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Lie and Jordan structures of differentially semiprime rings

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Abstract. Properties of Lie and Jordan rings (denoted respectively by R^L and R^J) associated with an associative ring R are discussed. Results on connections between the differentially simplicity (respectively primeness, semiprimeness) of R, R^L and R^J are obtained.

1. Introduction

Throughout here, R is an associative ring (with respect to the addition "+" and the multiplication " $\cdot$ ") with an identity, $\mathrm{Der}\, R$ is the set of all derivations in R. On the set R we consider two operations: the Lie multiplication "$[-,-]$" and the Jordan multiplication "$(-,-)$" defined by the rules

$$[a,b] = a \cdot b - b \cdot a$$

and

$$(a,b) = a \cdot b + b \cdot a$$

for any $a, b \in R$. Then

$$R^L = (R, +, [-,-])$$

is a Lie ring and

$$R^J = (R, +, (-,-))$$

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is a Jordan ring (see [13] and [14]) associated with the associative ring R. Recall that an additive subgroup A of R is called:

- *a Lie ideal* of R if
$$[a, r] \in A,$$
- *a Jordan ideal* of R if
$$(a, r) \in A$$

for all $a \in A$ and $r \in R$. Obviously, A is a Lie (respectively Jordan) ideal of R if and only if A^L (respectively A^J) is an ideal of R^L (respectively R^J).

In all that follows Δ will be any subset of $\operatorname{Der} R$ (in particular, $\Delta = \{0\}$) and $\delta \in \operatorname{Der} R$. A subset K of R is called Δ-*stable* if $d(a) \in K$ for all $d \in \Delta$ and $a \in K$. An ideal I of a (Lie, Jordan or associative) ring A is said to be a Δ-*ideal* if I is Δ-stable. A (Lie, Jordan or associative) ring A is said to be:

- *simple* (respectively Δ-*simple*) if there no two-sided ideals (respectively Δ-ideals) other 0 or A,
- *prime* (respectively Δ-*prime*) if, for all two-sided ideals (respectively Δ-ideals) K, S of A, the condition $KS = 0$ implies that $K = 0$ or $S = 0$ (if $\Delta = \{\delta\}$ and A is Δ-prime, then we say that A is δ-*prime*),
- *semiprime* (respectively Δ-*semiprime*) if, for any two-sided ideal (respectively Δ-ideal) K of A, the condition $K^2 = 0$ implies that $K = 0$,
- *primary* if, for any two-sided ideals K, S of A, the condition $KS = 0$ implies that $K = 0$ or S is nilpotent.

Every non-commutative Δ-simple ring is Δ-prime and every Δ-prime ring is Δ-semiprime. We say that R is $\mathbb{Z}$-*torsion-free* if, for any $r \in R$ and integers n, the condition $nr = 0$ holds if and only if $r = 0$. If the implication
$$2r = 0 \Rightarrow r = 0$$
is true for any $r \in R$, then R is said to be 2-*torsion-free*. Let
$$F_p(R) = \{a \in R \mid a \text{ has an additive order } p^k \\ \text{for some non-negative} k = k(a)\}$$
be the p-part of R, where p is a prime. Then $F_p(R)$ is a Δ-ideal of R. If R is Δ-semiprime, then
$$pF_p(R) = 0.$$

In particular, in a Δ-prime ring R it holds $F_p(R) = 0$ (and so the characteristic $\operatorname{char} R = 0$) or $F_p(R) = R$ (and therefore $\operatorname{char} R = p$). Obviously that the additive group R^+ of a Δ-prime ring R is torsion-free if and only if $\operatorname{char} R = 0$. Recall that a ring R is said to be *of bounded index* m, if m is the least positive integer such that $x^m = 0$ for all nilpotent elements $x \in R$. We say that a ring R satisfies *the condition* (X) if one of the following holds:

(1) R or $R/\mathbb{P}(R)$ is $\mathbb{Z}$-torsion-free, where $\mathbb{P}(R)$ is the prime radical of R,

(2) R is of bounded index m such that an additive order of every nonzero torsion element of R, if any, is strictly larger than m.

As noted in [16, p.283], a $\mathbb{Z}$-torsion-free δ-prime ring is semiprime. In this way we prove the following

Proposition 1. *For a ring R the following hold:*

(1) *if R is a Δ-semiprime ring with the condition (X), then it is semiprime,*

(2) *if R is both semiprime (respectively satisfies the condition (X)) and Δ-prime, then R is prime.*

Relations between properties of an associative ring R, a Lie ring R^L and a Jordan ring R^J was studied by I.N. Herstein and his students (see [7, 8, 11] and bibliography in [9] and [5]); he has obtained, for a ring R of characteristic different from 2, that the simplicity of R implies the simplicity of a Jordan ring R^J [7, Theorem 1], and also that every Lie ideal of a simple Lie ring R is contained in the center $Z(R)$ [7, Theorem 3]. K. McCrimmon [20, Theorem 4] has proved that R is a simple algebra if and only if R^J is a simple Jordan algebra. Our result is the following

Theorem 1. *For a 2-torsion-free ring R the following statements are true:*

(1) *R is a Δ-simple ring if and only if R^J is a Δ-simple Jordan ring,*

(2) *R is a Δ-prime ring if and only if R^J is a Δ-prime Jordan ring,*

(3) *R is a Δ-semiprime ring if and only if R^J is a Δ-semiprime Jordan ring.*

Let us $d \in \Delta$. Since $C(R)$ and $\operatorname{ann} C(R)$ are Δ-ideals, the rule

$$\overline{d} : R/\operatorname{ann} C(R) \ni r + \operatorname{ann} C(R) \mapsto d(r) + \operatorname{ann} C(R) \in R/\operatorname{ann} C(R)$$

determines a derivation $\overline{d}$ of the quotient ring $R/\operatorname{ann} C(R)$. Then

$$\overline{\Delta} = \{\overline{d} \mid d \in \Delta\} \subseteq \operatorname{Der}(R/\operatorname{ann} C(R)).$$

Inasmuch $d(Z(R)) \subseteq Z(R)$, the rule

$$\widehat{d} : R^L/Z(R) \ni r + Z(R) \mapsto d(r) + Z(R) \in R^L/Z(R)$$

determines a derivation $\widehat{d}$ of the Lie ring $R^L/Z(R)$. Then

$$\widehat{\Delta} = \{\widehat{d} \mid d \in \Delta\} \subseteq \operatorname{Der}(R^L/Z(R)).$$

Since the center $Z(R)$ is a nonzero Lie ideal of an associative ring R with an identity, a Lie ring R^L is not Δ-simple. Our next result is the following

Theorem 2. *Let R be a 2-torsion-free ring. Then the following are true:*

(1) *if the quotient ring $R^L/Z(R)$ is a $\widehat{\Delta}$-simple Lie ring, then R is non-commutative and $R/\operatorname{ann} C(R)$ is a $\overline{\Delta}$-simple ring,*

(2) *if R is a Δ-simple ring, then $R^L/Z(R)$ is a $\widehat{\Delta}$-simple Lie ring or R is commutative,*

(3) *if $R^L/Z(R)$ is a $\widehat{\Delta}$-semiprime Lie ring, then R is non-commutative and the quotient ring $R/\operatorname{ann} C(R)$ is a $\overline{\Delta}$-semiprime ring,*

(4) *if R is a Δ-semiprime ring, then $R^L/Z(R)$ is a $\widehat{\Delta}$-semiprime Lie ring or R is commutative,*

(5) *if $R^L/Z(R)$ is a $\widehat{\Delta}$-prime Lie ring, then R is non-commutative and $R/\operatorname{ann} C(R)$ is a $\overline{\Delta}$-prime ring,*

(6) *if R is a Δ-prime ring, then $R^L/Z(R)$ is a $\widehat{\Delta}$-prime Lie ring or R is commutative.*

Throughout, let $Z(R)$ denote the center of R, $[A, B]$ (respectively (A, B)) an additive subgroup of R generated by all commutators $[a, b]$ (respectively (a, b)), where $a \in A$ and $b \in B$, $C(R)$ the commutator ideal of R, $N(R)$ the set of nilpotent elements in R, $\operatorname{char} R$ the characteristic of R, $\operatorname{ann}_l I = \{a \in R \mid aI = 0\}$ the left annihilator of I in R, $\operatorname{ann}_r I = \{a \in R \mid Ia = 0\}$ the right annihilator of I in R, $\operatorname{ann} I = (\operatorname{ann}_r I) \cap (\operatorname{ann}_l I)$, $C_R(I) = \{a \in R \mid ai = ia \text{ for all } i \in I\}$ the centralizer of I in R and $\partial_a(x) = [a, x]$ for $a, x \in R$.

All other definitions and facts are standard and it can be found in [10], [17] and [19].

2. Differentially prime right Goldie rings

Let agree that

$$d^0 = \mathrm{id}_R$$

is the identity endomorphism for $d \in \Delta$.

Lemma 1. *The following conditions are equivalent:*

(1) *R is a Δ-semiprime ring,*

(2) *for any Δ-ideals A, B of R the implication*

$$AB = 0 \Rightarrow A \cap B = 0$$

is true,

(3) *if $a \in R$ is such that*

$$aR\delta_1^{m_1} \dots \delta_k^{m_k}(a) = 0$$

for any integers $k \geqslant 1$, $m_i \geqslant 0$ and derivations $\delta_i \in \Delta$ $(i = 1, \dots, k)$, then $a = 0$.

Proof. A simple modification of Proposition 2 from [17, §3.2]. □

Lemma 2. *The following conditions are equivalent:*

(1) *R is a Δ-prime ring,*

(2) *a left annihilator $\mathrm{ann}_l\, I$ of a left Δ-ideal I of R is zero,*

(3) *a right annihilator $\mathrm{ann}_r\, I$ of a right Δ-ideal I of R is zero,*

(4) *if $a, b \in R$ are such that*

$$aR\delta_1^{m_1} \dots \delta_k^{m_k}(b) = 0$$

for any integers $k \geqslant 1$, $m_j \geqslant 0$ and derivations $\delta_j \in \Delta$ $(j = 1, \dots, k)$, then $a = 0$ or $b = 0$.

Proof. A simple consequence of Lemma 2.1.1 from [10]. □

If I is an ideal of a ring R, then

$$\mathcal{C}_R(I) = \{x \in R \mid x + I \text{ is regular in the quotient ring } R/I\}$$

(see [19, Chapter 2, §1]). The next lemma extends Proposition 1 of [15].

Lemma 3. *Let R be a right Goldie ring and $\delta \in \operatorname{Der} R$. If R is δ-prime, then:*

(*a*) *the set $N = N(R)$ of nilpotent elements of R is its prime radical,*

(*b*) $\bigcap_{i=1}^{k} \delta^{-1}(N) = 0$ *for some integer k,*

(*c*) $\mathcal{C}_R(0) = \mathcal{C}_R(N)$.

Proof. From Theorem 2.2 of [16] (see the part $(ii) \Rightarrow (iii)$ of its proof), we obtain (a) and (b). By Proposition 4.1.3 of [19], $\mathcal{C}_R(0) \subseteq \mathcal{C}_R(N)$. By the same argument as in [16, p.284], we can obtain that $\mathcal{C}_R(0) = \mathcal{C}_R(N)$. □

Corollary 1. *If R is a commutative δ-prime Goldie ring and $\delta \in \operatorname{Der} R$, then $N(R)$ contains all zero-divisors of R.*

By Corollary 1.4 of [6], if I is a δ-prime ideal of a right Noetherian ring R and R/I has characteristic 0, then I is prime. The following lemma is an extension of Lemma 2.5 from [6].

Lemma 4. *Let R be a 2-torsion-free commutative Goldie ring and $\delta \in \operatorname{Der} R$. If R is δ-prime, then it is an integral domain.*

Proof. Assume that $a \in \operatorname{ann} N(R)$, $b \in N(R)$ and $r \in R$. Then

$$\begin{aligned} 0 &= \delta^2(arb) = \delta(\delta(a)rb + a\delta(r)b + ar\delta(b)) \\ &= \delta^2(a)rb + 2\delta(a)\delta(r)b + 2\delta(a)r\delta(b) + a\delta^2(r)b + 2a\delta(r)\delta(b) + ar\delta^2(b) \end{aligned}$$

and so

$$2\delta(a)R\delta(b) \subseteq N(R).$$

This means that $\delta(a) \in N(R)$ or $\delta(b) \in N(R)$. Hence $N(R)$ is δ-stable. By Lemma 3, $N(R)$ is a ideal and therefore $N(R) = 0$. By Lemma 1.2 of [4], R is prime and consequently it is an integral domain. □

Proof of Proposition 1.

(1) By Proposition 1.3 of [6] and Theorem 1 of [1], the prime radical $\mathbb{P}(R)$ is a Δ-ideal and so $\mathbb{P}(R) = 0$ is zero.

(2) Since $\mathbb{P}(R) = 0$, R is prime by Lemma 1.2 from [4]. □

By Theorem 4 of [22], a Δ-simple ring R of characteristic 0 is prime. Since every non-commutative Δ-simple ring is Δ-prime, in view of Proposition 1 we obtain the following

Corollary 2. *Let R be a semiprime ring (respectively a ring R satisfy the condition (X)). If R is Δ-simple, then it is prime.*

3. Differential analogues of Herstein's results

For the proof of Theorem 2 we need the next results. In the proofs below we use the same consideration, as in [12, Chapter 1, §1], and present them here in order to have the paper more self-contained. Let agree that everywhere in this section $k \geqslant 1$ and $m_i \geqslant 0$ are integers $(i = 1, \dots, k)$.

Lemma 5. *Let R be a Δ-semiprime ring, A and B its Δ-ideals. Then the following statements hold:*

(*i*) *if $AB = 0$, then $BA = 0$.*

(*ii*) $\operatorname{ann}_l A = \operatorname{ann}_r A$.

(*iii*) $A \cap \operatorname{ann}_r A = 0$.

Proof. (*i*) Indeed, BA is a Δ-ideal and $(BA)^2 = 0$ and so $BA = 0$.

(*ii*) We denote $(\operatorname{ann}_r A)A$ by X. Since X is a Δ-ideal and $X^2 = 0$, we deduce that $X = 0$. This means that

$$\operatorname{ann}_r A \subseteq \operatorname{ann}_l A.$$

The inverse inclusion we can prove similarly.

(*iii*) Since $A \cap \operatorname{ann}_r A$ is a nilpotent Δ-ideal, the assertion holds. □

Henceforth

$$X_a = \{[\delta_1^{m_1} \dots \delta_k^{m_k}(a), x] \mid x \in R,\ \delta_i \in \Delta,\ m_i \geqslant 0 \text{ and } k \geqslant 1 \text{ are integers } \ (i = 1, \dots, k)\}.$$

It is clear that $[a, x] \in X_a$.

Lemma 6. *Let R be a Δ-semiprime ring and $a \in R$. Then the following statements hold:*

(*i*) *if*

$$a[\delta_1^{m_1} \dots \delta_k^{m_k}(a), R] = 0$$

for any integers $k \geqslant 1$, $m_i \geqslant 0$ and derivations $\delta_i \in \Delta$ $(i = 1, \dots, k)$, then $a \in Z(R)$,

(*ii*) *if I is a right Δ-ideal of R, then $Z(I) \subseteq Z(R)$,*

(*iii*) *if I is a commutative right Δ-ideal of R and I is nonzero, then $I \subseteq Z(R)$. If, moreover, R is Δ-prime, then it is commutative.*

Proof. (i) Let $x, y \in R$ and $d, \delta \in \Delta$. Since

$$[b, xy] = [b, x]y + x[b, y] \tag{3.1}$$

for any $b \in X_a$ and $a[b, xy] = 0$, we conclude that $ax[b, y] = 0$. This gives that $ayx[b, y] = 0$ and $yax[b, y] = 0$ and consequently

$$(R[a, y]R)^2 = 0. \tag{3.2}$$

In addition,

$$0 = d(a[b, x]) = d(a)[b, x].$$

Multiplying (3.1) by $d(a)$ on left we get $d(a)x[b, y] = 0$. Moreover,

$$0 = \delta(ax[d(b), y]) = \delta(a)x[d(b), y]$$

and, by the similar argument, we obtain that

$$\delta_1^{m_1} \dots \delta_k^{m_k}(a)x[\delta_1^{m_1} \dots \delta_k^{m_k}(a), y] = 0$$

for any integers $k \geqslant 1$, $m_i \geqslant 0$ and derivations $\delta_i \in \Delta$ $(i = 1, \dots, k)$. As in the proof of the condition (3.2), we deduce that

$$(R[\delta_1^{m_1} \dots \delta_k^{m_k}(a), y]R)^2 = 0.$$

Then

$$I = \sum_{k=1}^{\infty} \sum_{\substack{\delta_1 \dots \delta_k \in \Delta \\ y \in R}} R[\delta_1^{m_1} \dots \delta_k^{m_k}(a), y]R$$

is a sum of nilpotent ideals and therefore it is a nil ideal. Since I is a Δ-ideal, we conclude that $I = 0$ and, as a consequence, $a \in Z(R)$.

(ii) Let $a \in Z(I)$ and $y \in R$. Then, for $\delta_1, \dots, \delta_k \in \Delta$, we have

$$\delta_1^{m_1} \dots \delta_k^{m_k}(a) \in Z(I)$$

and $ay \in I$. This gives that

$$a(\delta_1^{m_1} \dots \delta_k^{m_k}(a)y) = \delta_1^{m_1} \dots \delta_k^{m_k}(a)(ay) = a(y\delta_1^{m_1} \dots \delta_k^{m_k}(a)),$$

and thus

$$a[\delta_1^{m_1} \dots \delta_k^{m_k}(a), y] = 0.$$

By (i), $a \in Z(R)$ is central.

(*iii*) By (*ii*), $I \subseteq Z(R)$. Assume that R is Δ-prime, $u, v \in R$ and $a \in I$. Then $au \in I$ and so $au \in Z(R)$. Since

$$a(uv) = (au)v = v(au) = (va)u = a(vu),$$

we see that

$$[u, v] \in \operatorname{ann}_r I.$$

By Lemma 2(3), $[u, v] = 0$ and hence R is commutative. □

Lemma 7. *Let R be a Δ-prime ring and $a \in R$. If $a \in C_R(I)$ for some nonzero right Δ-ideal I of R, then $a \in Z(R)$.*

Proof. Let us $y \in R$ and $b \in I$. Then $by \in I$ and so $bay = a(by) = bya$. This yields that

$$I[a, y] = 0 = [a, y]I.$$

By Lemma 2(3), $[a, y] = 0$. Hence $a \in Z(R)$. □

Lemma 8. *The left annihilator $\operatorname{ann}_l(X_a)$ is a left Δ-ideal of R.*

Proof. Immediate from the definition. □

Lemma 9. *If R is a Δ-semiprime ring, then $C_R([R, R]) \subseteq Z(R)$.*

Proof. Let us $a \in C_R([R, R]), d, \delta \in \Delta$ and $x, y \in R$. Putting x for a and $xd(a)$ for xy in (3.1) we obtain

$$[x, xd(a)] = [x, x]d(a) + x[x, d(a)]$$

and, as a consequence, $[a, x[x, d(a)]] = 0$ and $[a, x][x, d(a)] = 0$. Then, by the same reasons as in the proof of Lemma 6(i), we obtain that $[a, x] \in \operatorname{ann}_l(X_a)$ and $A = \operatorname{ann}_l(X_a)$ is a Δ-ideal. Then

$$[\delta(a), x][d(a), x] = \delta([a, x][d(a), x]) = 0.$$

Since $A \cap \operatorname{ann}_l A = 0$, we deduce that is a nilpotent Δ-ideal and so $a \in Z(R)$. □

Lemma 10. *Let R be a 2-torsion-free Δ-semiprime ring. If $a \in R$ commutes with all elements of X_a, then $a \in Z(R)$.*

Proof. Let $r, x, y \in R$ and $d \in \Delta$. It is clear that $\partial_a^2(x) = 0$. From $\partial_a^2(xy) = 0$ it follows that

$$2\partial_a(x)\partial_a(y) = 0$$

and so $\partial_a(x)\partial_a(y) = 0$. Since

$$0 = \partial_a(x)\partial_a(rx) = \partial_a(x)\partial_a(r)x + \partial_a(x)r\partial_a(x) = \partial_a(x)r\partial_a(x),$$

we deduce that $\partial_a(x)R\partial_a(x) = 0$ and $(\partial_a(x)R)^2 = 0$. Moreover, $a[b, x] = [b, x]a$ for any $[b, x] \in X_a$ and therefore

$$d(a)[b, x] + a[d(b), x] + a[b, d(x)] = [b, x]d(a) + [d(b), x]a + [b, d(x)]a.$$

From this it holds that

$$d(a)[b, x] = [b, x]d(a).$$

This means that $C_R(X_a)$ is Δ-stable and $(\partial_{d(a)}(x)R)^2 = 0$. As a consequence,

$$I = \sum_{k=1}^{\infty} \sum_{\substack{x \in R \\ m_k \geqslant 0 \\ \delta_1, \ldots, \delta_k \in \Delta}} \partial_{\delta_1^{m_1} \ldots \delta_k^{m_k}(a)}(x)R$$

is a sum of nilpotent ideals and so I is a nil ideal. Since I is a Δ-ideal, we deduce that $I = 0$. Hence $a \in Z(R)$. □

The next lemma is an extension of Lemma 1 from [11] in the differential case.

Lemma 11. *Let R be a 2-torsion-free Δ-semiprime ring, T its Lie Δ-ideal. If $[T, T] \subseteq Z(R)$, then $T \subseteq Z(R)$.*

Proof. Let $x \in R$ and $t \in T$.

1) If $[T, T] = 0$, then $[t, x] \in T$ and so $[t, [t, x]] = 0$. By Lemma 10, $T \subseteq Z(R)$.

2) Now assume that $0 \neq [a, b] \in [T, T]$ for some $a, b \in T$. Then

$$\partial_a(b) \in Z(R) \text{ and } \partial_a^2(R) \subseteq Z(R).$$

Moreover, we have that

$$\begin{aligned} Z(R) \ni \partial_a^2(bx) &= \partial_a(\partial_a(b)x + b\partial_a(x)) \\ &= \partial_a^2(b)x + 2\partial_a(b)\partial_a(x) + b\partial_a^2(x) \\ &= 2\partial_a(b)\partial_a(x) + b\partial_a^2(x) \end{aligned}$$

and hence

$$[2\partial_a(b)\partial_a(x) + b\partial_a^2(x), b] = 0.$$

Then

$$\begin{aligned} 0 &= 2\partial_b(\partial_a(b))\partial_a(x) + 2\partial_a(b)\partial_b(\partial_a(x)) + \partial_b(b)\partial_a^2(x) + b\partial_b(\partial_a^2(x)) \\ &= 2\partial_a(b)\partial_b(\partial_a(x)) \end{aligned} \tag{3.3}$$

and

$$\partial_a(ba) = \partial_a(b)a + b\partial_a(a) = \partial_a(b)a.$$

Replacing ba for x in (3.3) we have

$$0 = 2\partial_a(b)\partial_b(\partial_a(b)a) = 2\partial_a(b)(\partial_b(\partial_a(b)) + \partial_a(b)\partial_b(a)) = -2\partial_a(b)^3$$

and thus $\partial_a(b)^3 = 0$. Then $R\partial_a(b)$ is a nilpotent ideal in R and, as a consequence,

$$\sum_{a,b\in T} R\partial_a(b)$$

is a nonzero nil Δ-ideal, a contradiction. □

Lemma 12. *If U is a Lie Δ-ideal of a ring R and $I(U) = \{u \in R \mid uR \subseteq U\}$, then $I(U)$ is the largest Δ-ideal of R such that $I(U) \subseteq U$.*

Proof. Let $u, v \in I(U)$, $x, y \in R$ and $\delta \in \Delta$. Clearly that $I(U)$ is an additive subgroup of R, $I(U) \subseteq U$ and $(ux)y = u(xy) \in (ux)R = u(xR) \subseteq uR \subseteq U$ that is $ux \in I(U)$. From

$$u(xy) - (yu)x = (ux)y - y(ux) = [ux, y] \in U$$

(and so $(yu)x \in U$) it holds that $yu \in I(U)$. Hence U is a two-sided ideal of R. Moreover,

$$\delta(u)x + u\delta(x) = \delta(ux) \in \delta(U) \subseteq U$$

and $u\delta(x) \in uR \subseteq U$. Therefore $\delta(u)x \in U$. This means that $I(U)$ is a Δ-ideal of R. If A is a Δ-ideal of R that is contained in U, then $AR \subseteq A \subseteq U$ and hence $A \subseteq I(U)$. □

Lemma 13. *Let U be a Lie Δ-ideal of R. If U is an associative subring of R, then $[U, U] = 0$ or U contains a nonzero Δ-ideal of R.*

Proof. Assume that $x \in R$ and $[U,U] \neq 0$. Then $[u,v] \neq 0$ for some $u, v \in U$ and

$$[u, vx] = u(vx) - (vx)u = (uv - vu)x + v(ux - xu).$$

Since $[u,x], [u,vx] \in U$ and $v[u,x] \in U$, we deduce that $[u,v]x \in U$. This means that $[u,v] \in I(U)$. In view of Lemma 12, $I(U)$ is a nonzero Δ-ideal of R that is contained in U. □

Proposition 2. *If U is a Lie Δ-ideal of R, then $[U,U] = 0$ or there exists a nonzero Δ-ideal I_U of R such that $[I_U, R] \subseteq U$.*

Proof. By Lemma 3 of [7],

$$T(U) = \{t \in R \mid [t,R] \subseteq U\}$$

is both a Lie ideal and an associative subring of R and $U \subseteq T(U)$. Moreover, for $\delta \in \Delta$, we have

$$[\delta(t), R] + [t, \delta(R)] = \delta([t,R]) \subseteq \delta(U) \subseteq U$$

and so $[\delta(t), R] \subseteq U$. Hence $T(U)$ is Δ-stable. If $[U,U] \neq 0$, then, by Lemmas 12 and 13,

$$I_U = I(T(U)) \subseteq T(U)$$

is a nonzero Δ-ideal of R such that $[I_U, R] \subseteq U$. □

Lemma 14. *Let U be a Lie Δ-ideal of a ring R. If $[U,U] = 0$, then the centralizer $C_R(U)$ is a Lie Δ-ideal and an associative subring of R.*

Proof. Is immediately. □

We extend Theorem 1.3 of [9] in the following

Proposition 3. *Let R be a Δ-simple ring of characteristic 2. If U is a Lie Δ-ideal of R, then one of the following holds:*

(1) $[R,R] \subseteq U$,

(2) $U \subseteq Z(R)$,

(3) *R contains a subfield P such that $U \subseteq P$ and $[P,R] \subseteq P$.*

Proof. If $[U, U] \neq 0$, then $[R, R] \subseteq U$ by Proposition 2. Therefore we assume that $[U, U] = 0$. By Lemma 14, $C_R(U)$ is a Lie Δ-ideal and an associative subring of R such that $U \subseteq C_R(U)$.

a) If $C_R(U)$ is non-commutative, then $C_R(U) = R$ by Lemma 13. Hence $U \subseteq Z(R)$.

b) Now assume that the centralizer $C_R(U)$ is commutative. If $c \in C_R(U)$ and $x \in R$, then

$$c^2 \in C_R(U) \text{ and } [c^2, x] = [[c, x], x] = 2c[c, x] = 0.$$

This gives that $c^2 \in Z(R)$. By Theorem 2 of [22], $Z(R)$ is a field. As a consequence, c^2 (and so c) is invertible in $C_R(U)$. Hence $C_R(U)$ is a field. □

Corollary 3. *Let R be a Δ-simple ring. If U is a Lie Δ-ideal of R, then one of the following holds:*

(1) $[R, R] \subseteq U$,

(2) $U \subseteq Z(R)$,

(3) $\operatorname{char} R = 2$ *and R contains a subfield P such that $U \subseteq P$ and* $[P, R] \subseteq P$.

4. Jordan properties

Lemma 15. *Let R be a Δ-simple ring of characteristic $\neq 2$, U its proper Jordan Δ-ideal and $a \in U$. If $[a, R] \subseteq U$, then $a = 0$.*

Proof. Let us $x, y \in R$. Since $[a, x] \in U$ and $(a, x) \in U$, we obtain that $2ax \in U$ and, as a consequence, $ax \in U$ and $(ax, y) \in U$. Moreover, from $axy \in U$ it follows that $yax \in U$. This means that $RaR \subseteq U$. Since $d(a) \in U$ for any $d \in \Delta$, in view of [21, Lemma 1.1] we obtain that

$$\sum_{k=1}^{\infty} \sum_{\substack{\delta_1, \ldots, \delta_k \in \Delta \\ (m_1, \ldots, m_k) \in \mathbb{N}^k}} R\delta_1^{m_1} \ldots \delta_k^{m_k}(a)R$$

is a proper Δ-ideal of R that is contained in U. Hence $a = 0$. □

Remark 1. Let R be a 2-torsion-free ring, U its Jordan Δ-ideal. If Δ contains all inner derivations of R, then U is an ideal of R.

In fact, we have

$$2xa = [a, x] + (a, x) \in U$$

for any $a, b, x \in U$ and so $xa \in U$. By the same argument, we can conclude that $ax \in U$.

Proof of Theorem 1.

(1) ($\Leftarrow$) If A is a nonzero proper Δ-ideal of a ring R, then A^J is a nonzero proper Δ-ideal of R^J, a contradiction.

($\Rightarrow$) Let U be a proper Jordan Δ-ideal of R, $a, b \in U$ and $x \in R$. By Lemma 1 of [7], $[(a, b), x] \in U$, and, by Lemma 15, we see that

$$(a, b) = 0. \tag{4.4}$$

In particular, $2a^2 = 0$ and, as a consequence, $a^2 = 0$ and $2axa = (a, (a, x)) = 0$. It follows that $axa = 0$. Since

$$0 = (a + b)x(a + b) = axb + bxa$$

and

$$0 = (b, (a, x)) = b(ax + xa) + (ax + xa)b = bax + bxa + axb + xba,$$

we deduce that $bax + xab = 0$. But $ab = -ba$ and so $bax - xba = 0$. This means that $ba \in Z(R)$. Then $(RabR)^2 = 0$. Since

$$I = \sum_{k=1}^{\infty} \sum_{\substack{a,b\in U,\ \delta_1,\dots,\delta_k\in\Delta \\ (m_1,\dots,m_k)\in\mathbb{N}^k}} Ra\delta_1^{m_1}\dots\delta_k^{m_k}(b)R$$

is a Δ-ideal of R that is a sum of nilpotent ideals, we obtain that $I = 0$. Therefore

$$0 = (b, x)a = (bx + xb)a = bxa + xba = 2bxa.$$

We conclude that $URU = 0$. From $(RUR)^2 = 0$ and $\delta(RUR) \subseteq RUR$ for any $\delta \in \Delta$ it holds that $U = 0$.

(2) ($\Leftarrow$) If A, B are Δ-ideals of R such that $AB = 0$, then $(BA)^2 = 0$ and so BA is a Jordan ideal of R satisfying the condition

$$(BA, BA) = 0.$$

Thus the condition (4.4) is true for $U = BA$. As in the proof of the part (1), we obtain that $BA = 0$. Then A^J, B^J are Δ-ideals of a Jordan ring R^J such that

$$(A^J, B^J) = 0.$$

Hence $A = 0$ or $B = 0$.

($\Rightarrow$) Let $a_1, a_2 \in A$ and $x, y \in R$. Suppose that R^J is not Δ-prime and therefore there exist nonzero Jordan Δ-ideals A, B of R such that

$$(A, B) = 0.$$

By the same reasons as above, we conclude that $A \cap B = 0$. Then, by Lemma 1 of [7], we have $[(a_1, a_2), x] \in A$ and hence

$$[(a_1, a_2), x] \pm ((a_1, a_2), x) \in A.$$

Therefore $x(a_1, a_2)y \in A$. Thus R contains Δ-ideals $R(A, A)R \subseteq A$ and $R(B, B)R \subseteq B$ such that

$$R(A, A)R(B, B)R \subseteq A \cap B = 0.$$

Hence $(A, A) = 0$ or $(B, B) = 0$ and this leads to a contradiction.

(3) ($\Leftarrow$) If A is a nonzero Δ-ideal of R such that $A^2 = 0$, then A^J is a nonzero Δ-ideal of the Jordan ring R^J such that

$$(A^J, A^J) = 0,$$

a contradiction.

($\Rightarrow$) Suppose that R has a nonzero Jordan Δ-ideal U such that

$$(U, U) = 0.$$

Then the condition (4.4) is true for any $a, b \in U$. As in the proof of the part (1), we obtain that $U = 0$. □

If R is a ring, then on the set R we can to define a left Jordan multiplication "$\langle -, - \rangle$" by the rule

$$\langle a, b \rangle = 2ab$$

for any $a, b \in R$. Then the equalities

$$\langle\langle\langle a, a\rangle, b\rangle, a\rangle = \langle\langle a, a\rangle, \langle b, a\rangle\rangle \quad \text{and} \quad \langle\langle a, b\rangle, a\rangle = \langle a, \langle b, a\rangle\rangle$$

are true and hence

$$R^{lJ} = (R, +, \langle -, - \rangle)$$

is a non-commutative Jordan ring (which is called *a left Jordan ring associated with an associative ring* R). It is clear that, for commutative ring R, we have

$$R^J = R^{lJ}.$$

If A is an additive subgroup of R that $\langle a, r\rangle, \langle r, a\rangle \in A$ for any $a \in A$ and $r \in R$, then A is called *an ideal* of R^{lJ}. If $\delta \in \Delta$ and $a, b \in R$, then

$$\delta(\langle a, b\rangle) = \delta(2ab) = 2\delta(a)b + 2a\delta(b) = \langle \delta(a), b\rangle + \langle a, \delta(b)\rangle$$

and therefore $\delta \in \mathrm{Der}(R^{lJ})$. By the other hand, if $\delta \in \mathrm{Der}(R^{lJ})$, then

$$2\delta(ab) = \delta(\langle a, b\rangle) = \langle \delta(a), b\rangle + \langle a, \delta(b)\rangle = 2(\delta(a)b + a\delta(b)).$$

If R is a 2-torsion-free ring, then $\delta \in \mathrm{Der}\, R$. Similarly, as in Theorem 1, we can prove the following

Proposition 4. *For a 2-torsion-free ring R the following conditions are true:*

(1) *R is a Δ-simple ring if and only if R^{lJ} is a Δ-simple Jordan ring,*

(2) *R is a Δ-prime ring if and only if R^{lJ} is a Δ-prime Jordan ring,*

(3) *R is a Δ-semiprime ring if and only if R^{lJ} is a Δ-semiprime Jordan ring.*

5. Proofs

The next lemma in the prime case is contained in [18, Lemma 7].

Lemma 16 ([2, Lemma 1.7])**.** *Let R be a ring. If $[[R, R], [R, R]] = 0$, then the commutator ideal $C(R)$ is nil.*

Corollary 4. *If R is a non-commutative Δ-semiprime ring, then $[R, R]$ is non-commutative.*

Proof of Theorem 2.

(1) It is clear that a ring R is non-commutative. If A is a nonzero proper Δ-ideal of R, then A^L is a nonzero proper Δ-ideal of R^L. Therefore $A \subseteq Z(R)$ and, as a consequence, $A \cdot C(R) = 0$.

(2) Suppose that a Δ-simple ring R is non-commutative and U is its nonzero proper Lie Δ-ideal. By Proposition 2, $[U,U]=0$. Then, by Lemma 11, $U \subseteq Z(R)$. Hence the quotient ring $R^L/Z(R)$ is $\widehat{\Delta}$-simple.

(3) Let A be a nonzero Δ-ideal of R such that $A^2=0$. Then A^L is a nonzero Δ-ideal of a Lie ring R^L and, moreover,

$$[A^L, A^L] = 0.$$

By Lemma 11, $A \subseteq Z(R)$ and hence $A \cdot C(R) = 0$.

(4) Suppose that R is non-commutative. Let A be a nonzero Lie Δ-ideal of R such that $[A,A]=0$. Then, by Lemma 11, $A \subseteq Z(R)$ and, as a consequence, the Lie ring $R^L/Z(R)$ is $\widehat{\Delta}$-semiprime.

(5) Let A, B be nonzero Δ-ideals of R such that $AB=0$. Obviously, $[A,B] \subseteq Z(R)$. Then $A \subseteq Z(R)$ or $B \subseteq Z(R)$.

(6) Assume that R is non-commutative and A, B are nonzero Lie Δ-ideals of R such that

$$[A,B]=0.$$

Then $A \cap B \subseteq Z(R)$. Since $A \cap B \subseteq \operatorname{ann} C(R)$ in a Δ-prime ring R, we have that the intersection $A \cap B = 0$ is zero. If $T(A) = R$ (see proof of Proposition 2), then $[R,R] \subseteq A$ and $B \subseteq C_R([R,R])$. By Lemma 9, $B \subseteq Z(R)$. So we assume that $T(A) \neq R$. If $[T(A), T(A)] = 0$, then $[A,A]=0$ and, by Lemma 11, $A \subseteq Z(R)$. Suppose that $[T(A),T(A)] \neq 0$. By Lemma 13, $T(A)$ contains a nonzero Δ-ideal I of R. Since

$$[I,B] \subseteq A \cap B = 0,$$

we conclude that $B \subseteq Z(R)$ by Lemma 7. □

The map

$$\partial_a : R \ni x \mapsto [a,x] \in R$$

is called *an inner derivation* of a ring R induced by $a \in R$. The set $\operatorname{IDer} R$ of all inner derivations of R is a Lie ring. Every prime Lie ring is primary Lie.

Lemma 17. *There is the Lie ring isomorphism*

$$\operatorname{IDer} R \ni \partial_a \mapsto a + Z(R) \in R^L/Z(R).$$

Proof. Evident. □

Corollary 5. *Let R be a ring. Then the following statements hold:*

(1) IDer R *is a simple Lie ring if and only if* $R^L/Z(R)$ *is a simple Lie ring,*

(2) IDer R *is a prime Lie ring if and only if* $R^L/Z(R)$ *is a prime Lie ring,*

(3) IDer R *is a semiprime Lie ring if and only if* $R^L/Z(R)$ *is a semiprime Lie ring,*

(4) IDer R *is a primary Lie ring if and only if* $R^L/Z(R)$ *is a primary Lie ring.*

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