Algebra and Discrete Mathematics
Volume **19** (2015). Number 1, pp. 39–43 RESEARCH ARTICLE

On a deformation diameter of Dynkin diagrams

Vitaliy M. Bondarenko, Vitaliy V. Bondarenko, Viktoria A. Lisykevych, Yulia M. Pereguda

Communicated by V. V. Kirichenko

Abstract. We introduce the notion of a deformation distance and calculate the diameter of Dynkin diagrams respect to this distance.

Introduction

Through the paper, all graphs are finite, undirected and simple (i.e. without loops and multiple edges) with more than one vertices. The vertex and edges sets of a graph G is denoted by G_0 and G_1, respectively. $\mathbb{R}^+$ denotes the set of positive real numbers.

By a *vertex-weighted* (respectively, *edges-weighted*) graph we mean a graph $G = (G_0, G_1)$ together with a weighted function $\varphi : G_0 \to \mathbb{R}^+$ (respectively, $\varphi : G_1 \to \mathbb{R}^+$); $\varphi(x)$ is called the weight of x.

If one talks on learning of metrics on graphs, the case of edges-weighted graphs is main. The φ-*distance* between vertices u and v of an edges-weighted graph, denoted $d_\varphi(u, v)$, is the weight $\sum_{i=1}^{s} \varphi(\lambda_i)$ of a shortest path $\lambda = \lambda_1\lambda_2 \cdots \lambda_s$ between them.

Each vertex-weighted graph $G = (G_0, G_1, \varphi)$ we associate the edges-weighted graph $G^+ = (G_0, G_1, \varphi^+)$, where, for any edge λ between vertices u and v, $\varphi^+(\lambda) = \varphi(u) + \varphi(v)$. In this paper we study the φ^+-distances on the Dynkin diagrams for φ determined by the so-called P-limiting

2010 MSC: 15A.

Key words and phrases: quadratic form, P-limiting number, graph, Dynkin diagram, weight, distance, diameter.

numbers of their vertices, which are connected with deformations of the corresponding quadratic Tits forms.

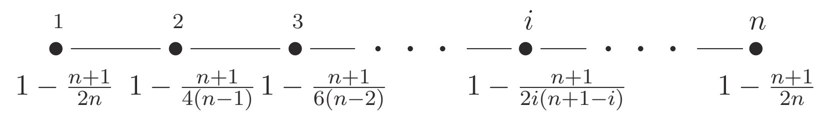
1. *P*-limiting numbers of Dynkin diagrams

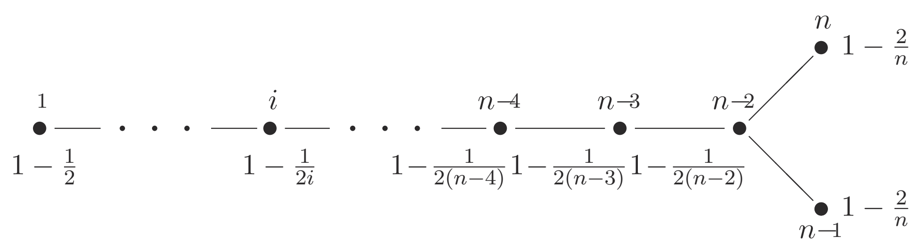
Let G be a Dynkin diagram and $s \in G_0$. The *s-th P-limiting number* of G is the greatest number $a \in \mathbb{R}$ for which the quadratic form

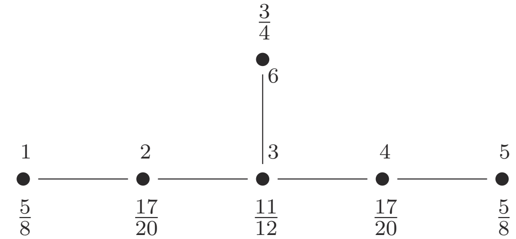
$$q_G^{(s)}(z,a) = q_Q(z_1, z_2, \ldots, z_n, a) := a z_s^2 + \sum_{i \in G_0 \setminus s} z_i^2 - \sum_{\{i-j\} \in G_1} z_i z_j$$

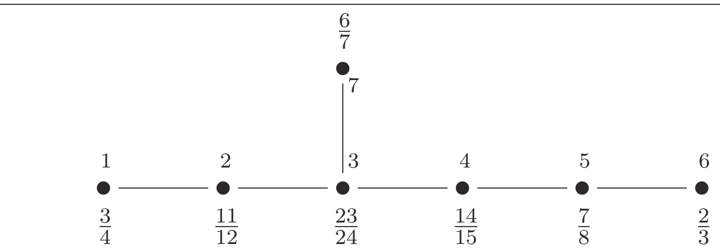
called the *s-th deformation* of the quadratic Tits form $q_G(z) \equiv q_G^{(s)}(z,1)$, is not positive; in this connection, see [1, 2]. The form $q_G^{(s)}(z,a)$ is called in [3] a *point-local deformation* of the form $q_G(z)$, and in this connection we denote the s-th P-limiting number by $\mathrm{pld}(s)$.

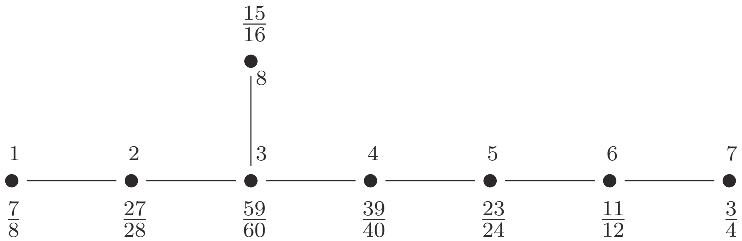
By [2, Theorem 5] the P-limiting numbers of the Dynkin diagrams A_n, D_n, E_6, E_7, E_8 are the following:











It is well-known that a (connected) graph G is a Dynkin diagram iff its quadratic Tits form $q_G(z)$ is positive. So any graph G with positive Tits form is associated the vertex weight pld. The corresponding edge weight pld$^+$ is determined by the following formula: $pld^+(\lambda) = pld(u) + pld(v)$ for any edge $\lambda = u - v$ (see Introduction).

2. Calculation of diameters

We consider the Dynkin diagrams $G = A_n, D_n, E_6, E_7, E_8$ with the weight pld^+ (by the above assumption, A_1 is excluded). The diameter of G (the maximum distance between two vertices) is denoted by $\mathcal{D}_{\mathrm{pld}^+}(G)$.

Theorem 1.

$$\mathcal{D}_{\mathrm{pld}^+}(G) = \begin{cases} \frac{2n^2-n-1}{n} - 2\sum\limits_{k=1}^{n-1} C_n^k \frac{(-1)^{k+1}}{k} & \textit{for } G = A_n, \\ \frac{4n^2-7n-4}{2n} - \sum\limits_{k=1}^{n-2} C_n^k \frac{(-1)^{k+1}}{k} & \textit{for } G = D_n, \\ 6\frac{29}{60} & \textit{for } G = E_6, \\ 8\frac{47}{60} & \textit{for } G = E_7, \\ 11\frac{37}{168} & \textit{for } G = E_8. \end{cases}$$

Proof. We write $d_+(i,j)$, $\mathcal{D}_+(G)$ instead of $d_{\mathrm{pld}^+}(i,j)$, $\mathcal{D}_{\mathrm{pld}^+}(G)$ and use that, for $i<j$,

$$\begin{aligned} d_+(i,j) &= 2\sum_{s=i}^{j} \mathrm{pld}^+(s) - \mathrm{pld}^+(i) - \mathrm{pld}^+(j) \\ &= 2\sum_{s=i}^{j-1} \mathrm{pld}^+(s) - \mathrm{pld}^+(i) + \mathrm{pld}^+(j). \end{aligned}$$

We let H_m denote the m-th harmonic number:

$$H_m = \sum_{k=1}^{m} C_m^k \,\frac{(-1)^{k+1}}{k}.$$

1) *The case of A_n.* Since

$$\begin{aligned} \sum_{i=1}^{n} \mathrm{pld}(i) &= \sum_{i=1}^{n}\left(1 - \frac{n+1}{2i(n+1-i)}\right) \\ &= n - \frac{1}{2}\left(\sum_{i=1}^{n}\frac{1}{i} + \sum_{i=1}^{n}\frac{1}{n+1-i}\right) \\ &= n - \frac{1}{2}\left(\sum_{i=1}^{n}\frac{1}{i} + \sum_{n+1-i=1}^{n}\frac{1}{n+1-i}\right) = n - H_n, \end{aligned}$$

we have that

$$\begin{aligned} \mathcal{D}_+(A_n) = d_+(1,n) &= 2\sum_{i=1}^{n}\mathrm{pld}(i) - 2\left(1 - \frac{n+1}{2n}\right) \\ &= 2n - 2H_n + \frac{1}{n} - 1 = 2n - 2H_{n-1} - \frac{1}{n} - 1 \\ &= \frac{2n^2 - n - 1}{n} - 2H_{n-1}. \end{aligned}$$

2) *The case of D_n.* It is easy to see that $\mathcal{D}_+(D_n) = d_+(1,n-1)$. Since $\sum_{i=1}^{n-2}\mathrm{pld}(i) = n-2-\frac{1}{2}H_{n-2}$, we have that

$$\begin{aligned} \mathcal{D}_+(D_n) &= 2\sum_{i=1}^{n-2}\mathrm{pld}(i) - \frac{1}{2} + 1 - \frac{2}{n} = 2n - 4 - H_{n-2} + \frac{1}{2} - \frac{2}{n} \\ &= \frac{4n^2 - 7n - 4}{2n} - H_{n-2}. \end{aligned}$$

3) *The case of E_s, $s = 6,7,8$.* Simple direct calculations show that $d_+(1,s-1) > d_+(1,s), d_+(s-1,s)$ and the diameter $\mathcal{D}_+(E_s) = d_+(1,s-1)$ is equal to the number indicated in the formulation of the theorem. $\square$

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Contact information

V. M. Bondarenko, V. V. Bondarenko Institute of Mathematics, Tereshchenkivska 3, 01601 Kyiv, Ukraine
E-Mail(s): `vit-bond@imath.kiev.ua`

V. A. Lisykevych Department of Mechanics and Mathematics, Kyiv National Taras Shevchenko Univ., Volodymyrska str., 64, 01033 Kyiv, Ukraine
E-Mail(s): `vikadrug@ukr.net`

Yu. M. Pereguda Department of Information and Telecommunication Systems, Zhytomyr Military Institute, Prospect Myru, 22, 10023 Zhytomyr, Ukraine
E-Mail(s): `pereguda.juli@rambler.ru`

Received by the editors: 18.03.2015
and in final form 18.03.2015.