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# A morphic ring of neat range one

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Abstract. We show that a commutative ring $R$ has neat range one if and only if every unit modulo principal ideal of a ring lifts to a neat element. We also show that a commutative morphic ring $R$ has a neat range one if and only if for any elements $a, b \in R$ such that $aR = bR$ there exist neat elements $s, t \in R$ such that $bs = c$, $ct = b$. Examples of morphic rings of neat range one are given.

The notion of principal ideals being uniquely generated first appeared in Kaplansky's classic paper [4]. He had raised the question of when a ring $R$ satisfies the property of being uniquely generated. He remarked that for commutative rings, the property holds for principal ideal rings and artinian rings. In the case of a left quasi morphic ring the property of being uniquely generated is equivalent to that a ring has stable range one. The concept of a neat range one ring is introduced by the first named author in [9]. In this paper we show that for a commutative morphic ring the condition of a neat range one is equivalent to the a uniquely generated weak condition relation with a neat elements.

Throughout this paper we assume that $R$ is a commutative ring with an identity element. To make the paper almost self-contained, we recall basic definitions and some results used later. We recall that:

(i) $R$ is a *Bezout ring*, if each finitely generated ideal of $R$ is principal, see [10].

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(ii) Two rectangular matrices $A$ and $B$ are *equivalent* if there exist invertible matrices $P$ and $Q$ of appropriate sizes such that $B = PAQ$, see [10].

(iii) The ring $R$ is *Hermite* if every rectangular matrix $A$ over $R$ is equivalent to an upper or a lower triangular matrix, see [10].

(iv) $R$ is *an elementary divisor ring* if every square $n$ by $n$ matrix $A$ with coefficients in $R$ can be converted to a diagonal matrix $diag(a_{11}, \dots, a_{nn})$ such that every $a_{ii}$ divides $a_{i+1,i+1}$, see [4].

(v) a ring $R$ is *a ring of stable range one*, if for any $a, b \in R$ such that $aR + bR = R$ there exists $t \in R$ such that $a + bt$ is a unit of $R$, see Bass [1].

(vi) An element $a \in R$ is defined to be a *clean* element of $R$, if $a$ can be written as the sum of a unit and an idempotent. The ring $R$ is defined to be a *clean ring*, if every element of $R$ is clean, see [10].

(vii) An element $a \in R$ is defined to be a *neat* element of $R$, if $R/aR$ is a clean ring. The ring $R$ is defined to be a *neat ring*, if every elements in a ring $R$ are neat, see [6].

(viii) $R$ is defined to be of *neat range one*, if for any $a, b \in R$ such that $aR + bR = R$ there exists $t \in R$ such that $a + bt$ is a neat element of $R$, see[9].

(ix) An element $a \in R$ is defined to be *morphic*, if $\mathrm{Ann}(a) \cong R/aR$, where $\mathrm{Ann}(a)$ denotes the annihilator of $a$ in $R$. The ring $R$ is defined to be morphic, if every its element is morphic, see [7].

We recall from [4] that every elementary divisior ring $R$ is both a Bezout ring and a Hermite ring. Note also that unity elements of $R$ are neat elements and, hence, every ring of stable range one is a ring of neat range one.

In our next result we need the following definition.

**Definition 1.** (a) An element $a \in R$ is a unit modulo a principal ideal $cR$ if $ax - 1 \in cR$ for some $x \in R$.

(b) A unit $a \in R$ modulo a principal ideal $cR$ lifts to a neat element, if $a - t \in bR$ for a neat element $t \in R$.

**Proposition 1.** *Let $R$ be a commutative ring. Then the following are equivalent:*

*1) $R$ has a neat rang one;*

*2) Every unit lifts to a neat element modulo every principal ideal.*

*Proof.* We assume that $R$ has neat range one. Let $a, b, c \in R$ be such that $ab - 1 \in cR$, i.e. $b$ is a unit modulo the principal ideal $cR$. We show that there exists a neat element $t \in R$ such that $b - t \in R$.

Let $x \in R$ be such that $ab - 1 = cx$. Then $ab - cx = 1$. Since $R$ has neat range one, there exists an element $s \in R$ and a neat element $t \in R$ such that $b - cs = t$. Therefore $b - t \in cR$ where $t$ is a neat element in $R$.

To prove the implication (2) $\Rightarrow$ (1), assume that every unity of $R$ lifts to a neat element modulo every principal ideal. We show that $R$ has a neat range one. Let $a, b, c \in R$ such that $ab + cd = 1$. Then $ab - 1 \in cR$. Therefore, by our hypothesis there exists a neat element $t \in R$ such that $b - t \in cR$. Thus $b - t = cx$ for some $x \in R$ i.e. $b + c(-x) = t$ is a neat element i.e. $R$ has neat range one. □

**Proposition 2.** *A morphic ring is a ring of neat range one if and only if for any pair of elements $a, b \in R$ such that $aR = bR$ there are neat elements $s, t \in R$ such that $as = b$ and $a = bt$.*

*Proof.* In view of Proposition 1 it suffices to show that every unit lifts to a neat element modulo every principal ideal in $R$.

Let $x$ be a unit that lifts to a neat element modulo the principal ideal $yR$, i.e there exists $z \in R$ such that $zx - 1 \in yR$. We would like to show that there exists a neat elements $t \in R$ such that $x - t \in yR$. Since $R$ is a morphic, there exists $a, b$ such that $yR = \mathrm{Ann}(a)$ and $xaR = \mathrm{Ann}(b)$.

Obviously, $xR \subset \mathrm{Ann}(ab)$ and $yR \subseteq \mathrm{Ann}(ab)$.

Since $zx-1 \in yR$, we have $xR+yR = R$ and $xR+yR = \mathrm{Ann}(ab)$. Then $ab = 0$ and $a \in \mathrm{Ann}(b)$. Also we have $\mathrm{Ann}(b) = xaR \subseteq aR$. Therefore $\mathrm{Ann}(b) = xaR = aR$. Under the assumption on the ring there exists a neat element $t \in R$ such that $xa = ta$. This implies that $(x - t)a = 0$. We have $x - t \in \mathrm{Ann}(a) = yR$. Thus from Proposition 1, the $R$ has neat range one.

Let $aR = bR$. Then there exist $x, y \in R$ such that $a = bx$, $b = ay$. Therefore $b = bxy$, $b(1 - xy) = 0$. This shows that $1 - xy \in \mathrm{Ann}(b)$.

Now $xy + (1 - xy) = 1$ where $xy \in xR$ and $1 - xy \in (1 - xy)R$. Therefore $xR + (1 - xy)R = R$. Since $R$ is assumed to have neat range one, there exists $s \in R$ such that $x + (1 - xt)s = t$ is a neat element in $R$. Since $1 - xy \in \mathrm{Ann}(b)$, we have $(x + (1 - xy)s)b = tb$, $xb = tb$ where $xb = a$. Thus $a = tb$ for some neat element $t \in R$. Similarly we have $b = sa$, for some neat element $s \in R$, which completes the proof. □

**Theorem 1.** *If $R$ is an elementary divisor ring, then $R$ is a ring of neat range one.*

*Proof.* By [8] for any elements $a, b, c \in R$ such that $aR + bR = R$ there exists an element $t \in R$ such that $s = a + bt = uv$, where $uR + cR = R$,

$vR + (1-c)R$, $uR + vR = R$. Let $\overline{u} = u + sR$, $\overline{v} = v + sR$. Since $uR + vR = R$, one has $ux + vy = 1$ and $\overline{u}^2\overline{x} = \overline{u}$, $\overline{v}^2\overline{y} = \overline{v}$, where $\overline{x} = x + sR$, $\overline{y} = y + sR$. Let $\overline{vy} = \overline{e}$, obviously $\overline{e}^2 = \overline{e}$ and $\overline{1} - \overline{e} = \overline{ux}$. Since $uR + cR = R$, we obtain $\overline{ce}\overline{\beta} = \overline{e}$, for some element $\overline{\beta} \in R/sR$. Similarly, $(\overline{1} - \overline{c})\overline{\alpha}(\overline{1} - \overline{e}) = \overline{1} - \overline{e}$ for some element $\overline{\alpha} \in R/sR$. We proved that for any element $\overline{c} = c + sR$ there exists an idempotent $\overline{e}$ such that $\overline{e} \in \overline{c}\overline{R}$ and $\overline{1} - \overline{e} \in (\overline{1} - \overline{c}\overline{R})$. We have proved that $R/sR$ is a clean ring [6] which completes the proof. □

As a consequence we obtain the following result.

**Theorem 2.** *If $R$ is an elementary divisor domain and $a \in R \setminus \{0\}$, then the factor-ring $R/aR$ is a morphic ring of neat range one.*

*Proof.* Since every elementary divisor domain is a Bezout ring [4], by [9] $R/aR$ is a morphic ring. Since every homomorphic image of an elementary divisor ring is an elementary divisor ring, by Theorem 3, $R/aR$ is a morphic ring of neat range one, which completes the proof. □

We say that $R$ has almost stable range one if every finite proper homomorphic image $R$ has stable range one. By [5] a Bezout ring of almost stable range one is an elementary divisor ring.

A well-known Henriksen example of a Bezout domain, namely $R = \mathbb{Z} + x\mathbb{Q}[x]$ (see [2]; for a general theorem on pullbacks of Bezout domains [3]), $R$ is an elementary divisor that does not have almost stable range one [8].

Let $R$ be an elementary divisor domain which is not of almost stable range one. Then there exists an element $a \in R$ such that in the factor-ring $\overline{R} = R/aR$ there exist elements $\overline{b}, \overline{c} \in \overline{R}$ such that $\overline{b}\overline{R} = \overline{c}\overline{R}$. There exist noninvertible neat elements $\overline{s}, \overline{t} \in R$ such that $\overline{b}\overline{s} = \overline{c}$, $\overline{c}\overline{t} = \overline{b}$.

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