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The endomorphisms monoids of graphs of order n with a minimum degree $n-3$

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Abstract. We characterize the endomorphism monoids, $End(G)$, of the generalized graphs G of order n with a minimum degree $n-3$. Criteria for regularity, orthodoxy and complete regularity of those monoids based on the structure of G are given.

Introduction and preliminaries

Endomorphism monoids of graphs are generalizations of automorphism groups of graphs. In recent years much attention has been paid to endomorphism monoids of graphs and many interesting results concerning graphs and their endomorphism monoids have been obtained. The techniques that are used in those studies connect semigroup theory to graph theory and establish relationships between graphs and semigroups.

We start with a review of results obtained about the regularity of endomorphism monoids of graphs.

A characterization of regular elements in $End(G)$ using endomorphic image and kernel was given by Li, in 1994, [9]. In 1996, the connected bipartite graphs whose endomorphism monoids are regular and orthodox

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were explicitly found by Wilkeit [22] and Fan [2], respectively. In 2003, [10], Li showed that the endomorphism monoids of $\overline{C}_{2n+1}$ $(n > 1)$ are groups, where $\overline{C}_n$ denotes the complement of a cycle C_n. Further, Pipattanajinda at al [17], proved that the endomorphism monoids of $\overline{C}_{2n}$ are completely regular. In 2008, Hou, Luo and Cheng proved that the endomorphism monoids of $\overline{P}_n$ are orthodox, see [5]. Furthermore, the cycle book graphs, generalized wheel graphs, N-prism graphs and splits graphs whose endomorphisms monoids are regular, orthodox and completely regular were studied in [3, 12–14, 16, 19–21].

In this paper we consider *finite simple graphs* G with *vertex set* $V(G)$ and *edge set* $E(G)$, with the number of vertices of G called the *order* of G. The graph with vertex set $\{1, \ldots, n\}$, with $n \geqslant 3$, $[n \geqslant 1]$ and edge set $\{\ \{i, i+1\} \mid i = 1, \ldots, n\} \cup \{1, n\}$, $[\{\ \{i, i+1\} \mid i = 1, \ldots, n\}]$ is called a *cycle* C_n, [a *path* P_n].

The *degree* of a vertex u in a graph G is the number of vertices adjacent to u and is denoted by $d_G(u)$ or simply by $d(u)$ if the graph G is clear from the context. The *minimum degree* of G is the minimum degree among the vertices of G and is denoted by $\delta(G)$. Thus, if G is a graph of order n and u is any vertex of G, then

$$0 \leqslant \delta(G) \leqslant d(u) \leqslant n - 1.$$

If $d(u) = r$ for every vertex u of G, where $0 \leqslant r \leqslant n-1$, then G is called *r-regular* graph. The *complement* (graph) $\overline{G}$ of G is a graph such that $V(\overline{G}) = V(G)$ and $\{u, v\} \in (\overline{G})$ if and only if $\{u, v\} \notin E(G)$ for any $a, b \in V(G), a \neq b$. A subgraph H of G is called an *induced subgraph*, if for any $u, v \in V(H)$, $\{u, v\} \in E(G)$ implies $\{u, v\} \in E(H)$. The induced subgraph H with $V(H) = S$ is also dented by $\langle S \rangle$. A *clique* of graph G is a maximal complete subgraph of G. The *clique number* of G, denoted by $\omega(G)$, is the maximal order among the cliques of G. Let G and H be two graphs. The *join* of G and H, denoted by $G + H$, is a graph such that $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{\{u, v\} | u \in V(G), v \in V(H)\}$.

Further, let $G, (H_x)_{x \in G}$ be graphs with $H_x = (V_x, E_x)$. The *generalized lexicographic product* $(G - join)$ of G with $(H_x)_{x \in G}$ is defined as

$$V(G[(H_x)_{x \in G}]) := \{(x, y_x) \mid x \in V(G), y_x \in V(H_x)\},$$

$$E(G[(H_x)_{x \in G}]) := \{\{(x, y_x), (x', y'_x)\} \mid (x, x') \in E(G)\}$$

$$\bigcup \{\{(x, y_x), (x, y'_x)\} \mid x \in V(G), (y_x, y'_x) \in E(H_x)\}.$$

Let G and H be graphs. A (graph) *homomorphism* from a graph G to a graph H is a mapping $f : V(G) \to V(H)$ which preserves edges, i.e. $\{u,v\} \in E(G)$ implies $\{f(u), f(v)\} \in E(H)$. A homomorphism f is called a (graph) *isomorphism* if f is a bijective and f^{-1} is a homomorphism. We call G *isomorphic* to H and write $G \cong H$, if there exists an isomorphism f from G onto H. A homomorphism from G into itself is called an (graph) *endomorphism* of G. An endomorphism f is said to be *strong* if for all $u, v \in V(G)$, $\{u,v\} \in E(G)$ if and only if $\{f(u), f(v)\} \in E(G)$. An endomorphism f is said to be *path strong* (resp. *cycle strong*) if for every path (or cycle) $f(v_1), f(v_2), \ldots, f(v_l)$ in $f(G)$ there exists a path (or cycle) $u_1, u_2, \ldots, u_l$ in G, where $u_i \in f^{-1}f(v_i)$, for all $i = 1, 2, \ldots, l$ and $f^{-1}(t)$ denotes the set of *preimages* of some vertex t of G under the mapping f. An isomorphism from G into itself is called an *automorphism*. In this paper we use the following notations:

- $End(G)$, the set of all endomorphisms of G,
- $End'(G)$, the set of all non-injective endomorphisms of G,
- $SEnd(G)$, the set of all strong endomorphisms of G, and
- $Aut(G)$, the set of all automorphisms of G.

The factor graph I_f of G under f which is a subgraph of G is called the *endomorphic image* of G under f. More precisely, I_f is a graph with $V(I_f) = f(V(G))$ and $\{f(u), f(v)\} \in E(I_f)$ if and only if there exist $u' \in f^{-1}f(u)$ and $v' \in f^{-1}f(v)$ such that $\{u', v'\} \in E(G)$. By ρ_f, we denote the *equivalence relation* on $V(G)$ induced by f, i. e. for any $u, v \in V(G)$, $(u, v) \in \rho_f$ if and only if $f(u) = f(v)$.

Let S be a semigroup (monoid respectively). An element a of S is called an *idempotent* if $a^2 = a$. An element a of S is called *regular* if $a = aa'a$ for some $a' \in S$, such a' is called a *pseudo inverse* to a. The semigroup S is called *regular* if every element of S is regular. A regular element a of S is called *completely regular* if there exists a pseudo inverse a' to a such that $aa' = a'a$. In this case we call a' a *commuting pseudo inverse* to a. The semigroup S is called *completely regular* if every element of S is completely regular. A regular semigroup S is called *orthodox* if the set of all idempotent elements of S (denoted by $Idpt(S)$) forms a semigroup under the operation of S.

Lemma 1 ([18]). *A semigroup S is completely regular if and only if S is a union of (disjoint) groups.*

A graph G is *endo-regular* (*endo-orthodox*, *endo-completely-regular*) if the monoid of all endomorphisms on G is regular (orthodox, completely regular respectively). Further, a graph G is *unretractive* if $End(G) = Aut(G)$.

The following results are used widely in this work:

Lemma 2 ([15]). *Let G be a graph and $f \in End(G)$. If f is a regular, then f is a path strong. Furthermore if f is a regular, then f is a cycle strong.*

Lemma 3 ([9]). *Let G be a graph and let $f \in End(G)$. Then f is a regular if and only if there exist idempotents $g, h \in End(G)$ such that $\rho_g = \rho_f$ and $I_f = I_h$.*

Lemma 4 ([10]). *Let G_1 and G_2 be graphs. If $G_1 + G_2$ is endo-regular, then G_1 and G_2 are both endo-regular.*

Lemma 5 ([4]). *Let G_1 and G_2 be two graphs. Then $G_1 + G_2$ is endo-orthodox if and only if*

(1) *$G_1 + G_2$ is endo-regular, and*

(2) *both of G_1 and G_2 are endo-orthodox.*

Lemma 6 ([7]). *Let G_1 and G_2 be graphs. The join G_1+G_2 is unretractive if and only if G_1 and G_2 are unretractive.*

Lemma 7 ([10]). *Let G be a graph and let $f \in End(G + s)$. Then there exists $g \in End(G + s)$ such that $g(s) = s$ with $\rho_g = \rho_f$ and $I_g = I_f$.*

Lemma 8 ([10]). *Let G be a graph. Then G is endo-regular if and only if $G + K_n$ is endo-regular for any $n \geqslant 1$, where K_n is the complete graph of n vertices.*

Lemma 9 ([15]). *For all positive integer n, the cycle C_{2n} is not endo-completely-regular.*

Lemma 10 ([10]). *$\overline{C}_{2n+1}$ is unretractive.*

Lemma 11 ([17]). *$\overline{C}_{2n}$ is endo-completely-regular.*

Lemma 12 ([5]). *The only clique of order $n + 1$ of $\overline{P}_{2n+1}$ is isomorphic to K_{n+1}.*

Lemma 13 ([5]). *$\overline{P}_n$ is endo-orthodox.*

Lemma 14 ([6]). *Complete r-partite graphs $\overline{K}_{n_1} + \ldots + \overline{K}_{n_r}$ is endo-regular,for $n_1, \ldots, n_r \in \boldsymbol{N}, r \in \boldsymbol{N}$.*

Lemma 15 ([21]). *Let G be a graph. Suppose $f \in End(G)$ and f is a regular. Then the following two statements are equivalent:*

(1) *f is a completely regular,*

(2) *there exists idempotent $g \in End(G)$ such that $f(G)=g(G)$ and $\rho_f = \rho_g$.*

Lemma 16 ([1]). *Let G be a graph of order $n \geqslant 3$. Then G is $(n-3)$-regular graph if and only if $G = \overset{r}{\underset{i=1}{+}} \overline{C}_{n_i}$ where $n = n_1 + \ldots + n_r$ and $r \geqslant 1$. In particular $r > 1$, implies $n \geqslant 6$.*

Let G be an $(n-3)$-regular graph of order n. We denote by G_x and G_E the sets of all induced subgraphs $\overline{C}_x$ and $\overline{C}_{2m}$ of G, respectively. Note that $G_x = \varnothing$, if G has no induced subgraphs $\overline{C}_x$.

Lemma 17 ([17]). *Let G be an $(n-3)$-regular graph of order n.*

(1) *G is endo-regular.*

(2) *G is endo-completely-regular if and only if $|G_3| = 0$ and $|G_{2m}| = 1$ for all induced subgraph $\overline{C}_{2m}$ of G.*

(3) *G is endo-orthodox and also unretractive if and only if $|G_3| = 0$ and $|G_{2m}| = 0$ for all induced subgraph $\overline{C}_{2m}$ of G, i.e. $G = \overset{r}{\underset{i=1}{+}} \overline{C}_{n_i}$ where $n_i \geqslant 5$, n_i is odd for all $i = 1, \ldots, r$.*

Observe that $\overline{C}_n$ and $\overline{P}_n$ are graphs with $d(u) \geqslant n-3$ for all $u \in V(\overline{C}_n), V(\overline{P}_n)$. From Lemmas 11 and 13, $\overline{C}_n$ is endo-completely-regular and $\overline{P}_n$ is endo-orthodox.

Next, we characterize the join of complements of paths and cycles - graphs of order n with minimum degree is $n-3$ that are endo-regular, endo-orthodox, endo-completely-regular. In section 1 we cover the cases $\delta(G) = n-1$ and $\delta(G) = n-2$ and in section 2 – the case $\delta(G) = n-3$.

1. Endo-regularity of graphs G of order n with $\delta(G) = n-2$

Clearly, the case $\delta(G) = n-1$ corresponds to the complete graph K_n, which is unretractive, so let us focus on graphs G of order n with $\delta(G) = n-2$. Such graphs have the following structure:

Lemma 18. *Let G be an $(n-2)$-regular graph of order n. Then G is the complete r-partite graph $\overline{K}_2 + \ldots + \overline{K}_2$, where $n = 2r$.*

Proof. Let $G \neq \overline{K}_2 + \ldots + \overline{K}_2$, where $n = 2r$. Then $\overline{G} \neq K_2 \cup \ldots \cup K_2$. Thus there exists $u \in V(\overline{G})$ such that $d_{\overline{G}}(u) > 1$. Since $n - 1 = d_G(u) + d_{\overline{G}}(u) > d_G(u) + 1$, we get $d_G(u) < n - 2$, a contradiction with the assumption on G. □

Next, it is easy to get that

Proposition 1. *Let G be an $(n-2)$-regular graph of order n. Then*

(1) $G = K_r[\overline{K}_2, \ldots, \overline{K}_2]$, *where* $n = 2r$,

(2) $End(G) = SEnd(G)$,

(3) $|End(G)| = 4^r r!$.

The proposition above follows directly from the following result from [8]:

Lemma 19 ([8])**.** (1) $SEnd(G[(H_x)_{x \in G}])$ *is regular.*

(2) $SEnd(G[(H_x)_{x \in G}])$ *is orthodox if and only if* $|V(H_x)| \leqslant 2$ *for all* $x \in G$.

From Proposition 1(1)–(2) and Lemma 19(1)–(2), it is easy to see also that,

Corollary 1. *A $(n-2)$-regular graph of order n is endo-regular. In particular, $(n-2)$-regular graph of order n is endo-orthodox.*

For completeness of the discussion let us show the condition for endo-complete-regularity of the $(n-2)$-regular graph of order n.

Proposition 2. *Let G be an $(n-2)$-regular graph of order n. Then G is endo-completely-regular if and only if $n = 2$.*

Proof. Necessity. From $\overline{K}_2 + \overline{K}_2 = C_4$. If $n > 2$, then we can rewrite $G = C_4 + \overline{K}_2 + \ldots + \overline{K}_2$, where $\overline{K}_2 + \ldots + \overline{K}_2$ is a complete $(n-2)$-partite graph. From Lemma 9, C_4 is not endo-completely-regular. Therefore, G is not endo-completely-regular.

Sufficiency. Straightforward. □

Next, we show that if the endomorphism monoid of the join of graphs is completely regular, then each endomorphism of the graphs is completely regular.

Proposition 3. *Let G and H be graphs. If $G + H$ is endo-completely-regular, then G and H are both endo-completely-regular.*

Proof. If G is not endo-completely-regular, then there exist $f \in End(G)$ such that f is not completely regular. Let $g : V(G+H) \to V(G+H)$ be define by $g|_G = f$ and $g|_H$ is identity map. Then $g \in End(G+H)$ and easy to see that g is not completely regular. Therefore, $G+H$ is not endo-completely-regular. □

Moreover, it is easy to show that for any graph G which is endo-orthodox (endo-completely-regular), the graph $G + K_n$ is endo-orthodox (endo-completely-regular).

Proposition 4. *Let G be a graph and $n \geqslant 1$. Then the following statements are true.*

(1) *G is endo-orthodox if and only if $G + K_n$ is endo-orthodox, and*

(2) *G is endo-completely-regular if and only if $G+K_n$ is endo-completely-regular.*

Proof. (1) Follows directly from Lemma 8 and Lemma 5.

(2) Necessity. Since $G + K_n = G + K_1 + \ldots + K_1$, we only need to prove that G is endo-completely-regular if and only if $G + s$ is end-completely-regular, where s is a singleton, that is $s = K_1$. Let G be endo-completely-regular. Consider $G + s$ and let $f \in End(G+s)$. From Lemma 7 there exists $g \in End(G+s)$ such that $g(s) = s$ with $\rho_g = \rho_f$ and $g(G+s) = f(G+s)$. Then $g(G) \subseteq V(G)$, implies $g|_G$, the restriction of g on G, is completely regular endomorphism of G. From Lemma 15, there exists idempotent $h \in End(G)$ such that $h(G) = g|_G(G)$ and $\rho_h = \rho_{g|_G}$. Thus $h' : V(G+s) \to V(G+s)$ define by $h'|_G = h$ and $h'(s) = s$ is idempotent endomorphism of $G+s$, and $h(G+s) = g(G+s)$ and $\rho_h = \rho_g$. So, $\rho_h = \rho_f$ and $h(G+s) = f(G+s)$, so f is completely regular. Therefore, $G + s$ is endo-completely-regular.

Sufficiency. Obvious from Proposition 3. □

Finally, let G be a graph of order n for which $\delta(G) = n-2$ and $d(u) = n-1$, for some $u \in V(G)$. Thus, the graph G is the join of the complete graph K_k and the complete r-partite graph $\overline{K}_2 + \ldots + \overline{K}_2$, where $n = k + 2r$. Using Lemma 1, Proposition 2 and Proposition 4, we get:

Proposition 5. *Let G be a graph of order n which $\delta(G) = n-2$. Then*

(1) *G is endo-orthodox;*

(2) *G is endo-completely-regular if and only if*
$|\{u \in V(G) \mid d(u) = n-2\}| = 2$, *i.e.* $G = K_k + \overline{K}_2$ *when* $n = k+2$.

2. Endo-regularity of graphs G of order n with $\delta(G) = n-3$

In this section we consider graphs G of order n with $\delta(G) = n-3$ and $d(u) > n-3$, for some $u \in V(G)$. Observe that $\overline{C}_{2n+1}$ is unretractive with $\omega(\overline{C}_{2n+1}) = n$, a fact that we use in the next Lemma.

Lemma 20. *Let G be a graph, $n \geqslant 2$ and $f \in Hom(\overline{C}_{2n+1}, G)$. If f is not an injective map, then $\omega(\overline{C}_{2n+1}) < \omega(G)$.*

Proof. Let $f \in Hom(\overline{C}_{2n+1}, G)$ be a non-injective homomorphism. Assume that $\omega(\overline{C}_{2n+1}) \geqslant \omega(G)$. Without loss of generality, suppose that $f(1) = f(2)$. Now $\langle\{1, 3, \ldots, 2n-1\}\rangle \cong \langle\{2, 4, \ldots, 2n\}\rangle \cong K_n$.
From $\langle\{f(1) = f(2), f(3), f(4), \ldots, f(2n-1), f(2n)\}\rangle \ncong K_{n+r}$, for all $r > 1$ and $\{x, y\} \in E(\overline{C}_{2n+1}) \Leftrightarrow y \neq x \pm 1$, implies that $f(3) = f(4), f(5) = f(6), \ldots, f(2n-1) = f(2n)$. Thus $\langle\{f(1), f(3), f(5), \ldots, f(2n-1), f(2n+1)\}\rangle \cong K_{n+1}$, a contradiction. Therefore, $\omega(\overline{C}_{2n+1}) < \omega(G)$. □

Corollary 2. *Let G be a graph, $n \geqslant 2$ and $f \in End(G + \overline{C}_{2n+1})$. If $f(\overline{C}_{2n+1}) = X \subseteq G$ is non-injective, then $\omega(\overline{C}_{2n+1}) < \omega(\langle X\rangle)$.*

Lemma 21. *For any integer n which $n \geqslant 2$ and the edge $e \in E(C_{2n+1})$, the only clique of order $n+1$ of $\overline{C}_{2n+1} + \{e\}$ is isomorphic to K_{n+1}.*

Proof. Since $\overline{C}_{2n+1} + \{e\} \cong \overline{P}_{2n+1}$, by Lemma 12, the only clique of order $n+1$ of $\overline{C}_{2n+1} + \{e\}$ is isomorphic to K_{n+1}. □

Lemma 22. *Let G be a graph, $n \geqslant 2$ and $f \in End'(G + \overline{C}_{2n+1})$. If $f(\overline{C}_{2n+1}) = X$, then $\langle X\rangle \cong \overline{C}_{2n+1}$.*

Proof. Suppose that the clique of G is isomorphic to K_s. Let $f(\overline{C}_{2n+1}) = X \neq \overline{C}_{2n+1}$. Assume that $\langle X\rangle \ncong \overline{C}_{2n+1}$.

1) If $f(\overline{C}_{2n+1}) \subseteq G$, then by Corollary 2 and Lemma 21, the clique of $\langle X\rangle$ is isomorphic to K_m and $n < m$.
2) If $f(\overline{C}_{2n+1}) \nsubseteq G$, then there exists $x \in V(\overline{C}_{2n+1})$ such that $f(x) \in V(\overline{C}_{2n+1})$, but $f(x+1) \in V(G)$. Thus $\{f(x), f(x+1)\} \in E(G + \overline{C}_{2n+1})$. So the clique of $\langle X\rangle$ is isomorphic to K_m and $n < m$.

From the above two observations, the cliques of $G \setminus \overline{C}_{2n+1}$ and $G \setminus X$ are isomorphic to K_{s-n} and K_{s-m}, respectively. But since $s - n > s - m$, it is impossible that $f|_{G\setminus\overline{C}_{2n+1}}$ is a homomorphism from $G \setminus \overline{C}_{2n+1}$ to $G \setminus X$. □

Lemma 23. *If G is a graph of order n with $\delta(G) = n - 3$, then G is a join of a complete graph, complements of cycles and complements of paths.*

Proof. If G is not as described, that is $G \ncong K_k + \overline{C}_{i_1} + \ldots + \overline{C}_{i_r} + \overline{P}_{j_1} + \ldots + \overline{P}_{j_s}$, where $k + i_1 + \ldots + i_r + j_1 + \ldots + j_s = n$, then $\overline{G} \ncong \overline{K}_k \cup C_{i_1} \cup \ldots \cup C_{i_r} \cup P_{j_1} \cup \ldots \cup P_{j_s}$. Thus there exists $u \in V(\overline{G})$ such that $d_{\overline{G}}(u) \geqslant 3$. Since $d_G(u) + d_{\overline{G}}(u) = n - 1$, $d_G(u) \leqslant n - 4$, which is a contradiction. □

In [10] (see Lemma 4 in this work), Li has shown that the regularity of the endomorphism monoid of the join of graphs implies that the endomorphism monoid of each of the graphs is regular. We use Lemma 22 and Lemma 23, to show that the converse of Lemma 4 is true in the case of a join of a graph G of order n with $\delta(G) = n - 3$, joined with the complement of an odd cycle, $\overline{C}_{2m+1}$, $m \geqslant 2$.

Proposition 6. *Let G be a graph of order n with $\delta(G) = n - 3$, and let $m \geqslant 2$. Then $G + \overline{C}_{2m+1}$ is endo-regular if and only if G is endo-regular.*

Proof. Necessity. Directly from Lemma 4, since $\overline{C}_{2m+1}, m \geqslant 2$ is unretractive.

Sufficiency. Let G be endo-regular and $f \in End(G + \overline{C}_{2m+1})$, where $m \geqslant 2$. Using Lemma 23, suppose that $G = G' + \overline{C}_{odd}$, where $\overline{C}_{odd}$ denotes the join of all complements of odd cycles which length is more than 3. Then $G + \overline{C}_{2m+1} = G' + \overline{C}_{odd} + \overline{C}_{2m+1}$, where $\overline{C}_{odd} + \overline{C}_{2m+1}$ is unretractive. From Lemma 22, we have $f(G + \overline{C}_{2m+1}) = f(G') \cup f(\overline{C}_{odd} + \overline{C}_{2m+1})$, when $f(G') \cap f(\overline{C}_{odd} + \overline{C}_{2m+1}) = \varnothing$. From Lemma 4, G' is endo-regular. Thus $f = f|_{G'} \cup f|_{\overline{C}_{odd} + \overline{C}_{2m+1}}$ is regular. Therefore, $G + \overline{C}_{2m+1}$ is endo-regular. □

Similarly to Proposition 6, we get the next result.

Corollary 3. *Let G be a graph of order n with $\delta(G) = n - 3$, and let $m \geqslant 2$. Then $G + \overline{C}_{2m+1}$ is endo-orthodox (endo-completely-regular) if and only if G is endo-orthodox (endo-completely-regular).*

Take any two graphs G_1 and G_2, where G_1 is a subgraph of G_2 ($G_1 \subseteq G_2$). Recall that G_1 is an *induced subgraph* of G_2, if for any $u, v \in V(G_1), \{u, v\} \in E(G_2)$ implies $\{u, v\} \in E(G_1)$. We write $G_1 < G_2$, in this case.

Now, let G be a graph of order n with $\delta(G) = n - 3$. From Lemma 8 and Proposition 6, G is endo-regular if and only if $G + K_k$ and $G + \overline{C}_{2m+1}$

are endo-regular for any $k \geqslant 1$ and $m \geqslant 2$. Then we consider only the graph G of order n with $\delta(G) = n-3$ such that $K_k \not< G$ and $\overline{C}_{2m+1} \not< G$, where $k \geqslant 1$ and $m \geqslant 2$. So, $G = \overline{C}_{2i_1} + \ldots + \overline{C}_{2i_r} + \overline{P}_{j_1} + \ldots + \overline{P}_{j_s}$, where $j_1, \ldots, j_s > 1$ and $2i_1 + \ldots + 2i_r + j_1 + \ldots + j_s = n$.

Lemma 24. *If G contains one of following induced subgraphs:*

(1) $\overline{P}_{2n} + \overline{P}_m$ *(except $K_{2,2}$),*

(2) $\overline{P}_m + \overline{C}_3$ *(except $K_{2,3}$),*

(3) $\overline{P}_{2n} + \overline{C}_{2m}$,

then G is not endo-regular.

Proof. First, we show that each of the graphs $\overline{P}_{2n} + \overline{P}_m$ (except $K_{2,2}$), $\overline{C}_3 + \overline{P}_m$ (except $K_{3,2}$) and $\overline{P}_{2n} + \overline{C}_{2m}$ are not endo-regular. To do that, let us recall (see Lemma 2), that every regular endomorphism is a path strong endomorphism. Thus, it suffices to show that for for each of the listed graphs there exist an endomorphism which is not a path strong.

(1) $\overline{P}_{2n} + \overline{P}_m$ (except $K_{2,2}$) is not endo-regular.

There are two possible cases:

(1.1) Let $\overline{P}_{2n} + \overline{P}_m = \overline{P}_{(2n)_1} + \overline{P}_{2_2}$ $(2n \geqslant 4)$ and let $f \in End(\overline{P}_{(2n)_1} + \overline{P}_{2_2})$ be defined by

$$f = \begin{pmatrix} 1_1 & 2_1 & 3_1 & 4_1 & \ldots & (2n-1)_1 & (2n)_1 \\ 1_2 & 2_2 & 3_1 & 4_1 & \ldots & (2n-1)_1 & (2n)_1 \end{pmatrix} \begin{pmatrix} 1_2 & 2_2 \\ 1_1 & 1_1 \end{pmatrix}.$$

Then $f^{-1}(2_2) = \{2_1\}$, $f^{-1}(3_1) = \{3_1\}$ and $\{2_2, 3_1\} \in E(\overline{P}_{(2n)_1} + \overline{P}_{m_2})$. But $\{2_1, 3_1\} \notin E(\overline{P}_{(2n)_1} + \overline{P}_{m_2})$, that is f is not a path strong.

(1.2) Let $\overline{P}_{2n} + \overline{P}_m = \overline{P}_{(2n)_1} + \overline{P}_{m_2}$ $(m_2 \geqslant 3)$ and let $f \in End(\overline{P}_{(2n)_1} + \overline{P}_{m_2})$ be defined by

$$f = \begin{pmatrix} 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & \ldots & (2n-1)_1 & (2n)_1 \\ 1_2 & 1_2 & 4_1 & 4_1 & 6_1 & 6_1 & \ldots & (2n)_1 & (2n)_1 \end{pmatrix}$$
$$\begin{pmatrix} 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 & \ldots & (m-1)_2 & m_2 \\ 1_1 & 2_1 & 3_2 & 4_2 & 5_2 & 6_2 & \ldots & (m-1)_2 & m_2 \end{pmatrix}.$$

Then $f^{-1}(2_1) = \{2_2\}$, $f^{-1}(3_2) = \{3_2\}$ and $\{2_1, 3_2\} \in E(\overline{P}_{(2n)_1} + \overline{P}_{m_2})$. But $\{2_2, 3_2\} \notin E(\overline{P}_{(2n)_1} + \overline{P}_{m_2})$, that is f is not a path strong.

(2) $\overline{P}_{(2n)_1} + \overline{C}_{3_2}$ $(2n \geqslant 4)$ is not endo-regular.

Consider the endomorphism $f \in End(\overline{P}_{(2n)_1} + \overline{C}_{3_2})$ defined by

$$f = \begin{pmatrix} 1_1 & 2_1 & 3_1 & 4_1 & \dots & (2n-1)_1 & (2n)_1 \\ 1_2 & 2_2 & 3_1 & 4_1 & \dots & (2n-1)_1 & (2n)_1 \end{pmatrix} \begin{pmatrix} 1_2 & 2_2 & 3_2 \\ 1_1 & 1_1 & 1_1 \end{pmatrix}.$$

Similarly to case (1), f is not a path strong.

(3) $\overline{P}_{(2n)_1} + \overline{C}_{(2m)_2}$ is not endo-regular.

Consider the map $f \in End(\overline{P}_{(2n)_1} + \overline{C}_{(2m)_2})$ defined by

$$f = \begin{pmatrix} 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & \dots & (2n-1)_1 & (2n)_1 \\ 1_2 & 1_2 & 4_1 & 4_1 & 6_1 & 6_1 & \dots & (2n)_1 & (2n)_1 \end{pmatrix}$$
$$\begin{pmatrix} 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 & \dots & (2m-1)_2 & (2m)_2 \\ 1_1 & 2_1 & 3_2 & 3_2 & 5_2 & 5_2 & \dots & (2m-1)_2 & (2m-1)_2 \end{pmatrix}.$$

Clearly, $f^{-1}(1_1) = \{1_2\}$, $f^{-1}(2_1) = \{2_2\}$, so $(2_1, 3_2, 5_2, \dots, (2m-1)_2, 1_1)$ is a paths of length $m+1$ in $f(\overline{P}_{(2n)_1} + \overline{C}_{(2m)_2})$. Assume that f is a path strong. Then there exists a path $(2_2, x^{(3)}, x^{(5)}, \dots, x^{(2m-1)}, 1_2)$ in $\overline{P}_{(2n)_1} + \overline{C}_{(2m)_2}$, where $x^{(y)} \in f^{-1}(y_2)$.

(1) From $\{2_2, x^{(3)}\} \in E(\overline{P}_{(2n)_1} + \overline{C}_{(2m)_2})$ and $f^{-1}(3_2) = \{3_2, 4_2\}$ it follows that $x^{(3)} = 4_2$.

(2) From $\{4_2, x^{(5)}\} \in E(\overline{P}_{(2n)_1} + \overline{C}_{(2m)_2})$ and $f^{-1}(5_2) = \{5_2, 6_2\}$ it follows that $x^{(5)} = 6_2$.

$\vdots$

(m-1) From $\{(2m-2)_2, x^{(2m-1)}\} \in E(\overline{P}_{(2n)_1} + \overline{C}_{(2m)_2})$ and $f^{-1}((2m-1)_2) = \{(2m-1)_2, (2m)_2\}$ it follows that $x^{(2m-1)} = (2m)_2$.

(m) But $\{(2m)_2, 1_2\} \notin E(\overline{P}_{(2n)_1} + \overline{C}_{(2m)_2})$. A contradiction with the properties of path strong maps.

Thus f is not a path strong. □

In [6] (see Lemma 14), Knauer has shown that a complete r-partite graph $\overline{K}_{n_1} + \ldots + \overline{K}_{n_r}$ is endo-regular. We use Lemma 24 and Lemma 14, to show conditions for the graph G, such that $G + \overline{P}_2$ and $G + \overline{C}_3$ are endo-regular.

Corollary 4. *$G + \overline{P}_2$ is endo-regular if and only if $G = \overline{C}_{3_1} + \ldots + \overline{C}_{3_r} + \overline{P}_{2_1} + \ldots + \overline{P}_{2_k}$, where $r \in \mathbb{Z}^+$ and $k \in \mathbb{Z}^+ \cup \{0\}$.*

Proof. Necessity. Let $G \neq \overline{C}_{3_1} + \ldots + \overline{C}_{3_r} + \overline{P}_{2_1} + \ldots + \overline{P}_{2_k}$. Then either $\overline{P}_m < G$ for some $m \geqslant 3$, or $\overline{C}_{2m} < G$ for some $m \geqslant 2$. Thus either $\overline{P}_m + \overline{P}_2 < G + \overline{P}_2$ for some $m \geqslant 3$, or $\overline{C}_{2m} + \overline{P}_2 < G + \overline{P}_2$ for some $m \geqslant 2$. Therefore, $G + \overline{P}_2$ is not endo-regular, by Lemma 24.

Sufficiency. By assumption, $G + \overline{P}_2$ is the complete $(r + k + 1)$-partite graph $\overline{C}_{3_1} + \ldots + \overline{C}_{3_r} + \overline{P}_{2_1} + \ldots + \overline{P}_{2_k} + \overline{P}_2$. Then $G + \overline{P}_2$ is endo-regular, by Lemma 14. □

Corollary 5. *$G + \overline{C}_3$ is endo-regular if and only if G is either $\overline{C}_{3_1} + \ldots + \overline{C}_{3_r} + \overline{P}_{2_1} + \ldots + \overline{P}_{2_k}$ or $\overline{C}_{3_1} + \ldots + \overline{C}_{3_r} + \overline{C}_{2m_1} + \ldots + \overline{C}_{2m_k}$, where $r \in \mathbb{Z}^+$ and $k \in \mathbb{Z}^+ \cup \{0\}$.*

Proof. Necessity. If $\overline{P}_m < G$ $(m > 2)$, then $\overline{P}_m + \overline{C}_3 < G + \overline{C}_3$. Thus $G + \overline{C}_3$ is not endo-regular, by Lemma 24. Let G contains both induce subgraphs $\overline{P}_2$ and $\overline{C}_{2m}$ i.e. $\overline{P}_2 + \overline{C}_{2m} < G$. Then $\overline{P}_2 + \overline{C}_{2m} < G + \overline{C}_3$. It is not endo-regular, by Lemma 24.

Sufficiency. If G is $\overline{C}_{3_1} + \ldots + \overline{C}_{3_r} + \overline{P}_{2_1} + \ldots + \overline{P}_{2_k}$, or $\overline{C}_{3_1} + \ldots + \overline{C}_{3_r} + \overline{C}_{2m_1} + \ldots + \overline{C}_{2m_k}$, then $G + \overline{C}_3$ is endo-regular, by Lemma 14 and Theorem 17(1). □

2.1. $+\overline{P}_{2n_k+1}$ is endo-regular

Let us denote by $+\overline{P}_{2n_k+1}$, the graph which consists of k joins of complements of odd paths with length $\geqslant 3$, that is, $+\overline{P}_{2n_k+1} = \overline{P}_{(2n_1+1)_1} + \overline{P}_{(2n_2+1)_2} + \ldots + \overline{P}_{(2n_k+1)_k}$, for some $n_i \in \mathbb{Z}^+, i = 1, \ldots, k$. Note that, $+\overline{P}_{2n_k+1} = \varnothing$, if $k = 0$.

Now we introduce series of set notations that we use in the proofs in this section.

Let $B^{(n_i)} = \{1_i, 3_i, \ldots, (2n_i + 1)_i\}$ be the set of all odd vertices of $\overline{P}_{(2n_i+1)_i}$ and let $B^{(n_i)}_{0_i,0_i} = \{2_i, 4_i, \ldots, (2n_i)_i\}$ be the set of all even vertices of $\overline{P}_{(2n_i+1)_i}$. It is not hard to see that $V(\overline{P}_{(2n_i+1)_i}) = B^{(n_i)} \cup B^{(n_i)}_{0_i,0_i}$, that the induced subgraph $\langle B^{(n_i)} \rangle$ of $\overline{P}_{(2n_i+1)_i}$ is isomorphic to the complete graph K_{n_i+1} and that the induced subgraph $\langle B^{(n_i)}_{0_i,0_i} \rangle$ of $\overline{P}_{(2n_i+1)_i}$ is isomorphic to the complete graph K_{n_i}.

We construct a family of vertex sets, generally denoted by $B^{(n_i)}_{r_i,s_i}$, where $r, s \in \{0, 1, \ldots, n\}$ in the following fashion:

Given the set $B^{(n_i)}_{0_i,0_i}$ – the set of all even vertices of $\overline{P}_{(2n_i+1)_i}$, let us consider the elements of $B^{(n_i)}_{0_i,0_i}$ ordered by the natural order. For $1 \leqslant r \leqslant n$ and $0 \leqslant s \leqslant n$, let the set $B^{(n_i)}_{r_i,s_i}$ equal to the set $B^{(n_i)}_{0_i,0_i}$ with the first

r and the last s elements on $B^{(n_i)}_{0_i,0_i}$ replaced by their odd predecessors. That is, for $x = 1, \ldots, r$ we replace $(2x)_i \in B^{(n_i)}_{0_i,0_i}$ by $(2x-1)_i$, and for $y = s, \ldots, n$ we replace $(2y)_i \in B^{(n_i)}_{0_i,0_i}$ by $(2y+1)_i$. Thus, we have:

$$\begin{aligned} B^{(n_i)}_{r_i,s_i} &= \{1_i, 3_i, 5_i, \ldots, (2r-1)_i, (2r+2)_i, (2r+4)_i, \ldots, (2(s-2))_i, \\ &\quad (2(s-1))_i, (2s+1)_i, (2s+3)_i, (2s+5)_i, \ldots, (2n_i+1)_i\}. \end{aligned}$$

Finally, let $X^{(n_k)} = B^{(n_1)} \cup \ldots \cup B^{(n_k)}$ and $X^{(n_k)}_{(r_1,s_1)\cdots(r_k,s_k)} = B^{(n_1)}_{(r_1,s_1)} \cup \ldots \cup B^{(n_k)}_{(r_k,s_k)}$. Then $V(+\overline{P}_{2n_k+1}) = X^{(n_k)} \cup X^{(n_k)}_{(0_1,0_1)\cdots(0_k,0_k)}$, the induced subgraph $\langle X^{(n_k)}\rangle$ of $+\overline{P}_{2n_k+1}$, is isomorphic to $K_{n_1+1} + \ldots + K_{n_k+1}$ and the induced subgraph $\langle X^{(n_k)}_{(r_1,s_1)\cdots(r_k,s_k)}\rangle$ of $+\overline{P}_{2n_i+1}$ is isomorphic to $K_{n_1} + \ldots + K_{n_k}$.

Corollary 6. *There exists a homomorphism $\overline{P}_{2m+1} \to \overline{P}_{2n+1}$ if and only if $m \leqslant n$.*

Lemma 25. *The following statements are true:*

(1) *There exists a unique maximal complete subgraph $K_{n_1+1}+\ldots+K_{n_k+1}$ of $+\overline{P}_{2n_k+1}$ of order $\sum_{i=1}^{k} n_i+k$, where $V(K_{n_1+1}+\ldots+K_{n_k+1})=X^{(n_k)}$.*

(2) *There exist exactly $\prod_{i=1}^{k}(3n_i+1)$ complete subgraphs $K_{n_1} + \ldots + K_{n_k}$ of $+\overline{P}_{2n_k+1}$, where $V(K_{n_1} + \ldots + K_{n_k}) = X^{(n_k)}_{(r_1,s_1)\cdots(r_k,s_k)}$ for some $0_i \leqslant r_i \leqslant n_i$ and $0_i \leqslant s_i \leqslant n_i, i = 1, \ldots, k$ such that $r_i + s_i \leqslant n_i$.*

Now, for each graph $+\overline{P}_{2n_k+1}$ consider a map $f \in End'(+\overline{P}_{2n_k+1})$ such that $f(X^{(n_k)}) = X^{(n_k)}$, and $f(X^{(n_k)}_{(0_1,0_1)\cdots(0_k,0_k)}) = X^{(n_k)}_{(r_1,s_1)\cdots(r_k,s_k)}$, where $r_i, s_i \in \{0_i, 1_i, \ldots, n_k\}, r_i + s_i \leqslant n_i$, for all $i = 1, \ldots, k$.

Lemma 26. *Let $x, y \in V(+\overline{P}_{2n_k+1})$. If $f(x) = f(y)$, then $x, y \in V(\overline{P}_{(2n_i+1)_i})$, for some $i \in \{1, \ldots, k\}$ and $x = y-1$ or $x = y+1$.*

Next, we construct an additional family of vertex sets needed for the discussion going further.

Let r_i, s_i be positive integers where $0 \leqslant r_i \leqslant n_i$ and $0 \leqslant s_i \leqslant n_i$ such that $r_i + s_i \leqslant n_i$. We denote by $B^{(n_i)}_{odd}, B^{(n_i)}_{even}$ and $B^{(n_i)}_{both}$ subsets of $B^{(n_i)}_{r_i,s_i}$ as follows:

$$\begin{aligned} B^{(n_i)}_{odd} &= \{2x+1 \in V(\overline{P}_{(2n_i+1)_i}) \mid 2x+1 \in B^{(n_i)} \cap B^{(n_i)}_{r_i,s_i}\}, \\ B^{(n_i)}_{even} &= \{2x \in V(\overline{P}_{(2n_i+1)_i}) \mid 2x \in B^{(n_i)} \cap B^{(n_i)}_{r_i,s_i}\} \\ B^{(n_i)}_{both} &= (B^{(n_i)} \cup B^{(n_i)}_{r,s}) \setminus B^{(n_i)}_{odd}. \end{aligned}$$

Note from the notation of $B^{(n_i)}_{both}$, that its elements are $B^{(n_i)}_{both} = \{x, x+1, \ldots, x+t-1, x+t\}$, where x and $x+t$ denotes the end vertices of $B^{(n_i)}_{both}$. In particular x and $x+t$ are odd. We use these notations and the observations that followed to prove the next result.

Lemma 27. *If $x, y \in B^{(n_i)}_{both}$, for some $i \in \{1, \ldots, k\}$, then $x, y \in f(\overline{P}_{(2n_j+1)_j})$, for some $j \in \{1, \ldots, k\}$.*

Proof. Assume that there exist two elements in $B^{(n_i)}_{both}$ such that one element belongs to $f(\overline{P}_{(2n_j+1)_j})$ and the other belongs to $f(\overline{P}_{(2n_s+1)_s})$, where $j \neq s$. Then, there exist $x, x+1$ such that $x \in f(\overline{P}_{(2n_t+1)_t})$, and $x+1 \in f(\overline{P}_{(2n_{t'}+1)_{t'}})$, where $t \neq t'$. But the elements of $\overline{P}_{(2n_t+1)_t}$ and $\overline{P}_{(2n_{t'}+1)_{t'}}$ are adjacent, a contradiction to x and $x+1$ being not adjacent. $\square$

Proposition 7. *For each $\overline{P}_{(2n_i+1)_i}$, $f(\overline{P}_{(2n_i+1)_i})$ contains only one $B^{(n_j)}_{both}$, for some $j \in \{1, \ldots, k\}$.*

Proof. First, let $\overline{P}_{(2n_i+1)_i} < +\overline{P}_{2n_k+1}$ such that $f(\overline{P}_{(2n_i+1)_i})$ does not contain $B^{(n_i)}_{both}$, where $i \in \{1, \ldots, k\}$. Then $|f^{-1}f(x_i)| = 2$, for all $x_i \in V(\overline{P}_{(2n_i+1)_i})$. This contradicts the fact that $V(\overline{P}_{(2n_i+1)_i})$ contains only odd elements.

Next, suppose that $f(\overline{P}_{(2n_i+1)_i})$ contains $B^{(n_j)}_{both}$ and $B^{(n_{j'})}_{both}$, for some $j, j' \in \{1, \ldots, k\}$ with $j \neq j'$. From Lemma 27, there exist $\overline{P}_{(2n_t+1)_t}$ such that $f(\overline{P}_{(2n_t+1)_t})$ does not contain any $B^{(n_{t'})}_{both}$, a contradiction to the first step. $\square$

Lemma 28. *If $x, x+t$ are the end vertices of $B^{(n_i)}_{both}$, for some $i \in \{1, \ldots, k\}$, then $f(y+1) = x+1, \ldots, f(y+t) = x+t$, or the mapping is in the reversed order. In particular y and $y+t$ are odd.*

Proof. Let $f(y+1) = x+1$ and $f(y+t) = x+t$. Then, there exist $x+s$ and $x+s+1$ such that $f(y+s) = x+s$, but $f(y+s+1) \neq x+s+1$. Suppose that $f(y+s') = x+s+1$, where $s' \neq s+1$, then we get a contradiction to $x+s$ and $x+s+1$ being not adjacent, but $y+s$ and $y+s'$ are.

Observe also that, since x and $x+t$ are odd, y and $y+t$ are also odd. $\square$

From Lemma 28, we get directly

Corollary 7. *Let $f(x_i), f((x+t)_i)$ be the end vertices of $B^{(n_j)}_{both}$, for some $i, j \in \{1, \ldots, k\}$. Then*

(1) $f(1_i) = f(2_i), \ldots, f((x-2)_i) = f((x-1)_i)$, *and*

(2) $f((x+t+1)_i) = f((x+t+2)_i), \ldots, f((2n_i)_i) = f((2n_i+1)_i)$.

Next, recall that for every endomorphism f we have the induced equivalence relation ϱ_f defined as $u\varrho_f v$ if and only if $f(u) = f(v)$. Let us provide an example for such relation.

Example 1. Let $f : V(\overline{P}_{7_1} + \overline{P}_{5_2} + \overline{P}_{5_3}) \to V(\overline{P}_{7_1} + \overline{P}_{5_2} + \overline{P}_{5_3})$ be a non-injective endomorphism, that is $f \in End'(\overline{P}_{7_1} + \overline{P}_{5_2} + \overline{P}_{5_3})$ defined by

$$f = \begin{pmatrix} 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & 7_1 & 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 1_3 & 2_3 & 3_3 & 4_3 & 5_3 \\ 1_2 & 1_2 & 2_3 & 2_3 & 3_2 & 1_1 & 1_1 & 3_1 & 4_1 & 5_1 & 4_3 & 4_3 & 5_2 & 5_2 & 7_1 & 7_1 & 5_3 \end{pmatrix}.$$

The induced congruence relation ϱ_f can be visualized as in Figure 1.

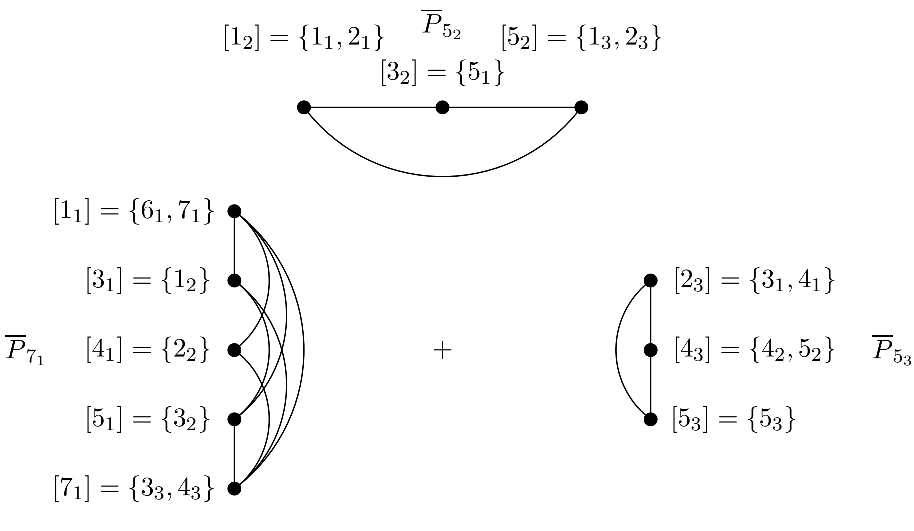

FIGURE 1. Congruence relation on $\overline{P}_{7_1} + \overline{P}_{5_2} + \overline{P}_{5_3}$ induced by f

Lemma 29. *Let $f \in End(+\overline{P}_{2n_k+1})$. Then*

(1) $I_f = I_g$, *for some idempotent* $g \in End(+\overline{P}_{2n_k+1})$,

(2) $\rho_f = \rho_h$, *for some idempotent* $h \in End(+\overline{P}_{2n_k+1})$.

Proof. For each $\overline{P}_{(2n_j+1)_j}$, let $f(x_i), f((x+t)_i)$ be the end vertices of $B_{both}^{(n_j)}$, for some $i, j \in \{1, \ldots, k\}$. Lemma 28 implies that $f(x_i) = y_j$, $f((x+1)_i) = (y+1)_j, \ldots, f((x+t)_i) = (y+t)_j$ [or in exactly reversed order].

(1) Let us define $g : V(+\overline{P}_{2n_k+1}) \to V(+\overline{P}_{2n_k+1})$ by

- $g(y_j) = y_j, g((y+1)_j) = (y+1)_j, \ldots, g((y+t)_j) = (y+t)_j$,
- $g(1_j) = g(2_j) = 1_j, g(3_j) = g(4_j) = 3_j, \ldots, g((y-2)_j) = g((y-1)_j) = (y-2)_j$,
- $g((y+t+1)_j) = g((y+t+2)_j) = (y+t+2)_j, g((y+t+3)_j) = g((y+t+4)_j) = (y+t+4)_j, \ldots, g((2n_j)_j) = g((2n_j+1)_j) = (2n_j+1)_j$.

Further, let $x_i, y_j \in V(+\overline{P}_{2n_k+1})$ and $\{x_i, y_j\} \in E(+\overline{P}_{2n_k+1})$.

1) If $i \neq j$, then $g(x_i) \in V(\overline{P}_{(2n_i+1)_i})$ and $g(y_j) \in V(\overline{P}_{(2n_j+1)_j})$. Thus $\{g(x_i), g(y_j)\} \in E(+\overline{P}_{2n_k+1})$.
2) If $i = j$, then $g(x_i), g(y_i) \in V(\overline{P}_{(2n_i+1)_i})$ with $y \neq x-1$ and $y \neq x+1$, or $\{x_i, y_i\} = \{1_i, (2n_i+1)_i\}$. From the definition of g we get that $\{g(x_i), g(y_j)\} \in E(+\overline{P}_{2n_k+1})$.

Therefore, $g \in End(+\overline{P}_{2n_k+1})$ is an idempotent and $I_f = I_g$.

(2) Now, let us define $h : V(+\overline{P}_{2n_k+1}) \to V(+\overline{P}_{2n_k+1})$ by

- $h(x_i) = x_i, h((x+1)_i) = (x+1)_i, \ldots, h((x+t)_i) = (x+t)_i$,
- $h(1_i) = h(2_i) = 1_i, \ldots, h((x-2)_i) = h((x-1)_i) = (x-2)_i$,
- $h((x+t+1)_i) = h((x+t+2)_i) = (x+t+2)_i, \ldots, h((2n_i)_i) = h((2n_i+1)_i) = (2n_i+1)_i$.

In a similar fashion as in case (1) we have $h \in End(+\overline{P}_{2n_k+1})$ is an idempotent and $\rho_f = \rho_h$. □

Theorem 1. *The graph $+\overline{P}_{2n_k+1}$ is endo-regular. In particular $+\overline{P}_{2n_k+1}$ is endo-orthodox.*

Proof. The endo-regularity follows directly from Lemma 3 and Lemma 29.

Further, assuming the endo-regularity of $+\overline{P}_{2n_k+1}$, by Lemma 13 it follows that each $\overline{P}_{2n_i+1}$ is endo-orthodox and thus by Lemma 5 if follows that $+\overline{P}_{2n_k+1}$ is endo-orthodox. □

Further, let us denote by $+\overline{C}_{2m_l}$, the graph which consists of l joins of complements of even cycles, that is, $+\overline{C}_{2m_l} = \overline{C}_{(2m_1)_1} + \overline{C}_{(2m_2)_2} + \ldots + \overline{C}_{(2m_l)_l}$, for some $m_j \in \mathbb{Z}^+, j = 1, \ldots, l$. Let us denote by $+\overline{P}_{2n_k+1} + +\overline{C}_{2m_l}$, the join of the graphs $+\overline{P}_{2n_k+1}$ and $+\overline{C}_{2m_l}$. In the result above, we established that $+\overline{P}_{2n_k+1}$ is endo-regular, and from Theorem 17(1), $+\overline{C}_{2m_l}$ is endo-regular. We show next that $+\overline{P}_{2n_k+1} + +\overline{C}_{2m_l}$ is not endo-regular.

Proposition 8. *The join graph $+\overline{P}_{2n_k+1} + +\overline{C}_{2m_l}$ is not endo-regular.*

Proof. Since $+\overline{P}_{2n_k+1} + +\overline{C}_{2m_l} = (\overline{P}_{n_1} + \overline{C}_{m_2}) + [(+\overline{P}_{2n_k+1} + +\overline{C}_{2m_l}) \setminus (\overline{P}_{n_1} + \overline{C}_{m_2})]$, by Lemma 4, $+\overline{P}_{2n_k+1} + +\overline{C}_{2m_l}$ is not endo-regular, if $\overline{P}_{n_1} + \overline{C}_{m_2}$ is not endo-regular. Thus, it suffices to show that $\overline{P}_{n_1} + \overline{C}_{m_2}$ is not endo-regular, with $n \geqslant 3$, $n-$odd and $m \geqslant 4$, $m-$even.

To achieve that, consider the map $f \in End(\overline{P}_{n_1} + \overline{C}_{m_2})$ defined as

$$f = \begin{pmatrix} 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & \ldots & (n-1)_1 & n_1 \\ 1_2 & 2_2 & 3_2 & 5_1 & 5_1 & \ldots & n_1 & n_1 \end{pmatrix}$$
$$\begin{pmatrix} 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 & \ldots & (m-1)_2 & m_2 \\ 1_1 & 1_1 & 3_1 & 3_1 & 5_2 & 5_2 & \ldots & (m-1)_2 & (m-1)_2 \end{pmatrix}.$$

Suppose that f is regular. From Lemma 3, there exist an idempotent $g \in End(\overline{P}_{n_1} + \overline{C}_{m_2})$ such that $I_f = I_g$. Then

1) $g(1_2) = 1_2, g(2_2) = 2_2, g(3_2) = 3_2$, and
2) $g(5_2) = g(6_2) = 5_2$, $g(7_2) = g(8_2) = 7_2$, $\ldots$, $f((m-3)_2) = f((m-2)_2) = (m-3)_2, f((m-1)_2) = f(m_2) = (m-1)_2$.

Since $1_2, 2_2, 3_2, 5_2, 7_2, \ldots, (m-3)_2, (m-1)_2 \in V(I_f)$ and $4_2 \notin V(I_f)$, we get $4_2 \notin V(I_g)$. Now, we have $g(4_2) \neq g(x)$, where $x \in \{1_2, 2_2, 6_2, 7_2, \ldots, m_2\}$, since $\{4_2, x\} \in E(\overline{C}_{m_2})$, for all x. Thus $g(4_2) = g(3_2)$ or $g(4_2) = g(5_2)$. But $g(5_2) = g(6_2)$, $g(4_2) \neq g(5_2)$. Then $g(4_2) = g(3_2) = 3_2$. This contradicts to $\{2_2, 4_2\} \in E(\overline{C}_{m_2})$, but $\{g(2_2), g(4_2)\} \notin E(\overline{C}_{m_2})$. Therefore, f is not regular. $\square$

2.2. Endo-regularity of graphs G of order n with $\delta(G) = n-3$

In this subsection we compile the results and provide complete characterization of the endomorphism monoid of graphs G of order n with a minimum degree $n-3$, based on the structure of G.

Everywhere further in the section G is a graph of order n with $\delta(G) = n-3$, and we denote by G^* the graph $G^* = G \setminus (K_t + (+\overline{C}_{2m_l+1}))$, where K_t is the maximal complete subgraph of G and $+\overline{C}_{2m_l+1}$ is the maximal induced subgraph of the joins of complements of odd cycles which length is more than 3. It is easy to see that G is unretractive if and only if $V(G^*)$ is empty.

First, we prove the following result.

Theorem 2. *G is endo-regular if and only if G^* is one of following graphs:*

(1) *a complement of path,*

(2) *an $(n-2)$-regular graph of order n,*

(3) *an $(n-3)$-regular graph of order n,*

(4) *the join of complements of odd paths.*

Proof. Necessity. Suppose that G^* is not one of the graphs (1) to (4). Then G contains one of following induced subgraphs:

1) $\overline{P}_{2n} + \overline{P}_m$ (except $K_{2,2}$),

2) $\overline{P}_n + \overline{C}_3$ (except $K_{2,3}$),

3) $\overline{P}_n + \overline{C}_{2m}$.

Therefore, G is not endo-regular, from Lemma 24 and Proposition 8.

Sufficiency. Obvious from Lemma 13, Corollary 1, Theorem 17(1) and Theorem 1. □

Next, we provide the results for endo-orthodoxy and endo-complete-regularity of G.

Theorem 3. *G is endo-orthodox if and only if G^* is one of following graphs:*

(1) *a complement of path,*

(2) *an $(n-2)$-regular graph of order n,*

(3) *the join of complements of odd paths.*

Proof. Necessity. Let G^* be an $(n-3)$-regular graph of order n. From Lemma 17(2), G^* is not endo-orthodox.

Sufficiency. Obvious from Lemma 13, Corollary 1 and Theorem 1. □

Theorem 4. *If $n \geqslant 4$, then $End(\overline{P}_n)$ is not completely regular.*

Proof. For each $n \geqslant 4$, let

$$f = \begin{pmatrix} 1 & 2 & 3 & \dots & n-3 & n-2 & n-1 & n \\ 3 & 4 & 5 & \dots & n-1 & n & 1 & 1 \end{pmatrix} \in End(\overline{P}_n).$$

Since $End(\overline{P}_n)$ is regular, let g be a pseudo inverse to f. Then exactly $g(3) = 1, g(4) = 2, \dots, g(n) = n-2$. Since $\{1, n\}, \{2, n\} \in E(\overline{P}_n)$, we have $g(1) = g(2) = n$. Thus $gf(2) = g(4) = 2$ and $fg(2) = f(n) = 1$. Therefore, $gf \neq fg$, g is not a commuting pseudo inverse to f. □

Lemma 30. *$\overline{P}_n$ is endo-completely-regular if and only if $n \leqslant 3$.*

Proof. Straightforward. □

Lemma 31. *Let $+\overline{P}_{3_k}$ be the k-joins of $\overline{P}_3$. Then $+\overline{P}_{3_k}$ is endo-completely-regular.*

Proof. Let $f \in End(+\overline{P}_{3_k})$. Then $f|_{X^{(3_k)}}$ is injective from $X^{(3_k)}$ - the set of all odd vertices of $+\overline{P}_{3_k}$ - onto itself. Consider $2_i \in V(\overline{P}_{3_i})$. If $f(2_i) = 2_j$ for some $2_j \in V(\overline{P}_{3_j})$, then it is not hard to get that either $f|_{\overline{P}_{3_i}} = \binom{1_i 2_i 3_i}{1_j 2_j 3_j}$ or $f|_{\overline{P}_{3_i}} = \binom{1_i 2_i 3_i}{3_j 2_j 1_j}$. If $f(2_i) \neq 2_j$ for all $2_j \in V(+\overline{P}_{3_k})$, then $f(2_i) = f(1_i)$ or $f(2_i) = f(3_i)$.

Let F_2 be the set of all $2_i \in V(+\overline{P}_{3_k})$ such that $f(2_i) = 2_j$ for some $2_j \in V(+\overline{P}_{3_k})$. Define $g : V(+\overline{P}_{3_k}) \to V(+\overline{P}_{3_k})$ by

1) $g|_{X^{(3_k)} \cup F_2} = f^{-1}|_{X^{(3_k)} \cup F_2}$,
2) $g(2_i) = g(1_i)$ or $g(2_i) = g(3_i)$, if $f(2_i) = f(1_i)$ or $f(2_i) = f(3_i)$, resp. for each $2_i \notin F_2$.

It is clear that $g \in End(+\overline{P}_{3_k})$, $fgf(x) = f(x)$ and $fg(x) = gf(x) = x$, for all $x \in X^{(3_k)} \cup F_2$.

Let $2_i \in V(+\overline{P}_{3_k})$ such that $f(2_i) = f(1_i)$. (The case $f(2_i) = f(3_i)$ can be proven similarly.) Then $g(2_i) = g(1_i)$. Therefore, $fgf(2_i) = fgf(1_i) = f(1_i) = f(2_i)$ and $fg(2_i) = fg(1_i) = gf(1_i) = gf(2_i)$. □

Theorem 5. *G is endo-completely-regular if and only if G^* is one of following graphs:*

(1) *a complement of the path of length 2,*

(2) *the join of complements of paths of length 3,*

(3) *an* $(n-3)$*-regular graph of order* n*, where* $|G_3^*| = 0$ *and* $|G_{2m}^*| = 1$ *for all induced subgraphs* $\overline{C}_{2m}$ *of* G^*.

Proof. First, if G^* is a $(n-3)$-regular graph of order n, by Theorem 17(3), we get that G^* is endo-completely-regular if and only if $|G_3^*| = 0$ and $|G_{2m}^*| = 1$ for all induced subgraphs $\overline{C}_{2m}$ of G^*.

Now suppose that G^* is not a $(n-3)$-regular graph of order n.

Necessity. Let G^* be a graph $\overline{P}_{2n}$ or $\overline{P}_{2n+1}$, where $n > 1$, by Lemma 30, G^* is not endo-completely-regular.

Sufficiency. Follows directly from Proposition 2 and Lemma 31. □

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