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RESEARCH ARTICLE

Quasi-Euclidean duo rings with elementary reduction of matrices

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Abstract. We establish necessary and sufficient conditions under which a class of quasi-Euclidean duo rings coincides with a class of rings with elementary reduction of matrices. We prove that a Bezout duo ring with stable range 1 is a ring with elementary reduction of matrices. It is proved that a semiexchange quasi-duo Bezout ring is a ring with elementary reduction of matrices iff it is a duo ring.

Introduction

The problem of the factorization of square matrices over rings was considered in the mid 1960's and was formulated in this way: (P) to characterize the integral domain R, under which an arbitrary invertible square matrix is a product of elementary matrices. *An elementary matrix* with elements of the ring R is understood as a square matrix of one of the following types:

(1) a diagonal matrix with invertible elements on the main diagonal;

(2) a matrix that differs from the unit matrix by the presence of any nonzero element outside the main diagonal.

If R is a field, according to the Gauss approach, arbitrary invertible matrix under it may be decomposed into the product of elementary matrices and the structure of the general linear group $GL_n(R)$ is thoroughly

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studied (see [5]). The investigation of the integral domain (particularly non-commutative) that satisfies the conditions of the problem (P), started in 1966 with the Cohn's fundamental work [3], who defined these domains as the *general Euclidean* (*GE-rings*, for shortness), due to the fact that Euclidean domains were the first well-known examples of GE-rings and are not the fields. Cohn's work became the reason of the thorough and detailed study of the general and special linear group's structure under different rings. In 1996 Zabavsky B. V. [13] analyzed the rings with the elementary reduction of matrices and set up a problem of investigation of such rings.

A ring R is called *a ring with elementary reduction of matrices* [13] in case of an arbitrary matrix over R possesses elementary reduction, i.e. for an arbitrary matrix A over the ring R there exist such elementary matrices over R , $P_1, \ldots, P_k, Q_1, \ldots, Q_s$ of respectful size that

$$P_1 \cdots P_k \cdot A \cdot Q_1 \cdots Q_s = diag(\varepsilon_1, \ldots, \varepsilon_r, 0, \ldots, 0), \tag{1}$$

where $R\varepsilon_{i+1}R \subseteq R\varepsilon_i \cap \varepsilon_i R$ for any $i = 1, \ldots, r-1$.

Since in 1949 Kaplansky [7] established the investigation of the elementary divisors rings (i. e. the rings under which arbitrary matrix resolves itself to the accepted diagonality (1) into invertible matrices of the appropriate sizes), so the problem of finding the necessary and sufficient conditions, whereby a given ring is a ring with elementary reduction of matrices is closely related to the problem of arbitrary square invertible matrices decomposition into the product of elementary ones.

Main results

A ring R is understood as an associative ring with nonzero unit element and $U(R)$ is understood as the group of invertible elements of a ring R. A group generated by elementary matrices of type (2) of order n is called a group of elementary matrices $E_n(R)$, while $GE_n(R)$ is understood as a group of elementary matrices of n order over R.

A right (left) Bezout ring is a ring in which every finitely generated right (left) ideal is principal. *A Bezout ring* [6] is a ring which is both right and left Bezout ring. A ring R is called right Hermite if, for any row (a, b), $a, b \in R$, there exists an invertible matrix P of order 2 over R so that $(a, b)P = (d, 0)$, where $d \in R$. Left Hermite rings can be defined by analogy. If the ring is left and right Hermite, then it is called *Hermite ring* [7]. A ring is said to be a right (left) duo ring if any right (left) ideal of this ring is a 2-sided ideal. If the ring is both left and right duo ring, then it is called *duo ring* [4].

A ring R is called a right (left) quasi-duo ring, if any right (left) maximal ideal in R is a two-sided ideal. If the ring is both left and right quasi-duo ring, then it is called *quasi-duo ring* [11].

We say that a ring R has quasi-algorithm, if the function $\varphi : R \times R \to W$ (where W is some ordinal) is given so that for any $a, b \in R$ $(b \neq 0)$ one can find elements $q, r \in R$ such as $a = bq + r$ and $\varphi(b, r) < \varphi(a, b)$. If one can find some quasi-algorithm on R then the ring R is called *quasi-Euclidean* [1].

A ring R is said to have *stable range 1*, if for any $a, b \in R$ satisfying $aR + bR = R$, there exists such $t \in R$ that $a + bt$ is an invertible element in R [12]. A ring R is said to have *idempotent stable range 1*, if for any $a, b \in R$ satisfying $aR + bR = R$, there exists such idempotent $e \in R$ that $a + be$ is invertible [2].

A ring R is called *an exchange ring* if for any element $a \in R$ there exists an idempotent $e \in R$ such that $e \in aR$ and $1 - e \in (1 - a)R$ [9].

Proposition 1. *A right quasi-Euclidean ring is right Hermite ring.*

Proof. By Theorem 8 [1] for any elements $a, b \in R$, $a \neq 0$, there exists a finite divisible chain, that

$$b = aq_1 + r_1,\ a = r_1q_2 + r_2,\ \ldots,\ r_{n-2} = r_{n-1}q_n + r_n,\ r_{n-1} = r_nq_{n+1}.$$

Then

$$(a, b)\begin{pmatrix}1 & -q_1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -q_2 & 1\end{pmatrix}\begin{pmatrix}1 & -q_3\\ 0 & 1\end{pmatrix}\cdots\begin{pmatrix}1 & -q_{n+1}\\ 0 & 1\end{pmatrix} = (r_n, 0).$$

So for any elements $a, b \in R$ there exists matrix $P \in GE_2(R)$, that $(a, b)P = (r_n, 0)$. Therefore, R is right Hermite ring. □

Lemma 1. *Let R be a duo ring. Then for any matrix $E \in E_n(R)$, there exists such matrix $E' \in E_n(R)$, that*

$$diag(d, \ldots, d) \cdot E = E' \cdot diag(d, \ldots, d).$$

Proof. The proof follows from the fact, that if R is a duo ring, for any element $a \in R$, there exists such an element $a' \in R$, that $da = a'd$. □

Proposition 2. *A quasi-Euclidean duo ring R is a ring with elementary reduction of matrices if and only if a matrix of the form*

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in M_2(R),$$

where $aR + bR + cR = R$ admits elementary reduction.

Proof. The necessity is obvious. To prove the sufficiency, we consider the case where $aR + bR + cR = dR$, $d \notin \mathcal{U}(R)$. By virtue of Proposition 1, there exist such elements $a_1, b_1, c_1 \in R$, that

$$a = da_1,\ b = db_1,\ c = dc_1 \quad \text{and} \quad a_1R + b_1R + c_1R = R.$$

Then

$$\begin{pmatrix} a & 0 \\ c & b \end{pmatrix} = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ b_1 & c_1 \end{pmatrix}.$$

Since the matrix $A = \begin{pmatrix} a_1 & 0 \\ b_1 & c_1 \end{pmatrix}$ admits elementary reduction, there exist such elementary matrices $P_1, \ldots, P_k, Q_1, \ldots, Q_s \in E_2(R)$ of respectful size, that

$$P_1 \cdots P_k \cdot A \cdot Q_1 \cdots Q_s = diag(\varepsilon_1, \varepsilon_2), \tag{2}$$

where $\varepsilon_1 R \cap R\varepsilon_1 \supseteq R\varepsilon_2 R$.

Multiply equation (2) by the matrix $diag(d, d)$ we obtain:

$$diag(d, d) \cdot P_1 \cdots P_k \cdot A \cdot Q_1 \cdots Q_s = diag(d\varepsilon_1, d\varepsilon_2).$$

According to Lemma 1 there exist such matrices $P_1', \ldots, P_k' \in E_2(R)$, that

$$P_1' \cdots P_k' \cdot diag(d, d) \cdot A \cdot Q_1 \cdots Q_s = diag(d\varepsilon_1, d\varepsilon_2).$$

Therefore, the matrix $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ also admits elementary reduction.

The proof is completed by induction of the order of matrices. □

Theorem 1. *Let R is quasi-Euclidean duo ring in which any noninvertible element belongs to most countable set of maximal ideals of R. Then R is a ring with elementary reduction of matrices.*

According to the Proposition 2, the proof of this theorem repeats the proof given in [14] in the case of commutative rings, therefore we do not give it.

Corollary 1. *A quasi-Euclidean duo ring in which the set of maximal ideals is at most countable is a ring with elementary reduction of matrices.*

Theorem 2. *A Bezout duo ring with stable range 1 is a ring with elementary reduction of matrices.*

Proof. By Theorem 2 [15] ring R is a right Hermite ring. It remains to be proven that the ring R is a ring with elementary reduction of matrices. According to the Proposition 1, it is sufficient to prove theorem for matrices

$$A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in M_2(R),$$

where $aR + bR + cR = dR$ for any element $d \in R$. Obviously, there exist such elements $a_1, b_1, c_1 \in R$, that

$$a = da_1, b = db_1, c = dc_1 \quad \text{and} \quad a_1R + b_1R + c_1R = R.$$

Then

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ b_1 & c_1 \end{pmatrix}.$$

Since R is a right Bezout ring of stable range 1, then for elements $a_1, b_1, c_1 \in R$ there exists such elements $s, t \in R$, that $a_1s + b_1 + c_1t = u \in U(R)$. Multiplying the last equality from the left side on the element d we get that $da_1s + db_1 + dc_1t = as + b + ct = du$. Since R is a duo ring, there exists such element $s' \in R$, that $as = s'a$, then $s^{'}a + b + ct = du$. Considering the matrices $P_1, P_2, P_3 \in GE_2(R)$

$$P_1 = \begin{pmatrix} 1 & s' \\ 0 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.$$

We have

$$P_1P_2AP_3 = \begin{pmatrix} 1 & s' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ b_1 & c_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & s' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b & c \\ a & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \begin{pmatrix} s'a + b + ct & c \\ a & 0 \end{pmatrix} = \begin{pmatrix} du & dc_1 \\ da_1 & 0 \end{pmatrix} = B$$

Then the matrix B and, hence, the matrix A obviously admits elementary reduction. Therefore, R is a ring with elementary reduction of matrices. □

Corollary 2. *A semilocal quasi-Euclidean duo ring is a ring with elementary reduction of matrices.*

Theorem 3. *Let R is Hermite duo ring and, for any $a, b \in R$ $(b \neq 0)$, there exists such $s \in R$, that $mspec(s) = mspec(a) \setminus mspec(b)$. Then R is a ring with elementary reduction of matrices.*

Proof. Let $a, b \in R$ be such elements, that $aR + bR = dR$, where $d \in R$. Note that the case $d \notin U(R)$ is irrelevant. Otherwise, there exist elements $a_1, b_1 \in R$, such that $a = da_1$, $b = db_1$ and $a_1R + b_1R = R$. As a result, we obtain $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$. By the fact that R is a duo ring, which would also imply the existence of mutually prime elements, thus there exists such $a_1', b_1' \in R$, that

$$(a, b) = (a_1', b_1') \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}.$$

Therefore, it is sufficient to prove the statement of the theorem for mutually prime elements. Thus, assume that $aR + bR = R$. It is obvious that

$$mspec(a) \cap mspec(b) = \{0\}. \qquad (3)$$

Using the statement of the theorem, there exists element $s \in R$, which belongs to all maximal ideals of the ring R, except for maximal ideals of the set $mspec(a)$, that is, we have the following equality

$$mspec(s) = mspec(0) \setminus mspec(a).$$

It is obvious that

$$mspec(s) \cap mspec(a) = \{0\}. \qquad (4)$$

Let us consider the element $a + bs \in R$ and assume that $a + bs \in \mathcal{M}$, where $\mathcal{M}$ is a maximal ideal of the ring R. There are the following possible cases:

1) $a \in \mathcal{M}$ and $b \in \mathcal{M}$ contradicts with the condition (3).
2) $a \in \mathcal{M}$ and $s \in \mathcal{M}$ contradicts with the condition (4).

Therefore, our initial assumption was incorrect and the condition $a + bs \in \mathcal{U}(R)$ is valid. It also implies that R is a ring of a stable range 1. Due to Theorem 2 a duo ring R is a ring with an elementary reduction of matrices. □

We will denote the Jacobson radical of a ring R by $J(R)$. A ring R is said to be a semiexchange ring [8] if the factor ring $R/J(R)$ is an exchange ring.

Theorem 4. *A semiexchange Bezout duo ring is a ring with elementary reduction of matrices.*

Proof. Let R be a semiexchange Bezout duo ring. Since all idempotent elements of a duo ring belong to its center, due to Theorem 12 [2], then $\bar{R} = R/J(R)$ is a ring with idempotent stable range 1. Since a stable range 1 lifts modulo $J(R)$, we obtain the result that a ring R also has a stable range 1. Then, according to Theorem 2, R is a ring with elementary reduction of matrices. □

Theorem 5. *Let R be semiexchange quasi-duo Bezout ring. Then R is a ring with elementary reduction of matrices if and only if it is a duo ring.*

Proof. As it was mentioned at the beginning and is proven in [10] being a quasi-duo elementary divisor ring implies the duo ring condition, so the necessity is proven.

Sufficiency follows from Theorem 4. □

Corollary 3. *A distributive semiexchange Bezout ring is a ring with elementary reduction of matrices if and only if it is a duo ring.*

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