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# Universal property of skew *PBW* extensions

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Abstract. In this paper we prove the universal property of skew $PBW$ extensions generalizing this way the well known universal property of skew polynomial rings. For this, we will show first a result about the existence of this class of non-commutative rings. Skew $PBW$ extensions include as particular examples Weyl algebras, enveloping algebras of finite-dimensional Lie algebras (and its quantization), Artamonov quantum polynomials, diffusion algebras, Manin algebra of quantum matrices, among many others. As a corollary we will give a new short proof of the Poincaré-Birkhoff-Witt theorem about the bases of enveloping algebras of finite-dimensional Lie algebras.

## 1. Introduction

Most of constructions in algebra are characterized by universal properties from which it is easy to prove important results about the constructed object. This is the case of the universal property of the tensor product; another well known example is the universal property for the localization of rings and modules by multiplicative subsets. A key example in non-commutative algebra is the skew polynomial ring $R[x;\sigma,\delta]$; the universal property in this case says that if $B$ is a ring with a ring homomorphism $\varphi : R \to B$ and in $B$ there exists and element $y$ such that $y\varphi(r) = \varphi(\sigma(r))y + \varphi(\delta(r))$ for every $r \in R$, then there exists an

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unique ring homomorphism $\widetilde{\varphi} : R[x;\sigma,\delta] \to B$ such that $\widetilde{\varphi}(x) = y$ and $\widetilde{\varphi}(r) = \varphi(r)$ (see [9]). In this paper we prove the universal property of skew $PBW$ extensions generalizing the universal property of skew polynomial rings. For this, we will prove first a theorem about the existence of skew $PBW$ extensions similar to the corresponding result on skew polynomial rings. As application we will get the Poincaré-Birkhoff-Witt theorem about the bases of enveloping algebras of finite-dimensional Lie algebras. This famous theorem says that if $K$ is a field and $\mathcal{G}$ is a finite-dimensional Lie algebra with $K$-basis $\{y_1,\dots,y_n\}$, then a $K$-basis of the universal enveloping algebra $\mathcal{U}(\mathcal{G})$ is the set of monomials $y_1^{\alpha_1}\cdots y_n^{\alpha_n}$, $\alpha_i \geqslant 0$, $1 \leqslant i \leqslant n$ (see [4], [6]).

Skew $PBW$ extensions were defined firstly in [7], and their homological and ring-theoretic properties have been studied in the last years (see [1], [3], [8], [10]). Skew polynomial rings of injective type, Weyl algebras, enveloping algebras of finite-dimensional Lie algebras (and its quantization), Artamonov quantum polynomials, diffusion algebras, Manin algebra of quantum matrices, are particular examples of skew $PBW$ extensions (see [8]). In this first section we recall the definition of skew $PBW$ extensions and some very basic properties needed for the proof of the main theorem.

**Definition 1.1.** Let $R$ and $A$ be rings. We say that $A$ is a *skew $PBW$ extension of $R$* (also called a $\sigma - PBW$ extension of $R$) if the following conditions hold:

(i) $R \subseteq A$.

(ii) There exist finite elements $x_1,\dots,x_n \in A$ such $A$ is a left $R$-free module with basis

$$\mathrm{Mon}(A) := \{x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n\}.$$

In this case it says also that $A$ *is a left polynomial ring over $R$* with respect to $\{x_1,\dots,x_n\}$ and $Mon(A)$ is the set of standard monomials of $A$. Moreover, $x_1^0\cdots x_n^0 := 1 \in Mon(A)$.

(iii) For every $1 \leqslant i \leqslant n$ and $r \in R - \{0\}$ there exists $c_{i,r} \in R - \{0\}$ such that

$$x_i r - c_{i,r} x_i \in R. \tag{1.1}$$

(iv) For every $1 \leqslant i,j \leqslant n$ there exists $c_{i,j} \in R - \{0\}$ such that

$$x_j x_i - c_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n. \tag{1.2}$$

Under these conditions we will write $A := \sigma(R)\langle x_1,\dots,x_n\rangle$.

The following proposition justifies the notation and the alternative name given for the skew $PBW$ extensions.

**Proposition 1.2.** Let $A$ be a skew $PBW$ extension of $R$. Then, for every $1 \leqslant i \leqslant n$, there exists an injective ring endomorphism $\sigma_i : R \to R$ and a $\sigma_i$-derivation $\delta_i : R \to R$ such that

$$x_i r = \sigma_i(r)x_i + \delta_i(r),$$

for each $r \in R$.

*Proof.* See [7], Proposition 3. □

Observe that if $\sigma$ is an injective endomorphism of the ring $R$ and $\delta$ is a $\sigma$-derivation, then the skew polynomial ring $R[x; \sigma, \delta]$ is a trivial skew $PBW$ extension in only one variable, $\sigma(R)\langle x\rangle$.

Some extra notation will be used in the rest of the paper.

**Definition 1.3.** Let $A$ be a skew $PBW$ extension of $R$ with endomorphisms $\sigma_i$, $1 \leqslant i \leqslant n$, as in Proposition 1.2.

(i) For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $\sigma^\alpha := \sigma_1^{\alpha_1} \cdots \sigma_n^{\alpha_n}$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$. If $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$, then $\alpha + \beta := (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$.

(ii) For $X = x^\alpha \in \mathrm{Mon}(A)$, $\exp(X) := \alpha$ and $\deg(X) := |\alpha|$.

(iii) If $f = c_1X_1 + \cdots + c_tX_t$, with $X_i \in Mon(A)$ and $c_i \in R - \{0\}$, then $\deg(f) := \max\{\deg(X_i)\}_{i=1}^t$.

The skew $PBW$ extensions can be characterized in a similar way as was done in [5] for $PBW$ rings.

**Theorem 1.4.** Let $A$ be a left polynomial ring over $R$ w.r.t. $\{x_1, \dots, x_n\}$. $A$ is a skew $PBW$ extension of $R$ if and only if the following conditions hold:

(a) For every $x^\alpha \in \mathrm{Mon}(A)$ and every $0 \neq r \in R$ there exist unique elements $r_\alpha := \sigma^\alpha(r) \in R - \{0\}$ and $p_{\alpha,r} \in A$ such that

$$x^\alpha r = r_\alpha x^\alpha + p_{\alpha,r}, \tag{1.3}$$

where $p_{\alpha,r} = 0$ or $\deg(p_{\alpha,r}) < |\alpha|$ if $p_{\alpha,r} \neq 0$. Moreover, if $r$ is left invertible, then $r_\alpha$ is left invertible.

(b) For every $x^\alpha, x^\beta \in \mathrm{Mon}(A)$ there exist unique elements $c_{\alpha,\beta} \in R$ and $p_{\alpha,\beta} \in A$ such that

$$x^\alpha x^\beta = c_{\alpha,\beta} x^{\alpha+\beta} + p_{\alpha,\beta}, \tag{1.4}$$

where $c_{\alpha,\beta}$ is left invertible, $p_{\alpha,\beta} = 0$ or $\deg(p_{\alpha,\beta}) < |\alpha + \beta|$ if $p_{\alpha,\beta} \neq 0$.

*Proof.* See [7], Theorem 7. □

## 2. Existence theorem for skew PBW extensions

If $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ is a skew $PBW$ extension of the ring $R$, then as was observed in the previous section, $A$ induces unique endomorphisms $\sigma_i : R \to R$ and $\sigma_i$-derivations $\delta_i : R \to R$, $1 \leqslant i \leqslant n$. Moreover, by (1.2), there exist $c_{ij}, d_{ij}, a_{ij}^{(k)} \in R$ such that $x_j x_i = c_{ij} x_i x_j + a_{ij}^{(1)} x_1 + \cdots + a_{ij}^{(n)} x_n + d_{ij}$, with $1 \leqslant i, j \leqslant n$. However, note that if $i < j$, since $Mon(A)$ is a $R$-basis, then $1 = c_{j,i} c_{i,j}$, i.e., for every $1 \leqslant i < j \leqslant n$, $c_{ji}$ is a right inverse of $c_{i,j}$ univocally determined. In a similar way, we can check that $a_{ji}^{(k)} = -c_{ji} a_{ij}^{(k)}$, $d_{ji} = -c_{ji} d_{ij}$. Thus, given $A$ there exist unique parameters $c_{ij}, d_{ij}, a_{ij}^{(k)} \in R$ such that

$$x_j x_i = c_{ij} x_i x_j + a_{ij}^{(1)} x_1 + \cdots + a_{ij}^{(n)} x_n + d_{ij}, \text{ for every } 1 \leqslant i < j \leqslant n. \tag{2.1}$$

**Definition 2.1.** Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew $PBW$ extension. $\sigma_i, \delta_i, c_{ij}$, $d_{ij}, a_{ij}^{(k)}$, $1 \leqslant i < j \leqslant n$, defined as before, are called the parameters of $A$.

Conversely, given a ring $R$ and parameters $\sigma_i, \delta_i, c_{ij}$, $d_{ij}, a_{ij}^{(k)}$, $1 \leqslant i < j \leqslant n$, we will construct in this section a skew $PBW$ extension with coefficient ring $R$ and satisfying the following equations

1) For $i < j$ in $I$ and $k$ in $I$, $x_j x_i = c_{ij} x_i x_j + \Sigma_k a_{ij}^{(k)} x_k + d_{ij}$,
2) For $i \in I$ and $r \in R$, $x_i r = \sigma_i(r) x_i + \delta_i(r)$,

where $I := \{1, \ldots, n\}$.

**Definition 2.2.** Let $R$ be a ring and $W$ be the free monoid in the alphabet $X \cup R$, with $X := \{x_i : i \in I\}$. Let $w$ be a word of $W$, the complexity of $w$, denoted $c(w)$, is a triple of nonnegative integers $(a, b, c)$, where $a$ is the number of $x$'s in $w$, $b$ is is the number of inversions involving only $x$'s, and $c$ is the number of inversions of the type $(x_i, r)$.

These triples are ordered with the lexicographic order, i.e., $(a,b,c) \leqslant (d,e,f)$ if and only if $a < d$, or, $a = d$ and $b < e$, or, $a = d$, $b = e$ and $c \leqslant f$. This is a well order. Let $T$ be the set of elements of $W$ such that $c(w) = (a,0,0)$ and $\mathbb{Z}T$ be the linear extension of $T$ in $\mathbb{Z}\langle X \cup R\rangle$ (the $\mathbb{Z}$-free algebra in the alphabet $X \cup R$).

**Definition 2.3.** Let $R$ be a ring, $\{c_{ij}\}_{i<j}$, $\{d_{ij}\}_{i<j}$ and $\{a_{ij}^{(k)}\}_{i<j,k}$ be elements of $R$ indexed by $i,j,k$ in $I$. Let $\sigma_i, \delta_i : R \to R$ be two functions for each $i \in I$. Suppose that $c_{ij}$ is left invertible and that $\sigma_i(r) \neq 0$ for $r \neq 0$. We define the function $p$

$$p : W \to \mathbb{Z}\langle X \cup R\rangle, \quad \text{with } X := \{x_i : i \in I\},$$

by induction in the complexity, as follows:

1) If $w \in T$ then $p(w) = w$.
2) If $w = v_1 x_i r v_2$, with $r \in R$, $v_1 \in W$ and $rv_2 \in T$ then

$$p(w) = p(v_1\sigma_i(r)x_i v_2) + p(v_1\delta_i(r)v_2).$$

3) If $w = v_1 x_j x_i v_2$, where $v_1 \in W$, $x_i v_2 \in T$ with $i < j$, then

$$p(w) = p(v_1 c_{ij} x_i x_j v_2) + \Sigma_k p(v_1 a_{ij}^{(k)} x_k v_2) + p(v_1 d_{ij} v_2).$$

The linear extension of $p$ to $\mathbb{Z}\langle X \cup R\rangle \to \mathbb{Z}\langle X \cup R\rangle$ is also denoted $p$. The image of $p$ is contained in $\mathbb{Z}T$. Let $Mon := \{\Pi_{k=1}^{n} x_{i_k} : i_1 \leqslant \cdots \leqslant i_n, n \geqslant 0\}$, and $F_R(Mon)$ be the left free $R-$module with basis $Mon$. We define $q : \mathbb{Z}T \to F_R(Mon)$ as the bilinear extension of $q(r_1 \dots r_m x_{i_1} \dots x_{i_n}) := (\Pi_{k=1}^{m} r_k) x_{i_1} \dots x_{i_n}$. Finally, we define $h : \mathbb{Z}\langle X \cup R\rangle \to F_R(Mon)$ as $h := qp$.

**Theorem 2.4** (Existence)**.** Let $R, I, X, a_{ij}^{k}, c_{ij}, \sigma_i, \delta_i, h, p, q$ be as in Definition 2.3. Then, there exists a skew $PBW$ extension $A$ of $R$ with variables $X := \{x_i : i \in I\}$ such that

(a) $x_i r = \sigma_i(r)x_i + \delta_i(r)$.
(b) $x_j x_i = c_{ij} x_i x_j + \Sigma_k a_{ij}^{(k)} x_k + d_{ij}$, for $i < j$ in $I$.

if and only if

(1) For every $i$ in $I$, $\sigma_i$ is a ring endomorphism of $R$ and $\delta_i$ is $\sigma_i$-derivation.

(2) $h(x_jx_ir) = h(p(x_jx_i)r)$, for $i < j$ in $I$ and $r \in R$.

(3) $h(x_kx_jx_i) = h(p(x_kx_j)x_i)$, for $i < j < k$ in $I$.

*Proof.* ($\Longrightarrow$) Numeral (1) is the content of Proposition 1.2. Conditions (2) and (3) follow from (a) and (b) and the associativity $x_j(x_ir) = (x_jx_i)r$ and $x_k(x_jx_i) = (x_kx_j)x_i$.

($\Longleftarrow$) Define $t : F_R(Mon) \to \mathbb{Z}\langle X \cup R\rangle$ as $t(\Sigma r_{\bar{x}}\bar{x}) := \Sigma r_{\bar{x}}\bar{x} \in \mathbb{Z}\langle X \cup R\rangle$, where $\Sigma r_{\bar{x}}\bar{x}$ is the unique expression of an element in $F_R(Mon)$ as a sum over a finite set, $\bar{x} \in Mon$ and $r_{\bar{x}} \neq 0$ is an element of $R$.

We define a product in $F_R(Mon)$ by

$$f \star g = h(t(f)t(g)), \qquad f, g \in F_R(Mon),$$

and we will prove in Lemma 2.8 below that $h(ab) = h(a) \star h(b)$, with $a, b \in \mathbb{Z}\langle X \cup R\rangle$. From this we get that $h : \mathbb{Z}\langle X \cup R\rangle \to F_R(Mon)$ is a surjection that preserves sums, products and $h(1) = 1$. This makes $F_R(Mon)$ a ring, which is a skew *PBW* extension of $R$ by the definition of the product $\star$.

To complete the proof we proceed to prove Lemma 2.8, but for this, we have to show first some preliminary propositions under the hypothesis (1)-(3). □

**Proposition 2.5.** For $a, b \in W$ and $r, s \in R$ the following equalities hold:

(i) $h(a0b) = 0$.

(ii) $h(a(-r)b) = -h(arb)$.

(iii) $h(a(r + s)b) = h(arb + asb)$.

(iv) $h(a1b) = h(ab)$.

(v) $h(a(rs)b) = h(arsb)$.

*Proof.* (i) and (ii) follow from (iii) since $r \mapsto h(arb)$ is a group homomorphism from the additive group of $R$ into $F_R(Mon)$.

(iii) is proven by induction on $c(a(r+s)b)$ and applying the definition of $h$. Here the conditions $\delta_i(a+b) = \delta_i(a)+\delta_i(b)$ and $\sigma_i(a+b) = \sigma_i(a)+\sigma_i(b)$ in the hipothesis (1) of Theorem 2.4 are used.

(iv) is proven by induction on $c(a1b)$ and making use of part (i). The relevant hypothesis are $\sigma_i(1) = 1$ and $\delta_i(1) = 0$ which are part of the hypothesis (1) in Theorem 2.4.

(v) This part is proven by induction on $c(a(rs)b)$ and making use of (iii). The relevant hypothesis are $\sigma_i(ab) = \sigma_i(a)\sigma_i(b)$ and $\delta_i(ab) = \sigma_i(a)\delta_i(b) + \delta_i(a)b$. □

**Proposition 2.6.** Let $y, z \in \mathbb{Z}\langle X \cup R\rangle$ and $a \in \mathbb{Z}T$. Then $h(yaz) = h(ytq(a)z)$.

*Proof.* This is because we can obtain $tq(a)$ from $a$ with a finite number of operations described in Proposition 2.5. Indeed if $a \in \mathbb{Z}T$ then by definition of $T$ we heave $a = \Sigma n_u u$ where the sum is over $u \in T$, $n_u \in \mathbb{Z}$ and $u = r_{1,u} \ldots r_{m,u} x_{j_1} \ldots x_{j_k}$ ($j_1, \ldots j_k$ and $m, k$ depend on $u$) here $r_s \in R$ and $1 \leqslant j_1 \leqslant \cdots \leqslant j_k \leqslant n$. Then by definition of $t, q$ we have $tq(a) = \Sigma_{x\in A} a(x)x$ where $A = \{x \in Mon(X) : a(x) \neq 0\}$, and $a(x) = \Sigma_{u\in B(x)} n_u \Pi_s r_{s,u} \in R$ where $B(x) = \{u \in T : x_{j_1} \ldots x_{j_k} = x\}$. Using the Proposition 2.5 (i) we obtain that

$$h(ytq(a)z) = h(y\Sigma_{x\in Mon(X)} a(x)xz).$$

Using that $h$ is linear we get

$$h(y\Sigma_{x\in Mon(X)} a(x)xz) = \Sigma_{x\in Mon(x)} h(ya(x)xz).$$

Using Proposition 2.5 (i),(ii),(iii) we get that

$$h(ya(x)xz) = \Sigma_{u\in B(x)} n_u h(y(\Pi_s r_{s,u})xz).$$

Further, using Proposition 2.5 (iv)(v) we get that

$$h(y(\Pi_s r_{s,u})xz) = h(yr_{1,u} \ldots r_{m,u}xz) = h(yuz). \qquad \square$$

**Proposition 2.7.** If $x, y, z \in \mathbb{Z}\langle X \cup R\rangle$ then $h(xp(y)z) = h(xyz)$.

*Proof.* The identity is linear in $x, y, z$, so we may assume they are words. Next we proceed by induction on $c(xyz)$. First assume that the first inversion from right to left in $xyz$ is in $y$, say $y = w_1 x_j s w_2$ with $s = x_i$ with $i < j$ or $s \in R$, and $sw_2 \in T$. Then

$$h(xyz) = h(xw_1 p(x_j s)w_2 z) = h(xp(w_1 p(x_j s)w_2)z) = h(xp(y)z)$$

by the definition of $p$ and induction.

Now assume that the first inversion of $xyz$ is not contained in $yz$, or $xyz \in T$, in this case $y \in T$ and $p(y) = y$.

Next, assume that the first inversion of $xyz$ is contained in $z$ say $z = w_1 x_j s w_2$ with $sw_2 \in T$ and $s = x_i$ with $i < j$ or $s \in R$. Then

$$h(xyz) = h(xyw_1 p(x_j s)w_2) = h(xp(y)w_1 p(x_j s)w_2) = h(xp(y)z)$$

by definition of $h$ and induction.

Now assume that the first inversion of $xyz$ has a part in $y$ and a part in $z$, say $y = y'x_j$ and $z = sz'$ with $z \in T$ and $s = x_i$ with $i < j$ or $s \in R$. Assume further that the first inversion of $y$ exists and is contained in $y'$, say $y' = w_1 x_k s' w_2$ with $s'w_2 \in T$ an $s' = x_i$ with $i < k$ or $s' \in R$. Then

$$\begin{aligned} h(xyz) &= h(xy'p(x_j s)z') = h(xp(y')p(x_j s)z') \\ &= h(xp(w_1 p(x_k s')w_2)p(x_j s)z') = h(xw_1 p(x_k s')w_2 p(x_j s)z') \\ &= h(xw_1 p(x_k s')w_2 x_j sz') = h(xp(w_1 p(x_k s')w_2 x_j)sz') \\ &= h(xp(y)z) \end{aligned}$$

by definition of $h$ and induction applied alternatively. So the last case is $y = y'x_k x_j$ with $k > j$ and $z = sz'$ with $s = x_i$ with $i < j$ or $s \in R$ and $z \in T$. In this case

$$h(xyz) = h(xy'x_k p(x_j s)z') = h(xy'p(x_k p(x_j s))z')$$

by definition of $h$ and induction, also observe

$$h(xy'p(x_k p(x_j s))z') = h(xy'p(p(x_k x_j)s)z')$$

because $qp(p(x_k x_j)s) = qp(x_k p(x_j s)$ by hipothesis (2) and (3) in Theorem 2.4, and also by Proposition 2.6. Also

$$h(xy'p(p(x_k x_j)s)z') = h(xy'p(x_k x_j)sz') = h(xp(y'p(x_k x_j))z)$$

by induction applied twice, and $h(xp(y'p(x_k x_j))z) = h(xp(y)z)$ by definition of $p$, as required. □

**Lemma 2.8.** $h(ab) = h(a) \star h(b)$, for $a, b \in \mathbb{Z}\langle X \cup R\rangle$.

*Proof.* $h(a) \star h(b) = h(tqp(a)tqp(b)) = h(p(a)p(b)) = h(ab)$, the first equality is from the definition of $\star$, the second equality is from Proposition 2.6 twice and the third equality is Proposition 2.7 twice. □

## 3. The universal property

In this section we will prove the main theorem about the characterization of skew *PBW* extensions by a universal property in a similar way as this is done for skew polynomial rings. This problem was studied in [2] where skew *PBW* extensions were generalized to infinite sets of generators.

**Theorem 3.1** (Main theorem: The universal property)**.**
Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew $PBW$ extension with parameters $\sigma_i, \delta_i, c_{ij}, d_{ij}, a_{ij}^{(k)}$, $1 \leqslant i, j \leqslant n$. Let $B$ be a ring with homomorphism $\varphi : R \to B$ and elements $y_1, \dots, y_n \in B$ such that

(i) $y_i\varphi(r) = \varphi(\sigma_i(r))y_i + \varphi(\delta_i(r))$, for every $r \in R$.

(ii) $y_jy_i = \varphi(c_{ij})y_iy_j + \varphi(a_{ij}^{(1)})y_1 + \cdots + \varphi(a_{ij}^{(n)})y_n + d_{ij}$.

Then, there exists an unique ring homomorphism $\widetilde{\varphi} : A \to B$ such that $\widetilde{\varphi}\iota = \varphi$ and $\widetilde{\varphi}(x_i) = y_i$, where $\iota$ is the inclusion of $R$ in $A$.

*Proof.* Since $A$ is a free $R$-module with basis $Mon(A)$, we define the $R$-homomorphism

$$\widetilde{\varphi} : A \to B, \quad r_1x^{\alpha_1} + \cdots + a_tx^{\alpha_t} \mapsto \varphi(r_1)y^{\alpha_1} + \cdots + \varphi(a_t)y^{\alpha_t},$$

where $y^\theta := y_1^{\theta_1} \cdots y_n^{\theta_n}$, with $\theta := (\theta_1, \dots, \theta_n) \in \mathbb{N}^n$. Note that $\widetilde{\varphi}(1) = 1$.

$\widetilde{\varphi}$ is multiplicative: In fact, applying induction on the degree $|\alpha + \beta|$ we have

$$\begin{aligned}
\widetilde{\varphi}(ax^\alpha bx^\beta) &= \widetilde{\varphi}(a[\sigma^\alpha(b)x^\alpha x^\beta + p_{\alpha,b}x^\beta]) \\
&= \widetilde{\varphi}[a\sigma^\alpha(b)[c_{\alpha,\beta}x^{\alpha+\beta} + p_{\alpha,\beta}] + ap_{\alpha,b}x^\beta] \\
&= \varphi(a)\varphi(\sigma^\alpha(b))\varphi(c_{\alpha,\beta})y^{\alpha+\beta} + \varphi(a)\varphi(\sigma^\alpha(b))\varphi(p_{\alpha,\beta})(y) \\
&\quad + \varphi(a)\varphi(p_{\alpha,b})(y)y^\beta,
\end{aligned}$$

where $\varphi(p_{\alpha,\beta})(y)$ is the element in $B$ obtained replacing each monomial $x^\theta$ in $p_{\alpha,\beta}$ by $y^\theta$ and every coefficient $c$ by $\varphi(c)$. In a similar way we have for $\varphi(p_{\alpha,b})(y)$ (observe that the degree of each monomial of $p_{\alpha,b}x^\beta$ is $< |\alpha + \beta|$). On the other hand, applying (i) and (ii) we get

$$\begin{aligned}
\widetilde{\varphi}(ax^\alpha)\widetilde{\varphi}(bx^\beta) &= \varphi(a)y^\alpha\varphi(b)y^\beta \\
&= \varphi(a)[\varphi(\sigma^\alpha(b))y^\alpha + \varphi(p_{\alpha,b})(y)]y^\beta \\
&= \varphi(a)\varphi(\sigma^\alpha(b))y^\alpha y^\beta + \varphi(a)\varphi(p_{\alpha,b})(y)y^\beta \\
&= \varphi(a)\varphi(\sigma^\alpha(b))[\varphi(c_{\alpha,\beta})y^{\alpha+\beta} + \varphi(p_{\alpha,\beta})(y)] \\
&\quad + \varphi(a)\varphi(p_{\alpha,b})(y)y^\beta \\
&= \varphi(a)\varphi(\sigma^\alpha(b))\varphi(c_{\alpha,\beta})y^{\alpha+\beta} + \varphi(a)\varphi(\sigma^\alpha(b))\varphi(p_{\alpha,\beta})(y) \\
&\quad + \varphi(a)\varphi(p_{\alpha,b})(y)y^\beta.
\end{aligned}$$

It is clear that $\widetilde{\varphi}\iota = \varphi$ and $\widetilde{\varphi}(x_i) = y_i$. Moreover, note that $\widetilde{\varphi}$ is the only ring homomorphism that satisfy these two conditions. $\square$

**Corollary 3.2.** Let $R$ be a ring and $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew $PBW$ extension of $R$ with parameters $\sigma_i, \delta_i, c_{ij}, d_{ij}, a_{ij}^{(k)}$, $1 \leqslant i, j \leqslant n$. Let $B$ be a ring with homomorphism $\varphi : R \to B$ and elements $y_1, \ldots, y_n \in B$ such that the conditions (i)-(ii) in Theorem 3.1 are satisfied with respect to the system of parameters $\sigma_i, \delta_i, c_{ij}, d_{ij}, a_{ij}^{(k)}$, $1 \leqslant i, j \leqslant n$, of the ring $R$. If $B$ satisfies the universal property, then $B \cong A = \sigma(R)\langle x_1, \ldots, x_n\rangle$. Moreover, the monomials $y_1^{\alpha_1} \cdots y_n^{\alpha_n}$, $\alpha_i \geqslant 0$, $1 \leqslant i \leqslant n$ are a $R$-basis of $B$.

*Proof.* By the universal property of $A$ there exists $\widetilde{\varphi}$ such that $\widetilde{\varphi}\iota = \varphi$; by the universal property of $B$ there exists $\widetilde{\iota}$ such that $\widetilde{\iota}\varphi = \iota$. Note that $\widetilde{\iota}\widetilde{\varphi}\iota = \iota$ and $\widetilde{\varphi}\widetilde{\iota}\varphi = \varphi$. The uniqueness gives that $\widetilde{\iota}\widetilde{\varphi} = i_A$ and $\widetilde{\varphi}\widetilde{\iota} = i_B$. Moreover, in the proof of Theorem 3.1 we observed that $\widetilde{\varphi}$ is not only a ring homomorphism but also a $R$-homomorphism, whence

$$\widetilde{\varphi}(Mon(A)) = \{y_1^{\alpha_1} \cdots y_n^{\alpha_n} | \alpha_i \geqslant 0, 1 \leqslant i \leqslant n\}$$

is a $R$-basis of $B$. □

**Corollary 3.3.** Let $R$ be a ring and $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension of $R$ with parameters $\sigma_i, \delta_i, c_{ij}, d_{ij}, a_{ij}^{(k)}$, $1 \leqslant i, j \leqslant n$. Let $B$ be a ring that satisfies the following conditions with respect to the system of parameters $\sigma_i, \delta_i, c_{ij}, d_{ij}, a_{ij}^{(k)}$, $1 \leqslant i, j \leqslant n$, of the ring $R$.

(i) There exists a ring homomorphism $\varphi : R \to B$.

(ii) There exist elements $y_1, \ldots, y_n \in B$ such that $B$ is a left free $B$-module with basis $Mon(y_1, \ldots, y_n)$, and the product is given by $r \cdot b := \varphi(r)b$, $r \in R, b \in B$.

(iii) The conditions (i) and (ii) in Theorem 3.1 hold.

Then $B \cong A = \sigma(R)\langle x_1, \ldots, x_n\rangle$.

*Proof.* According to the universal property of $A$, there exists a ring homomorphism $\widetilde{\varphi} : A \to B$ given by $r_1 x^{\alpha_1} + \cdots + a_t x^{\alpha_t} \mapsto \varphi(r_1) y^{\alpha_1} + \cdots + \varphi(a_t) y^{\alpha_t}$; from (ii) we get that $\widetilde{\varphi}$ is bijective. □

## 4. The Poincaré-Birkhoff-Witt theorem

Using the results of the previous sections, we will give now a new short proof of the Poincaré-Birkhoff-Witt theorem about the bases of enveloping algebras of finite-dimensional Lie algebras. Recall that if $K$ is a field and $\mathcal{G}$

is a Lie algebra with $K$-basis $Y := \{y_1, \dots, y_n\}$, the enveloping algebra of $\mathcal{G}$ is the associative $K$-algebra $\mathcal{U}(\mathcal{G})$ defined by $\mathcal{U}(\mathcal{G}) = K\{y_1, \dots, y_n\}/I$, where $K\{y_1, \dots, y_n\}$ is the free $K$-algebra in the alphabet $Y$ and $I$ the two-sided ideal generated by all elements of the form $y_j y_i - y_i y_j - [y_j, y_i]$, $1 \leqslant i, j \leqslant n$, where $[\,,\,]$ is the Lie bracket of $\mathcal{G}$ (see [9]).

**Theorem 4.1** (Poincaré-Birkhoff-Witt theorem)**.** The standard monomials $y_1^{\alpha_1} \cdots y_n^{\alpha_n}$, $\alpha_i \geqslant 0$, $1 \leqslant i \leqslant n$, conform a $K$-basis of $\mathcal{U}(\mathcal{G})$.

*Proof.* For the ring $K$ we consider the following system of variables and parameters:

$$\begin{aligned} X := \{x_1, \dots, x_n\}, \quad \sigma_i := i_K, \quad \delta_i := 0, \quad c_{i,j} := 1, \quad d_{ij} := 0, \\ [x_i, x_j] = a_{ij}^{(1)} x_1 + \cdots + a_{ij}^{(n)} x_n, \quad 1 \leqslant i, j \leqslant n. \end{aligned} \tag{4.1}$$

We want to prove that conditions (1)–(3) in Theorem 2.4 hold. Condition (1) trivially holds. For (2) we have

$$\begin{aligned} h(x_j x_i r) &= h(x_j r x_i) = h(r x_j x_i) = r x_i x_j + r[x_j, x_i]; \\ h(p(x_j x_i) r) &= h(x_i x_j r) + h([x_j, x_i] r) = h(x_i r x_j) + r[x_j, x_i] \\ &= r x_i x_j + r[x_j, x_i]. \end{aligned}$$

Condition (3) of Theorem 2.4 also holds: In fact,

$$\begin{aligned} h(p(x_k x_j) x_i) &= h(x_j x_k x_i) + h([x_k, x_j] x_i) \\ &= h(x_j x_i x_k) + h(x_j [x_k, x_i]) + h([x_k, x_j] x_i) \\ &= x_i x_j x_k + h([x_j, x_i] x_k) + h(x_j [x_k, x_i]) + h([x_k, x_j] x_i) \\ &= x_i x_j x_k + (h(x_k [x_j, x_i]) + h([[x_j, x_i], x_k])) + (h([x_k, x_i] x_j) \\ &\quad + h([x_j, [x_k, x_i]])) + (h(x_i [x_k, x_j]) + h([[x_k, x_j], x_i])) \\ &= h(x_k x_j x_i) + h([[x_j, x_i], x_k] + [x_j, [x_k, x_i]] + [[x_k, x_j], x_i]) \\ &= h(x_k x_j x_i). \end{aligned}$$

The last equality holds by the Jacobi identity, the second to the last equality follows regrouping the terms and applying the definition of $h$ to $h(x_k x_j x_i)$.

From Theorem 2.4 we conclude that there exists a skew $PBW$ extension $A = \sigma(K)\langle x_1, \dots, x_n \rangle$ that satisfies (4.1), in particular, the monomials $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\alpha_i \geqslant 0$, $1 \leqslant i \leqslant n$, conform a $K$-basis of $A$. But note that $\mathcal{U}(\mathcal{G})$ satisfies the hypothesis in Corollary 3.2, so $\mathcal{U}(\mathcal{G}) \cong A$ and $\mathcal{U}(\mathcal{G})$ has $K$-basis $y_1^{\alpha_1} \cdots y_n^{\alpha_n}$, $\alpha_i \geqslant 0$, $1 \leqslant i \leqslant n$. $\square$

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