

# Constructing $R$ -sequencings and terraces for groups of even order

M. A. Ollis

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**ABSTRACT.** The problem of finding  $R$ -sequencings for abelian groups of even orders has been reduced to that of finding  $R^*$ -sequencings for abelian groups of odd orders except in the case when the Sylow 2-subgroup is a non-cyclic non-elementary-abelian group of order 8. We partially address this exception, including all instances when the group has order  $8t$  for  $t$  congruent to 1, 2, 3 or 4 (mod 7). As much is known about which odd-order abelian groups are  $R^*$ -sequenceable, we have constructions of  $R$ -sequencings for many new families of abelian groups. The construction is generalisable in several directions, leading to a wide array of new  $R$ -sequenceable and terraceable non-abelian groups of even order.

## 1. Introduction

There are several problems, usually arising from methods to construct combinatorial objects, that require elements of a finite group to be listed in a way that satisfies various constraints. In this paper we consider  $R$ -sequenceability and terraceability, the combinatorial consequences of which include constructions of graph decompositions, quasi-complete Latin squares and neighbor-balanced designs, among others.

First we look at  $R$ -sequenceability and  $R^*$ -sequenceability; secondly we see how we can relax some of the constraints to give  $R^*$ -terraces, and

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from these terraces. These results allow us to construct R-sequencings and terraces for many groups that were not previously known to possess them, edging closer to answering the longstanding questions of exactly which groups are R-sequenceable or terraceable.

We often need to consider circular lists, where the first element is taken to be to the right of the last element. As in [10], in such a case we add a hooked arrow  $(a_1, a_2, \dots, a_n \leftrightarrow)$  and calculate subscripts modulo the length of the list. Some groups we will refer to throughout the paper: Let  $\mathbb{Z}_r$  be the additively written cyclic group on the symbols  $\{0, 1, \dots, r-1\}$ , let  $D_{2r}$  be the dihedral groups of order  $2r$  defined by

$$D_{2r} = \langle u, v : u^r = e = v^2, vu = u^{-1}v \rangle,$$

let  $Q_{4r}$  be the dicyclic group of order  $4r$  defined by

$$Q_{4r} = \langle u, v : u^{2r} = e, v^2 = u^r, vu = u^{-1}v \rangle,$$

and let  $A_4$  be the alternating group on 4 symbols.

Let  $G$  be a group of order  $n$  and let  $\mathbf{a} = (a_1, a_2, \dots, a_{n-1} \leftrightarrow)$  be a circular arrangement of the non-identity elements of  $G$ . Define  $\mathbf{b} = (b_1, b_2, \dots, b_{n-1} \leftrightarrow)$  by  $b_i = a_i^{-1}a_{i+1}$  for each  $i$ . If the elements of  $\mathbf{b}$  are also all of the non-identity elements of  $G$  then  $\mathbf{b}$  is a *rotational sequencing* or *R-sequencing* of  $G$  and  $\mathbf{a}$  is the corresponding *directed rotational terrace* or *directed R-terrace* of  $G$ . If  $G$  has an R-sequencing then it is said to be *R-sequenceable*. If, in addition, we have that  $a_2a_{n-1} = a_1 = a_{n-1}a_2$  then  $\mathbf{a}$  is a *directed R\*-terrace*,  $\mathbf{b}$  is an *R\*-sequencing* and  $G$  is said to be *R\*-sequenceable*.

Inspired by a map-colouring problem of Ringel, R-sequenceability was introduced by Friedlander, Gordon and Miller in [5]. Various different definitions, all equivalent to the above, are used in the literature; see, for example, [1, 5, 9, 12, 17].

**Example 1.** The following is a directed R\*-terrace for  $\mathbb{Z}_{11}$ :

$$(5, 6, 9, 3, 7, 4, 2, 1, 8, 10 \leftrightarrow).$$

Its R\*-sequencing is

$$(1, 3, 5, 4, 8, 9, 10, 7, 2, 6 \leftrightarrow).$$

Much is known about the R-sequenceability of abelian groups. Friedlander, Gordon and Miller [5] conjecture that the only abelian groups that

are not R-sequenceable are those with exactly one involution (which they prove cannot be R-sequenced). For even-order abelian groups the only groups for which the conjecture is open are those with non-cyclic Sylow 2-subgroups of order 8. Some new infinite families of groups whose Sylow 2-subgroups are isomorphic to  $\mathbb{Z}_4 \times \mathbb{Z}_2$  are shown to be R-sequenceable in Section 3, including those of order  $8t$  with an  $R^*$ -sequenceable subgroup of order  $t$  with  $t$  congruent to 1, 2, 3 or 4 (mod 7).

In the non-abelian case several infinite families of R-sequenceable groups are known, including dihedral and dicyclic groups  $D_{2n}$  and  $Q_{4n}$  when  $n$  is even and  $n > 2$ . See [11] for a recent survey of results. More are added in Section 3, including groups of the form  $H_1 \times H_2 \times \cdots \times H_s \times K$ , where each  $H_i$  is one of,  $\mathbb{Z}_2^2$ ,  $\mathbb{Z}_2^3$ ,  $D_8$ ,  $\mathbb{Z}_6 \times \mathbb{Z}_2$ ,  $D_{12}$  or  $A_4$  and  $K$  is  $R^*$ -sequenceable.

Again, let  $G$  be a group of order  $n$ , but now let  $\mathbf{a} = (a_1, a_2, \dots, a_n)$  be a linear arrangement of the elements of  $G$ . Define  $\mathbf{b} = (b_1, b_2, \dots, b_{n-1})$  by  $b_i = a_i^{-1} a_{i+1}$  for each  $i$ . If  $\mathbf{b}$  contains one occurrence of each involution of  $G$  and exactly two occurrences of elements from each set

$$\{g, g^{-1} : g^2 \neq e\}$$

then  $\mathbf{a}$  is a *terrace* for  $G$  and  $\mathbf{b}$  is its associated *2-sequencing*.

**Example 2.** The following is a terrace for  $\mathbb{Z}_{11}$ :

$$(0, 2, 1, 8, 10, 5, 6, 9, 3, 7, 4).$$

Its 2-sequencing is

$$(2, 10, 7, 2, 6, 1, 3, 5, 4, 8).$$

Terraces were introduced by Bailey [4] as a tool for constructing quasi-complete Latin squares; similar ideas had been used earlier by Williams [21] (restricted to cyclic groups) and Gordon [7] (in the case of *directed terraces*; those whose 2-sequencings have no repeated entries, in which case they are called *sequencings*). *Bailey's Conjecture* is that all groups other than non-cyclic elementary abelian 2-groups are terraced (it is known that non-cyclic elementary abelian 2-groups cannot be terraced [4]). This was proven for abelian groups in [16] and many nonabelian groups are known to have terraces. See [11] for more details on these topics. In Section 4 we add more groups, including direct products comprised of arbitrarily many non-cyclic, non-dicyclic groups of order 12, an  $R^*$ -sequenceable group and, optionally, a group of odd order.

For our constructions we are interested in direct and central factors of a group. If a group  $G$  can be written as a direct product  $H \times K$  then  $H$  is a *direct factor* of  $G$ . More generally, suppose  $H \trianglelefteq G$  and let

$$C_G(H) = \{g \in G : gh = hg \text{ for all } h \in H\}$$

be the centralizer of  $H$  in  $G$ . If  $G = HC_G(H)$  then  $H$  is a *central factor* of  $G$ . Direct factors are also central factors but central factors are not necessarily direct factors.

In the next section we give the main construction on which all the results rely. In Section 3 we see how it can be used to produce R-sequencings and in Section 4 we consider how it can be adapted to produce terraces.

## 2. The construction

We present the *main construction* for a circular sequence of the non-identity elements of our target group  $G$ , which has order  $n = 4mt$  and is of the form  $H \times K$  with  $|H| = 4m$  and  $|K| = t$ .

Given a circular sequence  $\mathbf{a} = (a_1, a_2, \dots, a_{4m-1} \leftrightarrow)$  of the non-identity elements of  $H$ , a permutation  $\sigma \in S_{4m-1}$ , and a circular sequence  $\mathbf{k} = (k_1, k_2, \dots, k_{t-1} \leftrightarrow)$  of the non-identity elements of  $K$  with  $k_{t-1}k_2 = k_1 = k_2k_{t-1}$ , we construct a sequence in  $H \times K$  from  $4m + 1$  subsequences. Let  $\mathbf{b} = (b_1, b_2, \dots, b_{4m-1} \leftrightarrow)$  be the quotients associated with  $\mathbf{a}$ ; that is,  $b_i = a_i^{-1}a_{i+1}$  for each  $i$ . Similarly, let  $\ell = (\ell_1, \ell_2, \dots, \ell_{t-1} \leftrightarrow)$  be the quotients associated with  $\mathbf{k}$ ; so  $\ell_i = k_i^{-1}k_{i+1}$  for each  $i$ .

In practice,  $\mathbf{a}$  will always be a directed R-terrace. In the next section  $\mathbf{k}$  will be a directed R\*-terrace and in Section 4 it will be a weaker object, an "R-terrace".

The first three subsequences each have distinct characteristics. These are followed by  $2m - 1$  that follow one pattern and then  $2m - 2$  that follow a slightly different one. The final subsequence has just one element. We define them in turn, noting the internal quotients that they generate as we go, and then consider the quotients generated at the joins.

Note that in calculating the quotients we make use of the condition  $k_{t-1}k_2 = k_1 = k_2k_{t-1}$ . In particular, we use that  $k_2 = \ell_{t-1}$  and  $k_{t-1}^{-1} = \ell_1$ . Also, recall that for circular lists subscripts are calculated modulo the length of the list.

The first subsequence is

$$(e, k_1), (e, k_2), \dots, (e, k_{t-2})$$

which has internal quotients

$$(e, \ell_1), (e, \ell_2), \dots, (e, \ell_{t-3}).$$

The second subsequence is

$$\begin{aligned} &(a_{\sigma(1)-2m}, k_{t-1}), (a_{\sigma(1)-2m+1}, k_1), (a_{\sigma(1)-2m+2}, k_1), \\ &\quad (a_{\sigma(1)-2m+3}, k_1), \dots, (a_{\sigma(1)}, k_1), (a_{\sigma(1)+1}, k_1), \\ &\quad (a_{\sigma(1)+2}, k_2), (a_{\sigma(1)+3}, k_3), \dots, (a_{\sigma(1)+t-2}, k_{t-2}) \end{aligned}$$

which has internal quotients

$$\begin{aligned} &(b_{\sigma(1)-2m}, \ell_{t-1}), (b_{\sigma(1)-2m+1}, e), (b_{\sigma(1)-2m+2}, e), \\ &\quad (b_{\sigma(1)-2m+3}, e), \dots, (b_{\sigma(1)}, e), (b_{\sigma(1)+1}, \ell_1), \\ &\quad (b_{\sigma(1)+2}, \ell_2), (b_{\sigma(1)+3}, \ell_3), \dots, (b_{\sigma(1)+t-3}, \ell_{t-3}). \end{aligned}$$

The third subsequence is

$$\begin{aligned} &(a_{\sigma(2)-2m+1}, k_{t-1}), (a_{\sigma(1)-2m+2}, e), (a_{\sigma(1)-2m+3}, e), \\ &\quad (a_{\sigma(1)-2m+4}, e), \dots, (a_{\sigma(2)}, e), (a_{\sigma(2)+1}, e), \\ &\quad (a_{\sigma(2)+2}, k_2), (a_{\sigma(3)+3}, k_3), \dots, (a_{\sigma(2)+t-2}, k_{t-2}) \end{aligned}$$

which has internal quotients

$$\begin{aligned} &(b_{\sigma(2)-2m+1}, \ell_1), (b_{\sigma(2)-2m+2}, e), (b_{\sigma(2)-2m+3}, e), \\ &\quad (b_{\sigma(2)-2m+4}, e), \dots, (b_{\sigma(2)}, e), (b_{\sigma(2)+1}, \ell_{t-1}), \\ &\quad (b_{\sigma(2)+2}, \ell_2), (b_{\sigma(2)+3}, \ell_3), \dots, (b_{\sigma(2)+t-3}, \ell_{t-3}). \end{aligned}$$

For  $i$  in the range  $4 \leq i \leq 2m + 2$ , the  $i$ th subsequence is

$$\begin{aligned} &(a_{\sigma(i-1)}, k_{t-1}), (a_{\sigma(i-1)+1}, e), (a_{\sigma(i-1)+2}, k_2), \\ &\quad (a_{\sigma(i-1)+3}, k_3), (a_{\sigma(i-1)+4}, k_4), \dots, (a_{\sigma(i-1)+t-2}, k_{t-2}) \end{aligned}$$

which has internal quotients

$$\begin{aligned} &(b_{\sigma(i-1)}, \ell_1), (b_{\sigma(i-1)+1}, \ell_{t-1}), (b_{\sigma(i-1)+2}, \ell_2), \\ &\quad (b_{\sigma(i-1)+3}, \ell_3), (b_{\sigma(i-1)+4}, \ell_4), \dots, (b_{\sigma(i-1)+t-3}, \ell_{t-3}). \end{aligned}$$

For  $i$  in the range  $2m + 3 \leq i \leq 4m$ , the  $i$ th subsequence is

$$(a_{\sigma(i-1)}, k_{t-1}), (a_{\sigma(i-1)+1}, k_1), (a_{\sigma(i-1)+2}, k_2), \\ (a_{\sigma(i-1)+3}, k_3), (a_{\sigma(i-1)+4}, k_4), \dots, (a_{\sigma(i-1)+t-2}, k_{t-2})$$

which has internal quotients

$$(b_{\sigma(i-1)}, \ell_{t-1}), (b_{\sigma(i-1)+1}, \ell_1), (b_{\sigma(i-1)+2}, \ell_2), \\ (b_{\sigma(i-1)+3}, \ell_3), (b_{\sigma(i-1)+4}, \ell_4), \dots, (b_{\sigma(i-1)+t-3}, \ell_{t-3}).$$

Note that the only difference in the structure of these subsequences compared to the previous ones is in the second coordinate of the second element, meaning that the only changes in structure in the quotients are in the second coordinates of the first and second elements. Also note that there are no subsequences of this form when  $m = 1$ .

The final subsequence consists of the single element  $(e, k_{t-1})$ . Of course, this gives rise to no internal quotients.

The quotients generated where the subsequences join are

$$(a_{\sigma(1)-2m}, \ell_{t-2}), (a_{\sigma(1)+t-1}^{-1} a_{\sigma(2)-2m+1}, \ell_{t-2}), \\ (a_{\sigma(2)+t-1}^{-1} a_{\sigma(3)}, \ell_{t-2}), (a_{\sigma(3)+t-1}^{-1} a_{\sigma(4)}, \ell_{t-2}), \dots, \\ (a_{\sigma(4m-2)+t-1}^{-1} a_{\sigma(4m-1)}, \ell_{t-2}), (a_{\sigma(4m-1)+t-1}^{-1}, \ell_{t-2})$$

(the fourth to the penultimate one, inclusive, are excluded when  $m = 1$ ). Finally,  $(e, \ell_{t-1})$  is the quotient generated between the last subsequence and the first.

When we come to prove that the main construction gives directed  $R^*$ -terraces and other similar objects, we will see that the permutation  $\sigma$  is responsible for lining up the subsequences in such a way that all of the properties we need are satisfied. In order to do this successfully, we also need constraints on the permutation.

Say that  $\sigma \in S_{4m-1}$  is *admissible* if  $\sigma(2) = \sigma(1) - 2m$  and

$$\{\sigma(3), \sigma(4), \dots, \sigma(2m + 1)\} = \{\sigma(2) + 1, \sigma(2) + 2, \dots, \sigma(2) + 2m - 1\}$$

where all calculations are performed modulo  $4m - 1$ .

For a positive integer  $t$ , the pair  $\mathbf{a}$  and  $\sigma$  are *t-compatible* if the following  $4m$  elements are distinct (i.e. are all of  $H$ ):

$$a_{\sigma(1)-2m}, a_{\sigma(1)+t-1}^{-1} a_{\sigma(2)-2m+1}, a_{\sigma(4m-1)+t-1}^{-1}$$

and

$$a_{\sigma(i)+t-1}^{-1} a_{\sigma(i+1)}$$

for each  $i$  with  $1 < i < 4m - 1$ .

### 3. R-sequencings

We can now prove the main results for R-sequencings. Theorem 1 gives the case where  $G$  has a direct factor of order a multiple of 4, which is sufficient for the abelian group case, and Theorem 4 gives the variant for a central factor.

**Theorem 1.** *Let  $G = H \times K$  with  $|H| = 4m$  and  $|K| = t$ . If  $H$  has an R-sequencing with a  $t$ -compatible  $\sigma \in S_{4m-1}$  and  $K$  is  $R^*$ -sequenceable then  $G$  is  $R^*$ -sequenceable.*

*Proof.* Let  $\mathbf{a}$  be the directed R-terrace of  $H$  and  $\mathbf{k}$  be the directed  $R^*$ -terrace of  $K$ , with the usual notation for their elements and quotients. Apply the main construction to get a circular sequence of elements in  $G$  and their quotients. We check that all elements of  $H$  appear with each element of  $K$  (with the exception that  $(e, e)$  does not appear) in each of the sequence and its quotients.

The elements that appear with  $k_1$  in the sequence are:

$$e, a_{\sigma(1)-2m+1}, a_{\sigma(1)-2m+2}, \dots, a_{\sigma(1)}, a_{\sigma(1)+1}, \\ a_{\sigma(2m+2)+1}, a_{\sigma(2m+3)+1}, \dots, a_{\sigma(4m-1)+1}.$$

When  $m > 1$ , the admissibility of  $\sigma$  implies that the two sets

$$\{\sigma(2m + 2), \sigma(2m + 3), \dots, \sigma(4m - 1)\}$$

and

$$\{\sigma(1) + 1, \sigma(1) + 2, \dots, \sigma(1) + 2m - 2\}$$

are equal as each has all of the numbers from 1 to  $4m - 1$  except for

$$\{\sigma(2), \sigma(2) + 1, \sigma(2) + 2, \dots, \sigma(2) + 2m\}$$

(recall that these calculations are performed modulo  $4m - 1$ ). Applying this to the last  $2m - 2$  elements we see that the sequence contains all of the elements of  $H$ . When  $m = 1$  we have the elements  $a_{\sigma(1)-1}, a_{\sigma(1)}, a_{\sigma(1)+1}$  which are distinct.

The elements that appear with  $k_j$ , for  $2 \leq j \leq t-2$  are:

$$e, a_{\sigma(1)+j}, a_{\sigma(2)+j}, \dots, a_{\sigma(4m-1)+j}$$

which comprise all of the elements of  $H$ .

The elements that appear with  $k_{t-1}$  are:

$$a_{\sigma(1)-2m}, a_{\sigma(2)-2m+1}, a_{\sigma(3)}, a_{\sigma(4)}, \dots, a_{\sigma(4m-1)}.$$

Applying the first clause of the admissibility definition to the first two elements we see that these are all of the non-identity elements of  $H$ .

The elements that appear with  $e$  are:

$$a_{\sigma(2)-2m+2}, a_{\sigma(2)-2m+3}, a_{\sigma(2)-2m+4}, \dots, a_{\sigma(2)},$$

$$a_{\sigma(2)+1}, a_{\sigma(3)+1}, a_{\sigma(4)+1}, \dots, a_{\sigma(2m+1)+1}.$$

Applying the second clause of the admissibility definition to the last  $2m-1$  elements we see that these are all of the non-identity elements of  $H$ .

Turning to the sequence of quotients, the elements that appear with  $\ell_1$  are:

$$e, b_{\sigma(1)+1}, b_{\sigma(2)-2m+1}, b_{\sigma(3)}, b_{\sigma(4)}, \dots, b_{\sigma(2m+1)},$$

$$b_{\sigma(2m+2)+1}, b_{\sigma(2m+3)+1}, \dots, b_{\sigma(4m-1)+1}.$$

The admissibility of  $\sigma$  implies that

$$\{\sigma(2), \sigma(2m+2), \sigma(2m+3), \dots, \sigma(4m-1)\} =$$

$$\{\sigma(1)+1, \sigma(2m+2)+1, \sigma(2m+3)+1, \dots, \sigma(4m-1)+1\}.$$

Coupled with the first clause of the admissibility definition applied to the third element we see that the sequence contains all of the elements of  $H$ .

The elements that appear with  $\ell_j$ , for  $2 \leq j \leq t-3$  are:

$$e, b_{\sigma(1)+j}, b_{\sigma(2)+j}, \dots, b_{\sigma(4m-1)+j}.$$

These are all of the elements of  $H$ .

The elements that appear with  $\ell_{t-2}$  are:

$$a_{\sigma(1)-2m}, a_{\sigma(1)+t-1}^{-1} a_{\sigma(2)-2m+1}, a_{\sigma(2)+t-1}^{-1} a_{\sigma(3)},$$

$$a_{\sigma(3)+t-1}^{-1} a_{\sigma(4)}, \dots, a_{\sigma(4m-2)+t-1}^{-1} a_{\sigma(4m-1)}, a_{\sigma(4m-1)+t-1}.$$



As  $\mathbf{a}$  and  $\sigma$  are  $t$ -compatible, these are all of the elements of  $H$ .

The elements that appear with  $\ell_{t-1}$  are:

$$b_{\sigma(1)-2m}, b_{\sigma(2)+1}, b_{\sigma(3)+1}, \dots, b_{\sigma(2m+1)+1},$$

$$b_{\sigma(2m+2)}, b_{\sigma(2m+3)}, \dots, b_{\sigma(4m-1)}, e.$$

We again use that

$$\{\sigma(2), \sigma(2m + 2), \sigma(2m + 3), \dots, \sigma(4m - 1)\} =$$

$$\{\sigma(1) + 1, \sigma(2m + 2) + 1, \sigma(2m + 3) + 1, \dots, \sigma(4m - 1) + 1\}$$

and the first clause of the admissibility definition, this time applied to the first element. Doing so, we see that the sequence contains all of the elements of  $H$ .

The elements that appear with  $e$  are:

$$b_{\sigma(1)-2m+1}, b_{\sigma(1)-2m+2}, \dots, b_{\sigma(1)}, b_{\sigma(2)-2m+2}, b_{\sigma(2)-2m+3}, \dots, b_{\sigma(2)}.$$

Using the first clause of the admissibility definition we see that these are all of the non-identity elements of  $H$ .

This shows that our sequence is a directed  $R$ -terrace. Finally, observe that the first two elements of our sequence are  $(e, k_1)$  and  $(e, k_2)$  and the last is  $(e, k_{t-1})$ . Therefore, that  $\mathbf{k}$  is a directed  $R^*$ -terrace of  $K$  implies that our sequence is a directed  $R^*$ -terrace of  $G$ .  $\square$

**Theorem 2.** *Let  $A$  be an abelian group such that  $A \cong S \times T$  where  $S$  is a Sylow 2-subgroup that is isomorphic to  $\mathbb{Z}_4 \times \mathbb{Z}_2$  and  $T$  has order congruent to 1, 2, 3 or 4 (mod 7). If  $T$  is  $R^*$ -sequenceable then  $A$  is  $R^*$ -sequenceable.*

*Proof.* To apply Theorem 1 for each desired value of  $t$  we require an  $R$ -sequencing for  $\mathbb{Z}_4 \times \mathbb{Z}_2$  along with a  $t$ -compatible  $\sigma$ . The following sequences and permutations do what is required:

$$t \equiv 1 \pmod{7}, \quad \sigma = (1, 6, 7),$$

$$\mathbf{a} = (0, 1), (2, 1), (1, 0), (2, 0), (3, 1), (3, 0), (1, 1).$$

$$t \equiv 2 \pmod{7}, \quad \sigma = (1, 4)(2, 7, 5),$$

$$\mathbf{a} = (1, 0), (2, 0), (1, 1), (0, 1), (2, 1), (3, 0), (3, 1).$$

$t \equiv 3 \pmod{7}$ ,  $\sigma = (1, 6, 7)$ ,

$$\mathbf{a} = (0, 1), (1, 0), (2, 0), (1, 1), (3, 0), (3, 1), (2, 1).$$

$t \equiv 4 \pmod{7}$ ,  $\sigma = (1, 7)(2, 3, 6)$ ,

$$\mathbf{a} = (1, 0), (1, 1), (2, 0), (3, 0), (2, 1), (0, 1), (3, 1).$$

We can therefore construct the R-sequencing for  $A$ . □

A computer search has shown that there are no satisfactory  $\mathbf{a}$  and  $\sigma$  for other values of  $t \pmod{7}$ . We can now find R-sequencings for new families of abelian groups whose Sylow 2-subgroups are non-cyclic of order 8, the only open cases in the even-order question for abelian groups:

**Corollary 1.** *Let  $K$  be an abelian group with  $|K| > 5$ . If  $|K|$  is congruent to 1, 3, 9 or 11  $\pmod{14}$  and the Sylow 3-subgroups of  $K$  are isomorphic to  $\mathbb{Z}_3^\alpha \times \mathbb{Z}_9^\alpha \times \mathbb{Z}_{27}^\beta \times \mathbb{Z}_{81}^\gamma$  or  $\mathbb{Z}_3^\alpha \times \mathbb{Z}_9^\alpha \times \mathbb{Z}_{27}^\beta \times \mathbb{Z}_{81}^\gamma \times \mathbb{Z}_{3^t}$ , where  $t > 1$  and  $t \equiv \alpha + \beta \pmod{2}$ , then  $\mathbb{Z}_4 \times \mathbb{Z}_2 \times K$  is  $R^*$ -sequenceable. In particular, if  $|K|$  is congruent to one of 1, 11, 17, 23, 25, 29, 31, or 37  $\pmod{42}$  then  $\mathbb{Z}_4 \times \mathbb{Z}_2 \times K$  is  $R^*$ -sequenceable.*

*Proof.* The group  $K$  is  $R^*$ -sequenceable [5, 10, 16] and hence we can apply Theorem 1 with  $H = \mathbb{Z}_4 \times \mathbb{Z}_2$ . The last sentence describes the cases where the Sylow 3-subgroups of  $K$  are trivial. □

Further, any progress on finding  $R^*$ -sequencings for odd-order groups with Sylow 3-subgroups other than those described in Corollary 1 can now be translated directly into solving more even-order cases by the same method. For example, it is known that for any abelian 3-group  $T$  there are infinitely many  $R^*$ -sequenceable abelian groups whose Sylow 3-subgroups are isomorphic to  $T$  [10].

Theorem 1 generalises the methods of [8] and [16] which are limited to the cases  $H = \mathbb{Z}_2^2$  and  $H = \mathbb{Z}_2^3$ . However, when  $m = 1$  and  $t \equiv 0 \pmod{3}$  it is impossible to achieve  $t$ -compatibility.

In this case, Headley [8] uses a slightly different construction which also works in our more general set-up; we will refer to this as the *Headley construction*. Given a circular sequence  $\mathbf{a} = [a_1, a_2, a_3]$  of the non-identity elements of  $H$  and a circular sequence  $\mathbf{k} = [k_1, k_2, \dots, k_{t-1}]$  of the elements of  $K$  with  $k_{t-1}k_2 = k_1 = k_2k_{t-1}$ , we again construct a sequence in  $H \times K$ . The first line, the second and third line combined, and the fourth line each have  $t$  elements and the fifth line has  $t - 1$ . Recall that subscripts

in  $\mathbf{a}$  are calculated modulo 3; we leave them unreduced in order to make the structure clearer:

$$\begin{aligned}
 &(e, k_1), (e, k_2), \dots, (e, k_{t-2}), (a_2, k_{t-1}), (a_1, e), \\
 &(a_3, k_2), (a_1, k_3), (a_2, k_4), \dots, (a_{t-6}, k_{t-4}), \\
 &(a_1, k_{t-3}), (a_3, k_{t-2}), (a_3, k_{t-1}), (a_2, k_1), (a_1, k_1), \\
 &(a_3, k_1), (a_2, k_2), (a_3, k_3), \dots, (a_{t-3}, k_{t-3}), (a_2, k_{t-2}), (a_1, k_{t-1}), (a_3, e), \\
 &(a_2, e), (a_1, k_2), (a_2, k_3), \dots, (a_2, k_{t-3}), (a_1, k_{t-2}), (e, k_{t-1}).
 \end{aligned}$$

**Theorem 3.** *Let  $G = H \times K$  with  $|H| = 4$  and  $|K| = t$ , where  $t \equiv 0 \pmod{3}$ . If  $H$  has an  $R$ -sequencing and  $K$  is  $R^*$ -sequenceable then  $G$  is  $R^*$ -sequenceable.*

*Proof.* Headley’s construction as described above gives the required directed  $R^*$ -terrace when  $\mathbf{a}$  is a directed  $R$ -terrace for  $H$  and  $\mathbf{k}$  is a directed  $R^*$ -terrace for  $K$ . Checking the sequence and the quotients is a similar (but more straightforward) process to the proof of Theorem 1.  $\square$

Following the approach of [15], we may relax the condition that  $H$  be a direct factor to it being a central factor if we add in conditions on the directed  $R^*$ -terrace of its quotient group.

**Theorem 4.** *Let  $G$  be a group of order  $4mt$  with central factor  $H$  of order  $4m$ . If  $H$  has a directed  $R$ -terrace  $\mathbf{a}$  with a  $t$ -compatible  $\sigma \in S_{4m-1}$  and  $G/H$  has a directed  $R^*$ -terrace  $[K_1, K_2, \dots, K_{t-1}]$  such that there are elements  $k_2 \in K_2$  and  $k_{t-1} \in K_{t-1}$  that commute, then  $G$  is  $R^*$ -sequenceable. If  $m = 1$  and  $t \equiv 0 \pmod{3}$  then the requirement for a  $t$ -compatible permutation  $\sigma$  may be dropped.*

*Proof.* Let  $k_1 = k_2k_{t-1}$  and for each  $i$  with  $3 \leq i \leq t - 2$  choose  $k_i \in K_i \cap C_G(H)$  (as  $HC_G(H) = G$ , the set  $K_i \cap C_G(H)$  must be non-empty). Each element of  $G$  is expressible in the form  $hk_i$  for a unique  $h \in H$  and we have that  $k_ih = hk_i$  for all  $i$  and all  $h \in H$ .

Now apply the main construction or Headley’s construction as appropriate to  $\mathbf{a}$  and  $[k_1, k_2, \dots, k_{t-1}]$ , with elements  $(h, k)$  of  $G$  replaced with  $hk$  throughout. Thanks to the commutativity of the elements of  $H$  with the  $k_i$ , the argument goes through exactly as in Theorem 1 or 3.  $\square$

We now turn to which groups it is possible to use in the role of  $H$  in Theorems 1 and 4. We present here one possible pair of directed  $R$ -terrace  $\mathbf{a}$  and permutation  $\sigma$  for the values of  $t$  modulo  $4m - 1$  for which they exist for the groups  $D_8, \mathbb{Z}_6 \times \mathbb{Z}_2, D_{12}$  and  $A_4$ . The group  $\mathbb{Z}_2^2$  with

$t \not\equiv 0 \pmod{3}$  is covered in [8] and  $\mathbb{Z}_2^3$  is covered in [16]. Recall that cyclic groups of even order and  $Q_8$  and  $Q_{12}$  are not R-sequenceable.

These directed R-terraces and permutations were found using the group-theory software package GAP [6].

**The case  $H = D_8$ .**

- $t \equiv 0 \pmod{7}, \sigma = (1, 6), \mathbf{a} = u, v, u^3, u^2, u^2v, uv, u^3v.$   
 $t \equiv 1 \pmod{7}, \sigma = (1, 6), \mathbf{a} = v, u^3v, u^2, u, uv, u^3, u^2v.$   
 $t \equiv 2 \pmod{7}, \sigma = (1, 7)(2, 3, 6), \mathbf{a} = u, v, u^2, u^3, u^3v, uv, u^2v.$   
 $t \equiv 3 \pmod{7}, \sigma = (1, 6, 7), \mathbf{a} = v, u^3v, u^2, u, uv, u^3, u^2v.$   
 $t \equiv 4 \pmod{7}, \sigma = (1, 7)(2, 3, 6), \mathbf{a} = v, u, u^2v, u^2, u^3, uv, u^3v.$   
 $t \equiv 5 \pmod{7}, \sigma = (1, 5, 2)(3, 4), \mathbf{a} = v, u^2v, u^3v, u^2, u^3, uv, u.$   
 $t \equiv 6 \pmod{7}, \sigma = (1, 5, 2), \mathbf{a} = v, u, u^2v, u^2, u^3, uv, u^3v.$

**The case  $H = \mathbb{Z}_6 \times \mathbb{Z}_2$ .**

- $t \equiv 0 \pmod{11}, \sigma = (1, 7, 2)(4, 5, 6)(10, 11),$   
 $\mathbf{a} = (2, 0), (4, 0), (2, 1), (3, 0), (5, 1), (3, 1), (4, 1), (1, 0), (1, 1), (0, 1), (5, 0).$   
 $t \equiv 1 \pmod{11}, \sigma = (1, 6, 3, 2, 11, 8, 9, 7)(4, 5),$   
 $\mathbf{a} = (1, 0), (0, 1), (2, 1), (5, 1), (2, 0), (4, 1), (3, 1), (4, 0), (5, 0), (3, 0), (1, 1).$   
 $t \equiv 2 \pmod{11}, \sigma = (1, 2, 7, 8, 6)(3, 10, 5, 11, 4, 9),$   
 $\mathbf{a} = (1, 0), (2, 1), (2, 0), (1, 1), (5, 1), (3, 0), (5, 0), (4, 0), (0, 1), (3, 1), (4, 1).$   
 $t \equiv 3 \pmod{11}, \sigma = (1, 6, 5, 2, 11, 9, 8, 10, 7, 3),$   
 $\mathbf{a} = (1, 0), (4, 0), (2, 1), (3, 1), (5, 0), (0, 1), (4, 1), (3, 0), (2, 0), (5, 1), (1, 1).$   
 $t \equiv 4 \pmod{11}, \sigma = (2, 6, 7, 8)(3, 9)(4, 11)(5, 10),$   
 $\mathbf{a} = (1, 0), (5, 0), (0, 1), (4, 0), (1, 1), (3, 1), (4, 1), (3, 0), (2, 0), (2, 1), (5, 1).$   
 $t \equiv 5 \pmod{11}, \sigma = (1, 10, 3, 8, 11)(2, 4, 9)(5, 7),$   
 $\mathbf{a} = (0, 1), (1, 0), (2, 0), (4, 1), (4, 0), (3, 1), (2, 1), (5, 1), (1, 1), (5, 0), (3, 0).$   
 $t \equiv 6 \pmod{11}, \sigma = (1, 8, 9, 10, 11)(3, 7, 6),$   
 $\mathbf{a} = (1, 0), (2, 0), (1, 1), (5, 0), (4, 0), (0, 1), (2, 1), (5, 1), (3, 1), (3, 0), (4, 1).$

$$t \equiv 7 \pmod{11}, \sigma = (1, 7, 5, 2)(3, 6, 4)(8, 11),$$

$$\mathbf{a} = (0, 1), (2, 0), (1, 0), (3, 0), (3, 1), (1, 1), (4, 1), (5, 1), (4, 0), (2, 1), (5, 0).$$

$$t \equiv 8 \pmod{11}, \sigma = (1, 3, 9, 5, 2, 8, 7, 11, 4)(6, 10),$$

$$\mathbf{a} = (1, 0), (0, 1), (5, 1), (3, 0), (4, 0), (1, 1), (4, 1), (2, 1), (2, 0), (3, 1), (5, 0)$$

$$t \equiv 9 \pmod{11}, \sigma = (1, 3, 2, 8, 4, 10, 6)(5, 9)(7, 11),$$

$$\mathbf{a} = (1, 0), (4, 0), (2, 1), (3, 1), (5, 0), (0, 1), (4, 1), (3, 0), (2, 0), (5, 1), (1, 1).$$

$$t \equiv 10 \pmod{11}, \sigma = (1, 2, 7, 8, 6, 9, 3, 11, 5)(4, 10),$$

$$\mathbf{a} = (2, 0), (2, 1), (0, 1), (3, 0), (4, 0), (5, 1), (4, 1), (1, 1), (5, 0), (1, 0), (3, 1).$$

**The case  $H = D_{12}$ .**

$$t \equiv 0 \pmod{11}, \sigma = (1, 7, 5, 4, 6, 3, 2)(8, 9, 11, 10),$$

$$\mathbf{a} = v, u, u^4v, uv, u^5v, u^3, u^4, u^2, u^2v, u^3v, u^5.$$

$$t \equiv 1 \pmod{11}, \sigma = (1, 4, 2, 9, 7, 11, 8, 5, 10, 6),$$

$$\mathbf{a} = u^3, v, u, u^3v, uv, u^2v, u^4, u^2, u^5, u^5v, u^4v.$$

$$t \equiv 2 \pmod{11}, \sigma = (2, 6, 10, 5, 11, 3, 8)(4, 7, 9),$$

$$\mathbf{a} = u^2, u, u^5, v, u^4v, u^3v, u^3, u^5v, u^2v, u^4, uv.$$

$$t \equiv 3 \pmod{11}, \sigma = (1, 6, 2, 11, 7, 5, 3, 4)(8, 10),$$

$$\mathbf{a} = v, u^4v, u, u^2v, u^4, u^5, u^2, uv, u^3v, u^3, u^5v.$$

$$t \equiv 4 \pmod{11}, \sigma = (1, 5)(2, 10, 9, 7, 4, 3, 11, 6),$$

$$\mathbf{a} = v, u^5v, u^2, u^4, u^3, u^2v, u^4v, uv, u, u^3v, u^5.$$

$$t \equiv 5 \pmod{11}, \sigma = (1, 10, 3, 6, 8, 11, 2, 4, 9),$$

$$\mathbf{a} = v, u^2, u^5, u^5v, u^4, u^3, u^2v, u^4v, u, u^3v, uv.$$

$$t \equiv 6 \pmod{11}, \sigma = (1, 10, 11, 3, 6, 5, 9, 2, 4, 8),$$

$$\mathbf{a} = u^2, u, u^3, v, u^4, u^5v, u^5, u^3v, u^2v, u^4v, uv.$$

$$t \equiv 7 \pmod{11}, \sigma = (2, 6, 9, 4, 8)(3, 10)(5, 11),$$

$$\mathbf{a} = v, u^5v, uv, u^3, u^3v, u^2, u^4v, u^2v, u^5, u^4, u.$$

$$t \equiv 8 \pmod{11}, \sigma = (2, 6, 10)(3, 11, 4, 9, 5, 8),$$

$$\mathbf{a} = u, v, u^2v, uv, u^4v, u^2, u^5v, u^4, u^3, u^3v, u^5.$$

$$t \equiv 9 \pmod{11}, \sigma = (1, 4, 10, 8, 5, 11, 6, 3, 2, 9, 7),$$

$$\mathbf{a} = u^2, u^4, v, u^5, u^3v, u^2v, u^3, u, uv, u^4v, u^5v.$$

$$t \equiv 10 \pmod{11}, \sigma = (2, 6, 8)(3, 9, 5, 7, 10)(4, 11),$$

$$\mathbf{a} = u^3, v, u^4v, uv, u^5, u^5v, u, u^2v, u^3v, u^4, u^2.$$

**The case  $H = A_4$ .**

$$t \equiv 0 \pmod{11}, \sigma = (1, 6, 5)(2, 11, 7, 3, 4)(8, 10, 9),$$

$$\mathbf{a} = (2, 3, 4), (1, 2)(3, 4), (1, 3, 2), (1, 3)(2, 4), (1, 4, 2), (1, 3, 4),$$

$$(1, 2, 3), (1, 4, 3), (1, 2, 4), (2, 4, 3), (1, 4)(2, 3).$$

$$t \equiv 1 \pmod{11}, \sigma = (1, 8)(4, 5, 6)(9, 10, 11),$$

$$\mathbf{a} = (2, 3, 4), (1, 2, 4), (1, 4, 3), (1, 3, 4), (1, 3, 2), (1, 2)(3, 4),$$

$$(1, 2, 3), (1, 3)(2, 4), (2, 4, 3), (1, 4, 2), (1, 4)(2, 3).$$

$$t \equiv 2 \pmod{11}, \sigma = (1, 3)(2, 8, 4, 9, 6, 10, 7)(5, 11),$$

$$\mathbf{a} = (1, 2)(3, 4), (2, 3, 4), (1, 3)(2, 4), (1, 3, 4), (1, 4, 2), (1, 2, 3),$$

$$(1, 4, 3), (1, 4)(2, 3), (1, 2, 4), (1, 3, 2), (2, 4, 3).$$

$$t \equiv 3 \pmod{11}, \sigma = (1, 9, 2, 3, 7, 4, 8, 11, 10)(5, 6),$$

$$\mathbf{a} = (2, 3, 4), (2, 4, 3), (1, 2, 4), (1, 4, 2), (1, 2, 3), (1, 4, 3),$$

$$(1, 3, 4), (1, 3, 2), (1, 4)(2, 3), (1, 3)(2, 4), (1, 2)(3, 4).$$

$$t \equiv 4 \pmod{11}, \sigma = (1, 5, 2, 10, 8, 6, 11, 7)(3, 4),$$

$$\mathbf{a} = (2, 3, 4), (1, 2, 4), (1, 3, 2), (1, 3)(2, 4), (1, 2, 3), (1, 4, 2), \\ (1, 4, 3), (1, 3, 4), (1, 4)(2, 3), (2, 4, 3), (1, 2)(3, 4).$$

$$t \equiv 5 \pmod{11}, \sigma = (1, 3)(2, 8, 4, 9, 7)(5, 11)(6, 10),$$

$$\mathbf{a} = (2, 3, 4), (1, 2, 3), (1, 3, 4), (1, 2)(3, 4), (2, 4, 3), (1, 3, 2), \\ (1, 4, 2), (1, 4, 3), (1, 2, 4), (1, 4)(2, 3), (1, 3)(2, 4).$$

$$t \equiv 6 \pmod{11}, \sigma = (1, 2, 7, 9, 4, 8, 6, 10, 3)(5, 11),$$

$$\mathbf{a} = (2, 3, 4), (1, 2, 4), (1, 2, 3), (1, 3, 4), (2, 4, 3), (1, 4, 3), \\ (1, 3)(2, 4), (1, 3, 2), (1, 4)(2, 3), (1, 4, 2), (1, 2)(3, 4).$$

$$t \equiv 7 \pmod{11}, \sigma = (1, 8, 11)(3, 7, 5)(4, 6),$$

$$\mathbf{a} = (2, 3, 4), (1, 2, 4), (1, 4, 2), (1, 2)(3, 4), (2, 4, 3), (1, 3)(2, 4), \\ (1, 2, 3), (1, 3, 4), (1, 4, 3), (1, 4)(2, 3), (1, 3, 2).$$

$$t \equiv 8 \pmod{11}, \sigma = (1, 5, 3)(2, 10, 6, 11, 8, 9, 7),$$

$$\mathbf{a} = (2, 3, 4), (2, 4, 3), (1, 2, 4), (1, 4, 2), (1, 3)(2, 4), (1, 3, 4), \\ (1, 2, 3), (1, 4, 3), (1, 3, 2), (1, 4)(2, 3), (1, 2)(3, 4).$$

$$t \equiv 9 \pmod{11}, \sigma = (1, 6, 2, 11, 10, 7, 4, 3),$$

$$\mathbf{a} = (1, 2)(3, 4), (2, 3, 4), (1, 3, 2), (1, 4)(2, 3), (1, 2, 3), (1, 4, 3), \\ (2, 4, 3), (1, 2, 4), (1, 3, 4), (1, 4, 2), (1, 3)(2, 4).$$

$$t \equiv 10 \pmod{11}, \sigma = (1, 7, 3, 4, 2)(9, 10),$$

$$\mathbf{a} = (1, 2)(3, 4), (2, 3, 4), (1, 3)(2, 4), (1, 3, 4), (2, 4, 3), (1, 2, 3), \\ (1, 4, 3), (1, 4)(2, 3), (1, 3, 2), (1, 2, 4), (1, 4, 2).$$

These directed R-terraces along with Theorems 1, 3 and 4 allow us to show the R-sequenceability of many new groups. The following result of Wang and Leonard extends the scope further still:

**Theorem 5.** [19] *If  $K$  is an  $R^*$ -sequenceable group of even order and  $N$  is a nilpotent group of odd order then  $K \times N$  is  $R^*$ -sequenceable.*

*Proof.* Follows immediately from Corollaries 2 and 6 of [19]. □

**Theorem 6.** *Groups of the form  $H_1 \times H_2 \times \cdots \times H_s \times K \times N$ , where each  $H_i$  is one of the groups  $\mathbb{Z}_2^2$ ,  $\mathbb{Z}_2^3$ ,  $D_8$ ,  $\mathbb{Z}_6 \times \mathbb{Z}_2$ ,  $D_{12}$  or  $A_4$ , the group  $K$  is  $R^*$ -sequenceable and  $N$  is a nilpotent group of odd order, are  $R^*$ -sequenceable.*

*Proof.* Repeatedly apply Theorem 1 and/or 3 to construct a directed  $R^*$ -terrace for  $H_1 \times H_2 \times \cdots \times H_s \times K$ . Apply Theorem 5 to complete the proof. □

Groups that are known to be  $R^*$ -sequenceable include: abelian groups with non-trivial non-cyclic Sylow 2-subgroups of orders other than 8 [5, 8]; abelian groups of odd order or with Sylow 2-subgroups isomorphic to  $\mathbb{Z}_2^3$  whose Sylow 3-subgroups are isomorphic to  $\mathbb{Z}_3^\alpha \times \mathbb{Z}_9^\alpha \times \mathbb{Z}_{27}^\beta$  or  $\mathbb{Z}_3^\alpha \times \mathbb{Z}_9^{\alpha+1} \times \mathbb{Z}_{27}^\beta$  [5, 16]; the abelian groups described in Corollary 1; nonabelian groups whose order is the product of two odd primes [20]; dihedral groups of order  $4k$ , unless  $k < 4$  or  $k \equiv 0$  or  $1 \pmod{6}$  [18]; and dicyclic groups of order congruent to 16 or 32  $\pmod{48}$  [18].

## 4. Terraces

In this section we follow the approach of [12, 15] whereby we relax the requirement of directedness in the  $R^*$ -terrace for  $K$  and see that  $R$ -terraces emerge from the construction. Further, if these  $R$ -terraces have an additional property then we may turn them into terraces.

Let  $G$  be a group of order  $n$  and let  $\mathbf{a} = (a_1, a_2, \dots, a_{n-1}, \leftrightarrow)$  be a circular arrangement of the non-identity elements of  $G$ . Define  $\mathbf{b} = (b_1, b_2, \dots, b_{n-1}, \leftarrow)$  by  $b_i = a_i^{-1}a_{i+1}$  for each  $i$ . If  $\mathbf{b}$  contains one occurrence of each involution of  $G$  and exactly two occurrences of elements from each set  $\{g, g^{-1} : g^2 \neq e\}$  then  $\mathbf{a}$  is a *rotational terrace* or  *$R$ -terrace* and  $\mathbf{b}$  is a *rotational 2-sequencing* or  *$R$ -2-sequencing*.

As in the directed definition, if  $a_{i-1}a_{i+1} = a_i = a_{i+1}a_{i-1}$  then  $\mathbf{a}$  is an  *$R^*$ -terrace* and  $\mathbf{b}$  is an  *$R^*$ -2-sequencing*. By re-indexing if necessary, we may assume this value of  $i$  in an  $R^*$ -terrace is 1, in which case the  $R^*$ -terrace is *standard*.

Given a standard  $R^*$ -terrace with this notation, suppose there is a value  $r$  such that  $b_r = a_{r+1}^{-1}$ . Then  $r$  is a *right match-point* of  $\mathbf{b}$ . We



will require standard  $R^*$ -terraces whose associated  $R^*$ -2-sequencings have a right match-point  $r$  with  $2 \leq r \leq n - 3$ . An equivalent object is an extendable terrace: a basic terrace  $(e, a_2, \dots, a_n)$  is *extendable* if  $a_n = a_2^2$  and  $a_{j-1}a_{j+1} = a_j = a_{j+1}a_{j-1}$  for some  $j$  with  $5 \leq j < n$ . The circular sequence  $(a_1, a_2, \dots, a_{n-1} \leftrightarrow)$  is a standard  $R^*$ -terrace whose  $R^*$ -2-sequencing has a right match-point  $r$  where  $2 \leq r \leq n - 3$  if and only if

$$(e, a_{r+1}, a_{r+2}, \dots, a_{n-1}, a_1, a_2, \dots, a_r)$$

is an extendable terrace [15]. This relationship is illustrated in Examples 1 and 2: the standard  $R^*$ -terrace in Example 1 has 5 as a match-point which can be used to give the extendable terrace in Example 2.

We can now give the main theorem for constructing terraces.

**Theorem 7.** *Let  $G = H \times K$  with  $|H| = 4m$  and  $|K| = t$ . If  $H$  has a directed  $R$ -terrace with a  $t$ -compatible  $\sigma \in S_{4m-1}$  and  $K$  has an extendable terrace then  $G$  has an extendable terrace.*

*Proof.* First, from the extendable terrace, construct an  $R^*$ -terrace  $\mathbf{k}$  for  $K$  that has a right match-point in position  $r$ , where  $2 \leq r \leq t - 3$ . Let  $\mathbf{a}$  be the directed  $R$ -terrace and apply the main construction to  $\mathbf{a}$  and  $\mathbf{k}$ . We claim that this gives an  $R^*$ -terrace for  $G$  that has a right match-point in position  $r$  and hence that  $G$  has an extendable terrace.

Compared to the proof of Theorem 1, all that changes is that some non-involutions  $g \in K$  may appear as  $\ell_i$  and  $\ell_j$ , with  $i \neq j$ , in the  $R^*$ -2-sequencing of  $K$  and, if this is the case for a given  $g$ , then  $g^{-1}$  does not appear in the  $R^*$ -2-sequencing of  $K$ .

The consequence for our purported  $R$ -2-sequencing is that, for any given non-involution  $h \in H$ , rather than having each of the four elements of the form  $(h^{\pm 1}, g^{\pm 1})$  once, we have  $(h, g)$  and  $(h^{-1}, g)$  twice and neither  $(h, g^{-1})$  nor  $(h^{-1}, g^{-1})$  appears. This does not break the constraints of being an  $R$ -2-sequencing. Similarly, if  $h \in H$  is an involution then we have  $(h, g)$  twice and  $(h, g^{-1})$  does not appear.

Finally, as the first  $t - 2$  elements of the  $R^*$ -terrace are

$$(e, k_1), (e, k_2), \dots, (e, k_{t-2}),$$

the right match-point at position  $r$  of the  $R^*$ -2-sequencing is maintained. □

As with the  $R$ -sequencing result, we have an analogue for central factors:

**Theorem 8.** *Let  $G$  be a group of order  $4mt$  with central factor  $H$  of order  $4m$ . If  $H$  has a directed  $R$ -terrace  $\mathbf{a}$  with a  $t$ -compatible  $\sigma \in S_{4m-1}$  and  $G/H$  has an extendable terrace  $(H, K_2, K_3, \dots, K_t)$  with  $j$  as the position of the element that is the product of its neighbours and such that there are elements  $k_{j-1} \in K_{j-1}$  and  $k_{j+1} \in K_{j+1}$  that commute, then  $G$  has an extendable terrace. If  $m = 1$  and  $t \equiv 0 \pmod{3}$  then the requirement for a  $t$ -compatible permutation  $\sigma$  may be dropped.*

*Proof.* Turn the extendable terrace for  $G/H$  into its equivalent standard  $R^*$ -terrace and the argument then mirrors that of Theorem 4. It is possible to ensure that the match-point condition is met by careful choice of the  $k_i$ . As in Theorem 7, the standard  $R^*$ -terrace for  $G$  that emerges has an equivalent extendable terrace.  $\square$

These results allow the construction of terraces for many infinite families of groups for which terraces were not previously known, even more so in conjunction with this powerful result for constructing new terraces from existing ones:

**Theorem 9.** [2, 3] *Let  $G$  be a group with a normal subgroup  $N$ . If  $N$  has odd order and  $G/N$  has a terrace then  $G$  has a terrace. If  $N$  has odd index and  $N$  has a terrace then  $G$  has a terrace.*

For example:

**Corollary 2.** *Let  $G$  be of the form  $H_1 \times H_2 \times \dots \times H_s \times K \times N$ , where each  $H_i$  is one of  $\mathbb{Z}_2^2$ ,  $\mathbb{Z}_2^3$ ,  $D_8$ ,  $\mathbb{Z}_6 \times \mathbb{Z}_2$ ,  $D_{12}$  or  $A_4$ , the group  $K$  has an extendable terrace, and  $|N|$  is odd. Then  $H_1 \times H_2 \times \dots \times H_s \times K$  has an extendable terrace and  $G$  has a terrace.*

*Proof.* To show that  $H_1 \times H_2 \times \dots \times H_s \times K$  has an extendable terrace, repeatedly apply Theorem 7 or, if  $H_i \cong \mathbb{Z}_2^2$ , Theorem 8 when necessary. Use Theorem 9 to complete the proof.  $\square$

Groups that are known to have an extendable terrace include:  $\mathbb{Z}_s$ , where  $s \geq 7$  and  $s$  is not twice an odd number [12, 14]; abelian 2-groups of order at least 8 that are not elementary abelian [12, 14];  $\mathbb{Z}_2^s \times \mathbb{Z}_p$  where  $s \geq 2$  and  $p$  is an odd prime [12–15]; non-abelian groups of order 12, 16 or 20 [15];  $D_{8s}$  for  $s > 1$  [15]; and these two additional families of groups of orders  $8s$ , with  $s > 1$  [15] (the first are the *semidihedral groups*; the second don't seem to have an accepted name in the literature):

$$\begin{aligned} SD_{8s} &= \langle u, v : u^{4s} = e = v^2, vu = u^{2s-1}v \rangle, \\ M_{8s} &= \langle u, v : u^{4s} = e = v^2, vu = u^{2s+1}v \rangle. \end{aligned}$$

In [15] it is suggested that groups with many involutions might be the most promising place to look for counterexamples to Bailey's Conjecture. Many new such groups are now known to be terraced; for example,  $\mathbb{Z}_2^r \times D_8^s \times D_{12}^t$  provided that  $r \neq 1$  and  $t > 0$ .

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## CONTACT INFORMATION

**M. A. Ollis**

Marlboro College,  
P.O. Box A, Marlboro, Vermont 05344, USA  
*E-Mail(s)*: [matt@marlboro.edu](mailto:matt@marlboro.edu)

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