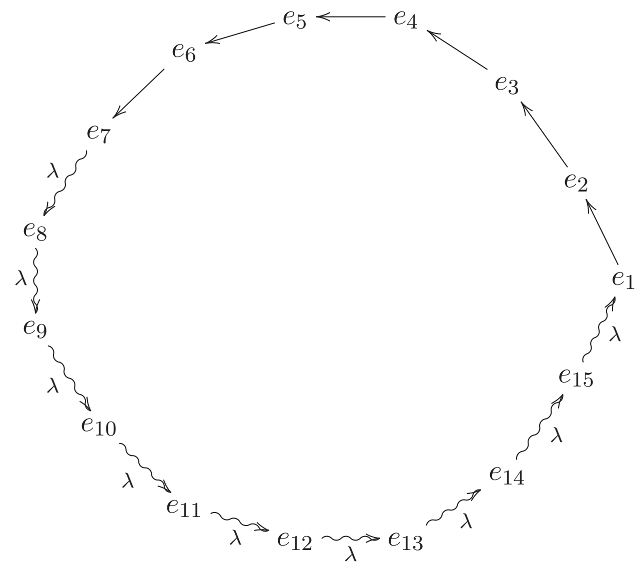
Let $\overline{e} = \{e_1, e_2, \ldots, e_{15}\}$ be the standard basis of the vector space F^{15} and let φ be the linear operator on this vector space determined (in the basis $\overline{e}$) by the matrix $M(\lambda, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1)$, which has, by definition, the form

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0
\end{pmatrix}.$$

Then

$$\begin{array}{lll}
\varphi(e_1) = e_2, & \varphi(e_2) = e_3, & \varphi(e_3) = e_4, \\
\varphi(e_4) = e_5, & \varphi(e_5) = e_6, & \varphi(e_6) = e_7, \\
\varphi(e_7) = \lambda e_8, & \varphi(e_8) = \lambda e_9, & \varphi(e_9) = \lambda e_{10}, \\
\varphi(e_{10}) = \lambda e_{11}, & \varphi(e_{11}) = \lambda e_{12}, & \varphi(e_{12}) = \lambda e_{13}, \\
\varphi(e_{13}) = \lambda e_{14}, & \varphi(e_{14}) = \lambda e_{15}, & \varphi(e_{15}) = \lambda e_1.
\end{array} \tag{2}$$

We can write these equalities in the form of the following diagram:

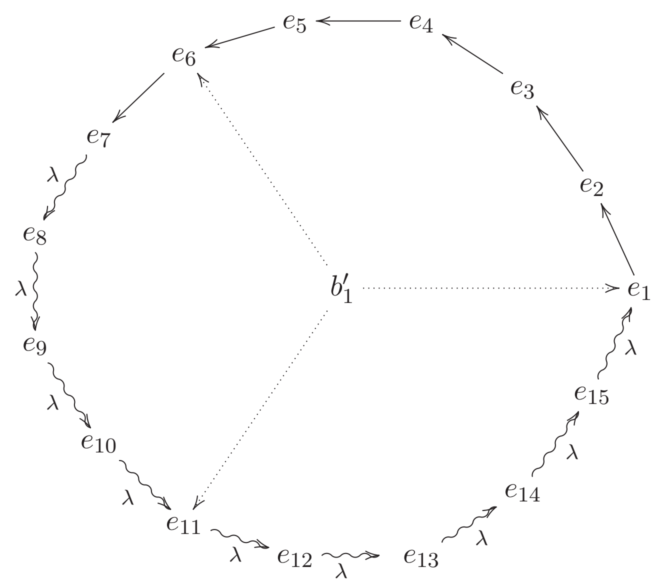


where $e_i \longrightarrow e_j$ and $e_i \underset{\lambda}{\rightsquigarrow} e_j$ mean respectively that $\varphi(e_i) = e_j$ and $\varphi(e_i) = \lambda e_j$.

Obviously, $\varphi^{15}(e_1) = \lambda^9 e_1$. Put

$$b_1' = \lambda^6 e_1 + \lambda^3 \varphi^5(e_1) + \varphi^{10}(e_1) = \lambda^6 e_1 + \lambda^3 e_6 + \lambda^4 e_{11}. \tag{3}$$

On the diagram



$b'_1 \dashrightarrow e_i$ indicate those e_i which appeared in (3). Then

$$\begin{aligned}\varphi^5(b'_1) &= \varphi^5(\lambda^6 e_1 + \lambda^3\varphi^5(e_1) + \varphi^{10}(e_1)) \\ &= \lambda^6\varphi^5(e_1) + \lambda^3\varphi^{10}(e_1) + \varphi^{15}(e_1) \\ &= \lambda^6\varphi^5(e_1) + \lambda^3\varphi^{10}(e_1) + \lambda^9 e_1 \\ &= \lambda^9 e_1 + \lambda^6\varphi^5(e_1) + \lambda^3\varphi^{10}(e_1) \\ &= \lambda^3(\lambda^6 e_1 + \lambda^3\varphi^5(e_1) + \varphi^{10}(e_1)) = \lambda^3 b'_1.\end{aligned}$$

Let us define $b'_2, \dots, b'_5$ by recursion

$$b'_2 = \varphi(b'_1), \qquad b'_3 = \varphi(b'_2), \qquad b'_4 = \varphi(b'_3), \qquad b'_5 = \varphi(b'_4). \tag{4}$$

Clearly

$$\begin{aligned} b'_1 & &&= \lambda^6 e_1 + \lambda^3 e_6 + \lambda^4 e_{11}, \\ b'_2 &= \varphi(b'_1) &&= \lambda^6 e_2 + \lambda^3 e_7 + \lambda^5 e_{12}, \\ b'_3 &= \varphi(b'_2) &&= \lambda^6 e_3 + \lambda^4 e_8 + \lambda^6 e_{13}, \\ b'_4 &= \varphi(b'_3) &&= \lambda^6 e_4 + \lambda^5 e_9 + \lambda^7 e_{14}, \\ b'_5 &= \varphi(b'_4) &&= \lambda^6 e_5 + \lambda^6 e_{10} + \lambda^8 e_{15}, \end{aligned}$$

and

$$\varphi(b'_5) = \lambda^6 e_6 + \lambda^7 e_{12} + \lambda^9 e_1 = \lambda^9 e_1 + \lambda^6 e_6 + \lambda^7 e_{12}.$$

Then $\varphi(b'_5) = \varphi^5(b'_1) = \lambda^3 b'_1$. From (2)–(4) it follows that

$$b'_j = \lambda^{\alpha_{1j}} e_j + \lambda^{\alpha_{2j}} e_{5+j} + \lambda^{\alpha_{3j}} e_{10+j}$$

for some integer $\alpha_{ij} \geq 0$; here $i = 1, 2, 3,\ j = 1, \dots, 5$. Put $\alpha_j = \min\limits_i\{\alpha_{ij}\}$. Then

$$\alpha_1 = 3,\ \alpha_2 = 3,\ \alpha_3 = 4,\ \alpha_4 = 5,\ \alpha_5 = 6,$$

whence $\alpha_j \geq \alpha_{j-1}$ $(j > 1)$, $\alpha_1 + 3 = \alpha_5$.

Let $\beta_{ij} = \alpha_{ij} - \alpha_j$ $(i = 1, 2, 3,\ j = 1, \dots, 5)$. Then $\beta_{ij} \geq 0$, and obviously that for any j there is $1 \leq \mu_j \leq 3$ such that $\beta_{\mu_j j} = 0$.

Put

$$b_j = \sum_{i=1}^{d} \lambda^{\beta_{ij}} e_{(i-1)5+j}$$

or more detail

$$\begin{aligned} &b_1 = \lambda^3 e_1 + e_6 + \lambda e_{11}, \quad && b_2 = \lambda^3 e_2 + e_7 + \lambda^2 e_{12}, \\ &b_3 = \lambda^2 e_3 + e_8 + \lambda^2 e_{13}, && b_4 = \lambda e_4 + e_9 + \lambda^2 e_{14}, \\ &b_5 = e_5 + e_{10} + \lambda^2 e_{15}. \end{aligned} \tag{5}$$

Then

$$\begin{aligned}
\lambda^{\alpha_1}b_1 &= \lambda^3 b_1 = \lambda^6 e_1 + \lambda^3 e_6 + \lambda^4 e_{11} = b_1',\\
\lambda^{\alpha_2}b_2 &= \lambda^3 b_2 = \lambda^6 e_2 + \lambda^3 e_7 + \lambda^5 e_{12} = b_2',\\
\lambda^{\alpha_3}b_3 &= \lambda^4 b_3 = \lambda^6 e_3 + \lambda^4 e_8 + \lambda^6 e_{13} = b_3',\\
\lambda^{\alpha_4}b_4 &= \lambda^5 b_4 = \lambda^6 e_4 + \lambda^5 e_9 + \lambda^7 e_{14} = b_4',\\
\lambda^{\alpha_5}b_5 &= \lambda^6 b_5 = \lambda^6 e_5 + \lambda^6 e_{10} + \lambda^8 e_{15} = b_5'.
\end{aligned}$$

It follows from (4) that $\varphi(b_{j-1}) = \lambda^{\alpha_j-\alpha_{j-1}}b_j$ $(j=2,\dots,5)$ and $\varphi(b_5) = \lambda^{3+\alpha_1-\alpha_5}b_1$ (since $\varphi(b_5') = \lambda^3 b_1'$). Then

$$\begin{aligned}
&\varphi(b_1) = \lambda^{3-3}b_2 = b_2, && \varphi(b_2) = \lambda^{4-3}b_2 = \lambda b_3,\\
&\varphi(b_3) = \lambda^{5-4}b_3 = \lambda b_4, && \varphi(b_4) = \lambda^{6-5}b_4 = \lambda b_5,\\
&\varphi(b_5) = \lambda^{3+3-6}b_1 = b_1.
\end{aligned} \tag{6}$$

Denote by $\overline{a}$ the following basis of F^{15}:

$$\overline{a} = \{e_6,\ e_7,\ e_8,\ e_9,\ e_5,\ e_1,\ e_2,\ e_3,\ e_4,\ e_{10},\ e_{11},\ e_{12},\ e_{13},\ e_{14},\ e_{15}\}.$$

The transition matrix from the basic $\overline{e}$ to the basis $\overline{a}$ is the (permutation) matrix

$$P = \begin{pmatrix}
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix} \in \mathrm{GL}_{15}(\mathbb{Z}).$$

From (5) it follows that

$$\overline{b} = \{b_1, b_2, b_3, b_4, b_5, e_1, e_2, e_3, e_4, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}, e_{15}\}$$

is a basis of F^{15}. The transition matrix from the basic $\overline{a}$ to the basis $\overline{b}$ is

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda^3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda^3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda^2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(belonging obviously to $\mathrm{GL}_{15}(\mathbb{Z}[\lambda])$).

Consider now the matrix $S = PC \in \mathrm{GL}_n(\mathbb{Z}[\lambda])$, which is the transition matrix from the basic $\overline{e}$ to the basis $\overline{b}$. It follows from (6) that

$$S^{-1}MS = \begin{pmatrix} A & D \\ 0 & B \end{pmatrix} \tag{7}$$

where

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 \end{pmatrix}$$

and B is an 10×10 matrix over the ring $\mathbb{Z}[\lambda]$. This completes the proof in our special case.

Now we proceed to the general case. Recall that $s = \sum_{i=1}^{n} s_i = dk$, $n = dm$.

Let φ be the linear operator on the vector space F^n determined by the matrix M in the standard basis $\overline{e} = \{e_1, e_2, \ldots, e_n\}$. Then $\varphi(e_1) = \lambda^{s_1} e_2$, $\varphi(e_2) = \lambda^{s_2} e_3, \ldots, \varphi(e_{n-1}) = \lambda^{s_{n-1}} e_n$, $\varphi(e_n) = \lambda^{s_n} e_1$. Thus

$$\varphi(e_i) = \begin{cases} \lambda^{s_i} e_{i+1}, & i < n, \\ \lambda^{s_i} e_1, & i = n. \end{cases} \tag{8}$$

Obviously, $\varphi^n(e_1) = \lambda^{\sum_{i=1}^n s_i} e_1 = \lambda^s e_1$. Let

$$b'_1 = \sum_{i=1}^{d} \lambda^{(d-i)k} \varphi^{(i-1)m}(e_1). \tag{9}$$

Then

$$\varphi^m(b'_1) = \varphi^m \left(\sum_{i=1}^{d} \lambda^{(d-i)k} \varphi^{(i-1)m}(e_1) \right) =$$

$$\sum_{i=1}^{d} \lambda^{(d-i)k} \varphi^{im}(e_1) = \sum_{i=1}^{d-1} \lambda^{(d-i)k} \varphi^{im}(e_1) + \varphi^{dm}(e_1) =$$

$$\varphi^{dm}(e_1) + \sum_{i=1}^{d-1} \lambda^{(d-i)k} \varphi^{im}(e_1) = \varphi^n(e_1) + \sum_{i=2}^{d} \lambda^{(d-i+1)k} \varphi^{(i-1)m}(e_1) =$$

$$= \lambda^s e_1 + \sum_{i=2}^{d} \lambda^{(d-i+1)k} \varphi^{(i-1)m}(e_1) = \lambda^{dk} e_1 + \sum_{i=2}^{d} \lambda^{(d-i+1)k} \varphi^{(i-1)m}(e_1) =$$

$$\sum_{i=1}^{d} \lambda^{(d-i+1)k} \varphi^{(i-1)m}(e_1) = \lambda^k \left(\sum_{i=1}^{d} \lambda^{(d-i)k} \varphi^{(i-1)m}(e_1) \right) = \lambda^k b'_1.$$

Let us define b'_j by the recursion

$$b'_j = \varphi(b'_{j-1}), \ (j = 2, \dots, m). \tag{10}$$

Then $\varphi(b'_m) = \varphi^m(b'_1) = \lambda^k b'_1$. From (8)–(10), it follows that

$$b'_j = \sum_{i=1}^{d} \lambda^{\alpha_{ij}} e_{(i-1)m+j}$$

for some $\alpha_{ij} \in \mathbb{Z}$ $(i = 1, \dots, d, \ j = 1, \dots, m)$. Moreover,

$$b'_j = \varphi(b'_{j-1}) = \sum_{i=1}^{d} \lambda^{\alpha_{ij-1}} \varphi(e_{(i-1)m+j-1}) =$$

$$= \sum_{i=1}^{d} \lambda^{\alpha_{ij-1} + s_{(i-1)m+j-1}} e_{(i-1)m+j} \ (j = 2, \dots, m).$$

Consequently, $\alpha_{ij} = \alpha_{ij-1} + s_{(i-1)m+j-1}$. Thus $\alpha_{ij} \geq \alpha_{i\,j-1}$ $(i = 1, \dots, d,$ $j = 2, \dots, m)$. Put $\alpha_j = \min_i \{\alpha_{ij}\}$ $(j = 1, \dots, m)$. Then $\alpha_j \geq \alpha_{j-1}$ $(j > 1)$. Since

$$\sum_{i=1}^{d} \lambda^{\alpha_{i1}+k} e_{(i-1)m+1} = \lambda^k b'_1 = \varphi(b'_m) = \sum_{i=1}^{d} \lambda^{\alpha_{i\,m}} \varphi(e_{(i-1)m+m}) =$$

$$= \sum_{i=1}^{d} \lambda^{\alpha_{i\,m}} \varphi(e_{im}) = \sum_{i=1}^{d-1} \lambda^{\alpha_{i\,m}} \varphi(e_{im}) + \lambda^{\alpha_{d\,m}} \varphi(e_{dm}) =$$

$$= \sum_{i=1}^{d-1} \lambda^{\alpha_{i\,m}+s_{im}} (e_{im+1}) + \lambda^{\alpha_{d\,m}+s_n} e_1 = \lambda^{\alpha_{d\,m}+s_n} e_1 + \sum_{i=1}^{d-1} \lambda^{\alpha_{i\,m}+s_{im}} (e_{im+1}) =$$

$$= \lambda^{\alpha_{d\,m}+s_n} e_1 + \sum_{i=2}^{d} \lambda^{\alpha_{i-1,m}+s_{(i-1)m}} (e_{(i-1)m+1})$$

we deduce that $\alpha_{i1} + k \geq \alpha_{i-1,m}$ $(i = 2, \ldots, d)$. Moreover, $\alpha_{11} + k \geq \alpha_{d\,m}$. Thus $\alpha_1 + k \geq \alpha_m$.

Put $\beta_{ij} = \alpha_{ij} - \alpha_j$ $(i = 1, \ldots, d,\ j = 1, \ldots, m)$. Then $\beta_{ij} \geq 0$ for all i, j and, obviously, for any j there is $1 \leq \mu_j \leq d$ such that $\beta_{\mu_j, j} = 0$. Let

$$b_j = \sum_{i=1}^{d} \lambda^{\alpha_{ij}-\alpha_j} e_{(i-1)m+j}.$$

Then

$$\lambda^{\alpha_j} b_j = \sum_{i=1}^{d} \lambda^{\alpha_{ij}} e_{(i-1)m+j} = b'_j,$$

and it follows from (10) that $\varphi(b_{j-1}) = \lambda^{\alpha_j - \alpha_{j-1}} b_j$ $(j = 2, \ldots, m)$. Since $\varphi(b'_m) = \lambda^k b'_1$, we deduce that $\varphi(b_m) = \lambda^{k+\alpha_1-\alpha_m} b_1$. Let $\beta(j-1) = \alpha_j - \alpha_{j-1}$, $\beta(m) = k + \alpha_1 - \alpha_m$. Clearly $\beta(j) \geq 0$ $(j = 1, \ldots, m)$. Moreover, $\varphi(b_{j-1}) = \lambda^{\beta(j-1)} b_j$ $(j = 2, \ldots, m)$, $\varphi(b_m) = \lambda^{\beta(m)} b_1$.

Consider the vectors $e_{(\mu_1-1)m+1}, e_{(\mu_2-1)m+2}, \ldots, e_{(\mu_m-1)m+m}$ (belonging to the original basic $\overline{e}$). They are distinct because their indices are not congruent modulo m. Therefore we can extend these vectors to a basis

$$\overline{a} = \{e_{(\mu_1-1)m+1}, e_{(\mu_2-1)m+2}, \ldots, e_{(\mu_m-1)m+m}, e_{i'_{m+1}}, \ldots, e_{i'_n}\}$$

of F^n which is equal, up to a permutation, to the basic $\overline{e} = \{e_1, \ldots, e_n\}$. The transition matrix from the basic $\overline{a}$ to a basis

$$\overline{b} = \{b_1, \ldots, b_m, e_{i'_{m+1}}, \ldots, e_{i'_n}\}$$

has the form

$$C = \begin{pmatrix} 1 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 1 & 0 & \ldots & 0 \\ \lambda^{\delta_{m+1,1}} & \ldots & \lambda^{\delta_{m+1,m}} & 1 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \lambda^{\delta_{n1}} & \ldots & \lambda^{\delta_{nm}} & 0 & \ldots & 1 \end{pmatrix}$$

where $\delta_{ij} \geq 0$ ($i = m+1, \ldots, n,\ j = 1, \ldots, m$). Obviously, $C \in \mathrm{GL}_n(\mathbb{Z}[\lambda])$ (as an matrix over $\mathbb{Z}[\lambda]$ with determinant 1).

Let $P \in \mathrm{GL}_n(\mathbb{Z})$ be a permutation matrix which is the transition matrix from the basic $\overline{e}$ to the basis $\overline{a}$. Then the matrix $S = PC$ is the transition matrix from the basic $\overline{e}$ to the basis $\overline{b}$, and since $\varphi(b_{j-1}) = \lambda^{\beta(j-1)} b_j$ ($j = 2, \ldots, m$), $\varphi(b_m) = \lambda^{\beta(m)} b_1$ we deduce that

$$S^{-1}MS = \begin{pmatrix} A & D \\ 0 & B \end{pmatrix},$$

where

$$A = \begin{pmatrix} 0 & 0 & \ldots & 0 & \lambda^{\beta(m)} \\ \lambda^{\beta(1)} & 0 & \ldots & 0 & 0 \\ 0 & \lambda^{\beta(2)} & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \lambda^{\beta(m-1)} & 0 \end{pmatrix},$$

and B is a $(n-m) \times (n-m)$ matrix. □

Theorem 1. *Let R be a commutative local ring, and let $n > 0$ and $s_1, \ldots, s_n \geq 0$ be integer numbers such that n and $s = \sum_{i=1}^{n} s_i$ are not coprime. Then for any common divisors $d > 1$ of n and s, and any $t \in R$ the matrix*

$$M(t, s_1, \ldots, s_n) = \begin{pmatrix} 0 & \ldots & 0 & t^{s_n} \\ t^{s_1} & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & t^{s_{n-1}} & 0 \end{pmatrix}$$

over the ring R is reducible.

The proof follows from Proposition 1 and the existence of a (unique) homomorphism of rings $f:\ \mathbb{Z}[\lambda] \to R$ such that $f(1) = 1,\ f(\lambda) = t$.

2. Irreducible matrices $M(t, s_1, \ldots, s_n)$

Throughout this section all matrices are considered over a commutative local ring R with Jacobson radical $\mathrm{Rad}(R) = tR, t \neq 0$, and their similarity are considered also over R. For a matrix M, we denote by $\overline{M}$ its reduction modulo the radical. As above n denotes a natural number.

Lemma 1. *Let $s_1, \ldots, s_n \in \{0,1\}$ with $s_i = 0$ for at least one $i \in \{1, \ldots, n\}$. Then the matrix $M = M(t, s_1, \ldots, s_n)$ is not similar to matrices of the form*

$$M_1 = \begin{pmatrix} tA & D \\ 0 & B \end{pmatrix}, \ M_2 = \begin{pmatrix} A & D \\ 0 & tB \end{pmatrix},$$

where A is an $r \times r$ matrix and B is an $(n-r) \times (n-r)$ matrix $(0 < r < n)$.

Proof. Suppose that

$$C^{-1}MC = M_1,$$

where $C = (c_{ij})_{1 \le i,j \le n} \in \mathrm{GL}_n(R)$, or equivalently,

$$MC = CM_1,$$

i. e.

$$\begin{pmatrix} 0 & \ldots & 0 & t^{s_n} \\ t^{s_1} & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & t^{s_{n-1}} & 0 \end{pmatrix} C = C \begin{pmatrix} tA & D \\ 0 & B \end{pmatrix}. \tag{11}$$

For $i, j \in \{1, \ldots, n\}$, the scalar equality $(MC)_{ij} = (CM_1)_{ij}$ is denoted by $(11, ij)$. Put $c_i = (c_{i1}, \ldots, c_{ir})$.

We write the equalities $(11, 1j)$, $(11, 2j), \ldots, (11, nj)$, where, in all cases, j runs from 1 to r, respectively in the form

$$t^{s_n} c_n = tc_1 A, \ \ t^{s_1} c_1 = tc_2 A, \ \ \ldots, \ \ t^{s_{n-1}} c_{n-1} = tc_n A. \tag{12}$$

Since $\mathrm{Rad}(R)$ is generated by $t \neq 0$, we have the following simple fact: if $t^s \gamma = t\delta$ for some $s \in \{0,1\}$, $\gamma, \delta \in R$ then either $s = 0$ and consequently $\gamma = t\delta$, or $s = 1$ and consequently $t(\gamma - \delta) = 0$; so, respectively, $\overline{\gamma} = 0$ or $\overline{\gamma} = \overline{\delta}$. If one put $c_{n+1} = c_1$, then by this fact $\overline{c}_i = 0$ or $\overline{c}_i = \overline{c}_{i+1}\overline{A}$ for any $i = 1, \ldots, n$. Because $s_i = 0$ for at least one i, we have that $\overline{c}_l = 0$ for some $l \in \{1, \ldots, n\}$. By (12) we successively obtain $\overline{c}_1 = \ldots \overline{c}_{l-1} = 0$, $\overline{c}_n = 0$ and $\overline{c}_{l+1} = \ldots \overline{c}_{n-1} = 0$. Thus the matrix C in not invertible modulo t and consequently is not invertible itself, a contradiction.

The case $C^{-1}MC = M_2$ is considered analogously. □

Lemma 2. *Let $s_1, \ldots, s_n \ge 0$ be integers with $s_i \neq 0$ for at least one $i \in \{1, \ldots, n\}$. The matrix $M = M(t, s_1, \ldots, s_n)$ is not similar to a matrix*

$$N = \begin{pmatrix} A & D \\ 0 & B \end{pmatrix},$$

where A is an $r \times r$ matrix and B is an $(n-r) \times (n-r)$ matrix $(0 < r < n)$, A or B is invertible.

The lemma follows at once from the nilpotency of $\overline{M}$.

Lemma 3. *The matrix $M(t, 0, 0, 0, 1, 1)$ is irreducible.*

Proof. Assume that the matrix $M(t, 0, 0, 0, 1, 1)$ is reducible. Then for some matrix $C \in \mathrm{GL}_5(R)$ we have

$$M(t, 0, 0, 0, 1, 1)C = C \begin{pmatrix} A & D \\ 0 & B \end{pmatrix}. \tag{13}$$

where A is an $s \times s$ matrix and B is a $(5-s) \times (5-s)$ matrix $(0 < s < 5)$.

Since the matrix $M(t, 0, 0, 0, 1, 1)$ is similar to its transpose, we can interchange the matrices A and B in (13), and therefore, without loss of generality, we can assume that $s \leq 5 - s$, i. e. $s = 1$ or $s = 2$.

In the case $s = 1$ either $A \in tR$ or $A \in R^*$, and we have a contradiction with Lemmas 1 or 2, respectively.

Let now $s = 2$ and let $C = (c_{ij})_{1 \leq i,j \leq 5}$, $c_p = (c_{p1}, c_{p2})$ $(p = 1, \ldots, 5)$. Then from the equality (13) we obtain

$$tc_5 = c_1 A, \quad c_1 = c_2 A, \quad c_2 = c_3 A, \quad c_3 = c_4 A, \quad tc_4 = c_5 A. \tag{14}$$

By Lemmas 1 and 2 $\operatorname{rank}(\overline{A}) \neq 0$ and $\operatorname{rank}(\overline{A}) \neq 2$. Consequently $\operatorname{rank}(\overline{A}) = 1$. Since the matrix $\overline{A}$ is nilpotent, we can assume, without loss of generality, that

$$A = \begin{pmatrix} t\alpha & 1 + t\beta \\ t\gamma & t\delta \end{pmatrix},$$

where $\alpha, \beta, \gamma, \delta \in R$. Substituting A in (14), we get the following equalities (which for convenience are written in pairs):

$$\begin{cases} (tc_{51}, \quad tc_{52}) & = & (c_{11}t\alpha + c_{12}t\gamma, \quad c_{11} + c_{11}t\beta + c_{12}t\delta), \\ (c_{11}, \quad c_{12}) & = & (c_{21}t\alpha + c_{22}t\gamma, \quad c_{21} + c_{21}t\beta + c_{22}t\delta), \\ (c_{21}, \quad c_{22}) & = & (c_{31}t\alpha + c_{32}t\gamma, \quad c_{31} + c_{31}t\beta + c_{32}t\delta), \\ (c_{31}, \quad c_{32}) & = & (c_{41}t\alpha + c_{42}t\gamma, \quad c_{41} + c_{41}t\beta + c_{42}t\delta), \\ (tc_{41}, \quad tc_{42}) & = & (c_{51}t\alpha + c_{52}t\gamma, \quad c_{51} + c_{51}t\beta + c_{52}t\delta). \end{cases} \tag{15}$$

From the equalities (15) we obtain that $\overline{c}_{11} = \overline{c}_{21} = \overline{c}_{31} = 0 = \overline{c}_{51}$.

Further, $\overline{c}_{12} = 0$ (since $c_{12} = c_{21} + c_{21}t\beta + c_{22}t\delta$ and $\overline{c}_{21} = 0$), and $\overline{c}_{22} = 0$ (since $c_{22} = c_{31} + c_{31}t\beta + c_{32}t\delta$ and $\overline{c}_{31} = 0$).

Finally, since $\overline{c}_{41} = \overline{c}_{51}\overline{\alpha} + \overline{c}_{52}\overline{\gamma}$ (by $tc_{41} = c_{51}t\alpha + c_{52}t\gamma$) and $\overline{c}_{52} = \overline{c}_{21}\overline{\alpha} + \overline{c}_{22}\overline{\gamma} + \overline{c}_{11}\overline{\beta} + \overline{c}_{12}\overline{\delta}$ (by $tc_{52} = c_{11} + c_{11}t\beta + c_{12}t\delta$ and $c_{11} = c_{21}t\alpha + c_{22}t\gamma$) we have that $\overline{c}_{41} = 0$ (taking into account the above equalities of the form $\overline{c}_{ij} = 0$).

Thus $\det(\overline{C}) = 0$, a contradiction. □

Lemma 4. *The matrix $M(t,0,0,1,1,1)$ is irreducible.*

Proof. Assume that $M(t,0,0,0,1,1)$ is reducible. Then for some matrix $C \in \mathrm{GL}_5(R)$ we have

$$M(t,0,0,1,1,1)C = C\begin{pmatrix} A & D \\ 0 & B \end{pmatrix}. \tag{16}$$

where A is an $s \times s$ matrix and B is a $(5-s)\times(5-s)$ matrix $(0 < s < 5)$.

As in the proof of Lemma 3, we can assume that $s \le 5-s$, i. e. $s = 1$ or $s = 2$. Since the case $s = 1$ is trivial, we consider only the case $s = 2$.

Let $C = (c_{ij})_{1\le i,j\le 5}$, $c_p = (c_{p1}, c_{p2})$ $(p = 1,\dots,5)$. Then from the equality (16) we obtain

$$tc_5 = c_1A, \quad c_1 = c_2A, \quad c_2 = c_3A, \quad tc_3 = c_4A, \quad tc_4 = c_5A. \tag{17}$$

By Lemmas 1 and 2 $\operatorname{rank}(\overline{A}) = 1$. Since the matrix $\overline{A}$ is nilpotent we can assume, without loss of generality, that

$$A = \begin{pmatrix} t\alpha & 1+t\beta \\ t\gamma & t\delta \end{pmatrix},$$

where $\alpha,\beta,\gamma,\delta \in R$. Substituting A in (17), we get the following equalities:

$$\left\{\begin{array}{rcl} (tc_{51}, \quad tc_{52}) & = & (c_{11}t\alpha + c_{12}t\gamma, \quad c_{11} + c_{11}t\beta + c_{12}t\delta), \\ (c_{11}, \quad c_{12}) & = & (c_{21}t\alpha + c_{22}t\gamma, \quad c_{21} + c_{21}t\beta + c_{22}t\delta), \\ (c_{21}, \quad c_{22}) & = & (c_{31}t\alpha + c_{32}t\gamma, \quad c_{31} + c_{31}t\beta + c_{32}t\delta), \\ (tc_{31}, \quad tc_{32}) & = & (c_{41}t\alpha + c_{42}t\gamma, \quad c_{41} + c_{41}t\beta + c_{42}t\delta), \\ (tc_{41}, \quad tc_{42}) & = & (c_{51}t\alpha + c_{52}t\gamma, \quad c_{51} + c_{51}t\beta + c_{52}t\delta). \end{array}\right. \tag{18}$$

From the equalities (18) we have $\overline{c}_{11} = \overline{c}_{21} = 0 = \overline{c}_{41} = \overline{c}_{51}$, and

$\overline{c}_{31} = \overline{c}_{42}\overline{\gamma}$ (since $tc_{31} = c_{41}t\alpha + c_{42}t\gamma$ and $\overline{c}_{41} = 0$),
$\overline{c}_{22} = \overline{c}_{31}$ (since $c_{22} = c_{31} + c_{31}t\beta + c_{32}t\delta$),
$\overline{c}_{12} = 0$ (since $c_{12} = c_{21} + c_{21}t\beta + c_{22}t\delta$ and $\overline{c}_{21} = 0$),
$\overline{c}_{52}\overline{\gamma} = 0$ (since $tc_{41} = c_{51}t\alpha + c_{52}t\gamma$ and $\overline{c}_{41} = \overline{c}_{51} = 0$),
$\overline{c}_{52} = \overline{c}_{31}\overline{\gamma}$ (since $tc_{52} = c_{11} + c_{11}t\beta + c_{12}t\delta$,
$c_{11} = c_{21}t\alpha + c_{22}t\gamma$ and $\overline{c}_{11} = \overline{c}_{21} = \overline{c}_{12} = 0$, $\overline{c}_{22} = \overline{c}_{31}$).

If $\overline{\gamma} = 0$ then $\overline{c}_{31} = 0$; if $\overline{\gamma} \neq 0$ then $\gamma \in R^*$ and hence $\overline{c}_{52} = 0$, $\overline{c}_{31} = 0$. Therefore, in both the cases $\det(\overline{C}) = 0$, a contradiction. □

Lemma 5. *The matrix $M(t, 0, 0, 1, 0, 1)$ is irreducible.*

Proof. Assume that $M(t, 0, 0, 1, 0, 1)$ is reducible. Then for some matrix $C \in \mathrm{GL}_5(R)$ we have

$$M(t,0,0,1,0,1)C = C\begin{pmatrix} A & D \\ 0 & B \end{pmatrix}. \tag{19}$$

where A is an $s \times s$ matrix and B is a $(5-s) \times (5-s)$ matrix $(0 < s < 5)$.

As in the proof of Lemma 3, we can assume that $s \leq 5 - s$, i. e. $s = 1$ or $s = 2$. Since the case $s = 1$ is trivial, we consider only the case $s = 2$. Let $C = (c_{ij})_{1 \leq i,j \leq 5}$, $c_p = (c_{p1}, c_{p2})$ $(p = 1, \ldots, 5)$. Then from the equality (19) we obtain

$$tc_5 = c_1A, \quad c_1 = c_2A, \quad c_2 = c_3A, \quad tc_3 = c_4A, \quad c_4 = c_5A. \tag{20}$$

By Lemmas 1 and 2 $\mathrm{rank}(\overline{A}) = 1$. Since the matrix $\overline{A}$ is nilpotent we can assume, without loss of generality, that

$$A = \begin{pmatrix} t\alpha & 1 + t\beta \\ t\gamma & t\delta \end{pmatrix},$$

where $\alpha, \beta, \gamma, \delta \in R$. Substituting A in equality (20), we get

$$\left\{\begin{array}{ccc} (tc_{51}, tc_{52}) & = & (c_{11}t\alpha + c_{12}t\gamma, c_{11} + c_{11}t\beta + c_{12}t\delta), \\ (c_{11}, c_{12}) & = & (c_{21}t\alpha + c_{22}t\gamma, c_{21} + c_{21}t\beta + c_{22}t\delta), \\ (c_{21}, c_{22}) & = & (c_{31}t\alpha + c_{32}t\gamma, c_{31} + c_{31}t\beta + c_{32}t\delta), \\ (tc_{31}, tc_{32}) & = & (c_{41}t\alpha + c_{42}t\gamma, c_{41} + c_{41}t\beta + c_{42}t\delta), \\ (c_{41}, c_{42}) & = & (c_{51}t\alpha + c_{52}t\gamma, c_{51} + c_{51}t\beta + c_{52}t\delta). \end{array}\right. \tag{21}$$

From equality (21) we have $\overline{c}_{11} = \overline{c}_{21} = \overline{c}_{41} = 0$, and

$\overline{c}_{12} = 0$ (since $c_{12} = c_{21} + c_{21}t\beta + c_{22}t\delta$ and $\overline{c}_{21} = 0$),

$\overline{c}_{51} = 0$ (since $tc_{51} = c_{11}t\alpha + c_{12}t\gamma$ and $\overline{c}_{11} = \overline{c}_{12} = 0$),

$\overline{c}_{42} = 0$ (since $c_{42} = c_{51} + c_{51}t\beta + c_{52}t\delta$ and $\overline{c}_{51} = 0$),

$\overline{c}_{31} = \overline{c}_{41}\overline{\alpha} + \overline{c}_{42}\overline{\gamma} = 0$ (since $tc_{31} = c_{41}t\alpha + c_{42}t\gamma$ and $\overline{c}_{41} = \overline{c}_{42} = 0$).

Therefore $\det(\overline{C}) = 0$, a contradiction. □

Lemma 6. *The matrix $M(t, 0, 1, 1, 0, 1)$ is irreducible.*

Proof. Assume that $M(t,0,1,1,0,1)$ is reducible. Then for some matrix $C \in \mathrm{GL}_5(R)$ we have

$$M(t,0,1,1,0,1)C = C\begin{pmatrix} A & D \\ 0 & B \end{pmatrix}. \tag{22}$$

where A is an $s \times s$ matrix and B is a $(5-s)\times(5-s)$ matrix $(0 < s < 5)$.

As in the proof of Lemma 3, we can assume that $s \le 5-s$, i. e. $s=1$ or $s=2$. Since the case $s=1$ is trivial, we consider only the case $s=2$.

Let $C = (c_{ij})_{1\le i,j\le 5}$, $c_p = (c_{p1}, c_{p2})$ $(p = 1,\ldots,5)$. Then from the equality (22) we obtain

$$tc_5 = c_1A, \quad c_1 = c_2A, \quad tc_2 = c_3A, \quad tc_3 = c_4A, \quad c_4 = c_5A. \tag{23}$$

By Lemmas 1 and 2 $\mathrm{rank}(\overline{A}) = 1$. Since the matrix $\overline{A}$ is nilpotent, we can assume, without loss of generality, that

$$A = \begin{pmatrix} t\alpha & 1+t\beta \\ t\gamma & t\delta \end{pmatrix},$$

where $\alpha, \beta, \gamma, \delta \in R$.

Substituting A in equality (23), we get

$$\left\{\begin{array}{rcl} (tc_{51}, \quad tc_{52}) & = & (c_{11}t\alpha + c_{12}t\gamma, \quad c_{11} + c_{11}t\beta + c_{12}t\delta), \\ (c_{11}, \quad c_{12}) & = & (c_{21}t\alpha + c_{22}t\gamma, \quad c_{21} + c_{21}t\beta + c_{22}t\delta), \\ (tc_{21}, \quad tc_{22}) & = & (c_{31}t\alpha + c_{32}t\gamma, \quad c_{31} + c_{31}t\beta + c_{32}t\delta), \\ (tc_{31}, \quad tc_{32}) & = & (c_{41}t\alpha + c_{42}t\gamma, \quad c_{41} + c_{41}t\beta + c_{42}t\delta), \\ (c_{41}, \quad c_{42}) & = & (c_{51}t\alpha + c_{52}t\gamma, \quad c_{51} + c_{51}t\beta + c_{52}t\delta). \end{array}\right. \tag{24}$$

From equality (24) we obtain that $\overline{c}_{11} = \overline{c}_{41} = 0 = \overline{c}_{31}$ and

$\overline{c}_{12} = \overline{c}_{21}$ (since $c_{12} = c_{21} + c_{21}t\beta + c_{22}t\delta$),

$\overline{c}_{51} = \overline{c}_{12}\overline{\gamma}$ (since $tc_{51} = c_{11}t\alpha + c_{12}t\gamma$ and $\overline{c}_{11} = 0$),

$\overline{c}_{21} = \overline{c}_{32}\overline{\gamma}$ (since $tc_{21} = c_{31}t\alpha + c_{32}t\gamma$ and $\overline{c}_{31} = 0$)

$\overline{c}_{42} = \overline{c}_{51}$ (since $c_{42} = c_{51} + c_{51}t\beta + c_{52}t\delta$),

$\overline{c}_{42}\overline{\gamma} = 0$ (since $tc_{31} = c_{41}t\alpha + c_{42}t\gamma$ and $\overline{c}_{31} = \overline{c}_{41} = 0$).

If $\overline{\gamma} = 0$ then $\overline{c}_{51} = 0$ and $\overline{c}_{21} = 0$. If $\overline{\gamma} \ne 0$ then $\gamma \in R^*$, $\overline{c}_{42} = 0$ and hence $\overline{c}_{51} = 0$, $\overline{c}_{12} = 0$ and $\overline{c}_{21} = 0$. Therefore, $\det(\overline{C}) = 0$, a contradiction. □

3. Main result

Theorem 2. *Let R be a commutative local ring with radical* $\operatorname{Rad}(R) = tR$, $t \neq 0$, *and let* $s_1, \dots, s_n \in \{0, 1\}$. *If* $0 < n \leq 6$, *then the matrix* $M(t, s_1, \dots, s_n)$ *over R is irreducible if and only if n and* $s = \sum_{i=1}^{n} s_i$ *are coprime.*

Proof. The necessity part follows from Theorem 1. Let now n and $\sum_{i=1}^{n} s_i$ are coprime. Then the matrix $M(t, s_1, \dots, s_n)$ is, up to cyclic permutations of s_i, one of the following:

$$\begin{gathered} M(t,0,1),\ M(t,0,0,1),\ M(t,0,0,0,1), \\ M(t,0,0,0,0,1),\ M(t,0,0,0,0,0,1), \end{gathered} \tag{25}$$

$$\begin{gathered} M(t,0,1,1),\ M(t,0,1,1,1), \\ M(t,0,1,1,1,1),\ M(t,0,1,1,1,1,1), \end{gathered} \tag{26}$$

$$\begin{gathered} M(t,0,0,0,1,1),\ M(t,0,0,1,1,1), \\ M(t,0,0,1,0,1),\ M(t,0,1,1,0,1). \end{gathered} \tag{27}$$

The irreducibility of the matrices (25) are obvios. The matrices (26) are irreducible by [6], and the matrices (27) are irreducible by Lemmas 3–6. $\square$

The last theorem does not hold if $n > 6$. For example, if R is a local ring of length 2 and $\operatorname{Rad}(R) = tR$ ($t \neq 0$, $t^2 = 0$), the matrix $M = M(t,0,0,0,0,1,1,1)$ (with $n = 7$ and $s_1 + \cdots + s_7 = 3$ to be coprime) is reducible over R because, for

$$C = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ t & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & t & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & t & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$C^{-1}MC = \left(\begin{array}{cc|ccccc} 0 & t & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & t \\ 0 & 0 & 1 & 0 & -t & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t & 0 \end{array}\right).$$

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Received by the editors: 20.10.2013
and in final form 20.10.2013.