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On the equivalence of matrices over commutative rings modulo ideals

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Abstract. We study connections between wildness of the problem of classifying the matrices over an integral domain up to equivalence modulo an ideal and properties of the set of prime elements of the domain.

Introduction

The paper is devoted to the problem of classifying up to equivalence the matrices over commutative rings not being principal ideal ones.

Let K be a commutative ring and $J \neq K$ be its ideal. For matrices P and Q over K, the notation $P \equiv Q \pmod J$ means that $P - Q$ is a matrix with entries from J.

Two $m \times n$ matrices A and B over K is said to be *equivalent modulo* J if, for some invertible $m \times m$ matrix Y and invertible $n \times n$ matrix X,

$$B \equiv Y^{-1}AX \pmod J.$$

Analogously, $n \times n$ matrices A and B over K is said to be *similar modulo* J if, for some invertible $n \times n$ matrix X,

$$B \equiv X^{-1}AX \pmod J.$$

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When $J = 0$, we have the classical definitions of equivalent and similar matrices.

The above definitions can be naturally generalized to any numbers of matrices, and in particular (in the case of similarity) for matrix representations of K-algebras.

Definition 1. Let $I, J \neq K$ be ideals of K and let $\Sigma = K\langle x, y\rangle$ denotes the free associative K-algebra generated by x, y. An $m \times n$ matrix M over Σ is said to be (I, J)-*perfect* if from the equivalence of the matrices $M \otimes T$ and $M \otimes T'$ over K modulo I, where T, T' are matrix representations of Σ over K, it follows that T and T' are equivalent modulo J. We say that the problem of classifying the matrices over K up to equivalence modulo I is *wild over K modulo J*, or simply *wild modulo J*, if there is an (I, J)-perfect representation over Σ.

Note that we do not require that K/J is a field, i. e. the notion of wildness is introduced by us not only concerning fields, as before for various classification problems, but also concerning rings (more detail see Section 5).

From now on K denotes, unless otherwise stated, an integral domain. Its non-unit element c is called a *prime element*, or simple a *prime*, if from $c|ab$ for some $a, b \in K$ it follows that $c|a$ or $c|b$. By different prime elements of K we mean pairwise non-associated ones.

Now we formulate the main results of this paper. In all statements, J denotes (as above) an ideal of K different from the ring itself.

Theorem 1. *Let K be an integral domain and J contains two different primes. Then the problem of classifying up to equivalence the matrices over K is wild modulo J.*

Theorem 2. *Let K be an integral domain, $t_1, t_2 \in J$ be different primes and $t \in J$ be non-zero. Then the problem of classifying the matrices over K up to equivalence modulo t_1t_2tK is wild modulo J.*

Theorem 3. *Let K be an integral domain, $t_1 \in J$ be a prime and $t \in J$ be non-zero. If J contains a prime t_2 different from t_1, then the problem of classifying the matrices over K up to equivalence modulo t_1^2tK is wild modulo J.*

From Theorems 2 and 3 it follows the following corollary.

Corollary 1. *Let K be an integral domain, and $t_1, t_2, t_3 \in J$ be primes that are not all equal. Then the problem of classifying the matrices over K up to equivalence modulo $t_1t_2t_3K$ is wild modulo J.*

1. Proof of Theorem 1

We need the following lemma.

Lemma 1. *Let p, q be different prime elements from J and let*

$$p^2a + q^2b + pqc = 0 \tag{1}$$

for some $a, b, c \in K$. Then $a, b, c \in J$.

Proof. From (1) it follows that, firstly,

$$q(qb + pc) = -p^2a$$

and, secondly,

$$p(pa + qc) = -q^2b$$

whence respectively $q|a$ and $p|b$. So $a = qa' \in J$ and $b = pb' \in J$ for some $a', b' \in K$. Then the equality (1) is equivalent to the equality

$$pq(pa' + qb' + c) = 0$$

whence $pa' + qb' + c = 0$ and consequently $c \in J$. □

It is natural to identify a matrix representations T of $\Sigma = K\langle x, y\rangle$ over K with the ordered pair of matrices $T(x), T(y)$; if these matrices are of size $n \times n$, we say that T is of K-dimension n. Then, for a matrix M over $K\langle x, y\rangle$ (see above the definition of wildness), the matrix $M \otimes T$ with T of K-dimension m is obtained from the matrix M by change x and y on the matrices $T(x)$ and $T(y)$, and $a \in K$ on the scalar matrix aE_n, where E_n is the identity $n \times n$ matrix.

Consider the following matrix M (of size 1×1) over Σ:

$$M = p^2x + q^2y + pq.$$

We prove that the matrix M is $(0, J)$-perfect.

Let $T = (A, B)$ and $T' = (A', B')$ be matrix representations of Σ over K of a K-dimension n. Then

$$M \otimes T = p^2A + q^2B + pqE_n,$$

$$M \otimes T' = p^2, A' + q^2B' + pqE_n.$$

Assume that the matrices $M \otimes T$ and $M \otimes T'$ (over K) are equivalent, i. e. there exists invertible $n \times n$ matrices X and Y over K such that $(M \otimes T)X = Y(M \otimes T')$. So we have the equality

$$(p^2 A + q^2 B + pqE_n)X = Y(p^2 A' + q^2 B' + pqE_n)$$

or, equivalently,

$$p^2(AX - YA') + q^2(BX - YB') + pq(X - Y) = 0.$$

By applying Lemma 1 to all scalar equalities of the last matrix equality, we have that

$$\begin{aligned} AX - YA' &\equiv 0 \pmod J, \\ BX - YB' &\equiv 0 \pmod J, \\ X - Y &\equiv 0 \pmod J, \end{aligned}$$

and consequently $AX \equiv XA' \pmod J$ and $BX \equiv XB' \pmod J$, as claimed.

Theorem 1 is proved.

2. Proof of Theorem 2

We need the following lemma.

Lemma 2. *Let $a, b, c \in K$ such that*

$$t_1^2 a + t_2^2 b + t_1 t_2 c \equiv 0 \,(\mathrm{mod}\, t_1 t_2 t K). \tag{2}$$

Then $a, b, c \in J$.

Proof. The comparison (2) means that there exists $u \in K$ such that

$$t_1^2 a + t_2^2 b + t_1 t_2 c = t_1 t_2\, tu. \tag{3}$$

From (3),

$$t_2(t_2 b + t_1 c - t_1 tu) = -t_1^2 a \tag{4}$$

whence $t_2 | a$ and therefore $a \in J$. Let $a = t_2 a'$. Then we have from (4) (after reducing by t_2 and elementary transformations) that

$$t_1(c + t_1 a' - tu) = -t_2 b$$

whence $t_1 | b$ and $t_2 | c + t_1 a' - tu$; consequently $b, c \in J$. □

Consider the following matrix M (of size 1×1) over Σ:

$$M = t_1^2 x + t_2^2 y + t_1 t_2.$$

We prove that the matrix M is $(t_1 t_2 t K, J)$-perfect.

Let $T = (A, B)$ and $T' = (A', B')$ be matrix representations of Σ over K of a K-dimension n. Then

$$M \otimes T = t_1^2 A + t_2^2 B + t_1 t_2 E_n,$$

$$M \otimes T' = t_1^2 A' + t_2^2 B' + t_1 t_2 E_n.$$

Assume that the matrices $M \otimes T$ and $M \otimes T'$ (over K) are equivalent modulo $t_1 t_2 t K$, i. e. there exists invertible $n \times n$ matrices X and Y over K such that $(M \otimes T)X \equiv Y(M \otimes T') \pmod{t_1 t_2 t K}$. So we have the comparison

$$(t_1^2 A + t_2^2 B + t_1 t_2 E_n)X \equiv Y(t_1^2 A' + t_2^2 B' + t_1 t_2 E_n) \pmod{t_1 t_2 t K}$$

or, equivalently,

$$t_1^2(AX - YA') + t_2^2(BX - YB') + t_1 t_2 (X - Y) \equiv 0 \pmod{t_1 t_2 t K}.$$

By applying Lemma 2 to all scalar comparisons of the last matrix one, we see that (as in the proof of Theorem 1) $AX \equiv XA' \pmod{J}$, and $BX \equiv XB' \pmod{J}$, as claimed.

Theorem 2 is proved.

3. Proof of Theorem 3

We need the following lemma.

Lemma 3. *Let $a, b, c \in K$ such that*

$$t_1^2 a + t_2^2 b + t_1 t_2 c \equiv 0 \pmod{t_1^2 t K}. \tag{5}$$

Then $a, b, c \in J$.

Proof. The comparison (5) means that there exists $u \in K$ such that

$$t_1^2 a + t_2^2 b + t_1 t_2 c = t_1^2\, t u. \tag{6}$$

From (6),

$$t_1(t_1 a + t_2 c - t_1 t u) = -t_2^2 b \tag{7}$$

whence $t_1|b$ and therefore $b \in J$. Let $b = t_1 b'$. Then we have from (7) (after reducing by t_1 and elementary transformations) that

$$t_1(a - tu) = -t_2(c + t_2 b')$$

whence $t_2|a - tu$ and $t_1|c + t_2 b'$; consequently $a, c \in J$. $\square$

We take as a $(t_1^2 tK, J)$-perfect matrix M over Σ the matrix of the same form as in the proof of Theorem 2 with t_2 to be any prime element different from t_1 (it exists by the condition of the theorem). Then the proof of Theorem 3 is analogously to that of Theorem 2, but it is need to use Lemma 3 instead of Lemma 2.

4. The case of factorial rings

A factorial ring K is an integral domain in which there exists a system of prime elements P such that every non-zero element $x \in K$ admits a unique representation

$$x = \varepsilon \prod_{p \in P} p^{s_p},$$

where ε is invertible and the integral exponents $s_p \geq 0$ are non-zero for only a finite number of elements. The number $l(x) = \sum_{p \in P} s_p$ is called the length of x.

From the above we have the following theorem.

Theorem 4. *Let K be a factorial ring and $J \neq K$ be its ideal having at least two primes. Then, for any element $v \in J$ of length $l(v) > 2$, the problem of classifying the matrices over K up to equivalence modulo vK is wild modulo J.*

Indeed, if $l(v) > 2$ then $v = t_1 t_2 t$, where t_1, t_2 are prime and t is not invertible. And consequently the statement of the theorem follows from Theorem 2 if $t_1 \neq t_2$ and from Theorem 3 if $t_1 = t_2$.

5. The case of Noether domain

The equivalence of matrices over Noether factorial and local rings were studied in [3]. We formulate here some consequences of our theorem in the case when the ring K is Noether.

Recall that according the main idea of wildness, a classification matrix problem is called wild if it involves the problem of classifying (up to similarity) the pairs of matrices over a field k; see precise definitions in [1, 2] (in particular, for a case with the ring of p-adic numbers [2, pp. 70-71]). The first authors proposed to generalize the traditional definition of wild matrix problems allowing to take k being a commutative ring (it is especially actual for matrix problems over non-Noether rings). An example of the realization of this idea is the definition of wildness modulo J of the problem of classifying the matrices over K up to equivalence modulo I (see Definition 1).

Definition 2. The problem of classifying the matrices over K up to equivalence modulo I is said to be *wild over a field*, or simply *wild*, if it wild modulo a maximal ideal J.

It is easy to see that from Theorems 1–3 and Corollary 1 we have respectively the following statements.

Theorem 5. *Let K be a Noether domain and p, q be its different primes such that $pK + qK \neq K$. Then the problem of classifying up to equivalence the matrices over K is wild.*

Theorem 6. *Let K be a Noether domain and t_1, t_2, t be its non-zero elements such that $t_1K + t_2K + tK \neq K$. If t_1 and t_2 are different primes, then the problem of classifying the matrices over K up to equivalence modulo t_1t_2tK is wild.*

Theorem 7. *Let K be a Noether domain and t_1, t_2, t be its non-zero elements such that $t_1K + t_2K + tK \neq K$. If t_1 and t_2 are different primes, then the problem of classifying the matrices over K up to equivalence modulo t_1^2tK is wild.*

Corollary 2. *Let K be a Noether domain and t_1, t_2, t_3 be its primes that are not all equal, and such that $t_1K + t_2K + t_3K \neq K$. Then the problem of classifying the matrices over K up to equivalence modulo $t_1t_2t_3K$ is wild.*

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