

Cyclic left and torsion-theoretic spectra of modules and their relations

Marta Maloid-Glebova

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ABSTRACT. In this paper, strongly-prime submodules of a cyclic module are considered and their main properties are given. On this basis, a concept of a cyclic spectrum of a module is introduced. This spectrum is a generalization of the Rosenberg spectrum of a noncommutative ring. In addition, some natural properties of this spectrum are investigated, in particular, its functoriality is proved.

Introduction

In this paper, we consider strongly-prime ideals and modules. The concept of strongly-prime ideal was introduced by Beachy in [1]. Also in that paper the author introduced and investigated the concept of a strongly-prime module. Independently, the concept of strongly-prime module and submodule were introduced and investigated by Dauns in his paper [3]. Also, the strongly-prime modules were investigated by Algirdas Kaučikas in [2], where the author studied strongly-prime submodules of cyclic modules, but he did not study the concept of the Rosenberg spectrum for modules. The concept of pre-order on ideals was introduced by Rosenberg, and this concept is a basic one in the definition of cyclic spectrum, whose functoriality is investigated in this paper. Also we consider the notion and some properties of torsion-theoretic spectra of rings and modules. The notion and main properties of torsion-theoretic spectra were introduced by Golan in [5]. The main result of this paper is the

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proof of the fact that there exists mapping from the cyclic spectrum to the torsion-theoretic spectrum of module is continuous and surjective.

1. Strongly-prime ideals and modules

Let R be an associative ring with $1 \neq 0$. To have a reference, recall some necessary concepts from the ring theory that are related to the concept of spectrum of a noncommutative ring.

A left ideal \mathfrak{p} of a ring R is called *prime*, if for every $x, y \in R$, $xRy \subseteq \mathfrak{p}$ implies $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. Clearly, any left prime ideal is two-sided if and only if it is prime in the classical way. Set of all two-sided prime ideals is denoted by $\text{Spec}(R)$ and is called a (prime) *spectrum* of a ring R .

Recall the definition of a *pre-order* \leq on the set of left ideals of ring R in the following way: $\mathfrak{a} \leq \mathfrak{b}$ for left R -ideals \mathfrak{a} and \mathfrak{b} if and only if there exists a finite subset V of ring R such that $(\mathfrak{a} : V) \subseteq \mathfrak{b}$. A left prime ideal \mathfrak{p} of a ring R is called a *left Rosenberg point* if $(\mathfrak{p} : x) \leq \mathfrak{p}$ for any $x \in R \setminus \mathfrak{p}$, [8]. The set of all left Rosenberg points of a ring R is called a *left Rosenberg spectrum* of R and is denoted by $\text{spec}(R)$.

The space $\text{spec}(R)$ may be defined in another way: this is the set of all strongly prime left ideals. Recall that left ideal \mathfrak{p} of the ring R is called *strongly-prime*, if for every $x \in R \setminus \mathfrak{p}$ there exist a finite set V of ring R such that $(\mathfrak{p} : Vx) = \{r \in R : rVx \subseteq \mathfrak{p}\} \subseteq \mathfrak{p}$. Clearly, every strongly-prime left ideal of a ring R is a prime left ideal and every maximal left ideal is strongly-prime. It is known that if R is noetherian, then $\text{Spec}(R) \subseteq \text{spec}(R)$.

Now let us recover the information about corresponding analogues of the above concepts for left modules over a ring R .

The concept of strongly-prime module can be given in two ways.

A nonzero left module M over a ring R is called *strongly-prime*, if for any nonzero $x, y \in M$ there exists a finite subset $\{a_1, a_2, \dots, a_n\} \subseteq R$ such that $\text{Ann}_R\{a_1x, a_2x, \dots, a_nx\} \subseteq \text{Ann}_R\{y\}$, $(ra_1x = ra_2x = \dots = ra_nx = 0)$, $r \in R$ implies $ry = 0$.

In [1], the authors introduced such a concept of strongly-prime submodule. A nonzero left module M over a ring R is called *strongly-prime*, if for any nonzero $x \in M$ there exists a finite subset $\{a_1, a_2, \dots, a_n\} \subseteq R$ such that $\text{Ann}_R\{a_1x, a_2x, \dots, a_nx\} = 0$. If in this concept we put $M = R$, we obtain the concept of a strongly-prime ring. Such strongly-prime rings were studied in [4].

A submodule P of some module M is called *strongly-prime*, if the quotient module M/P is a strongly-prime R -module. The set of all

strongly-prime submodules of module M is called the left prime spectrum of M and is denoted by $\text{spec}(M)$. In particular, a left ideal $\mathfrak{p} \subset R$ is called *strongly-prime* if the quotient module R/\mathfrak{p} is a strongly-prime R -module. In terms of elements, left ideal $\mathfrak{p} \subset R$ is strongly-prime if for every $u \notin \mathfrak{p}$ there exists such elements $\{a_1, \dots, a_n\} \subseteq R$ and a natural number $n = n(u)$ such that $ra_1u, \dots, ra_nu \in \mathfrak{p}$, $r \in R$ implies $r \in \mathfrak{p}$.

2. Preorder on the set of modules and cyclic left spectrum of module

It is easy to see that if R is a left noetherian ring and $\mathfrak{p} \in \text{Spec}(R)$, then R/\mathfrak{p} is a left noetherian prime ring. This implies that it is sufficient to prove that in a left noetherian prime ring R zero ideal belongs to $\text{spec}(R)$. But taking into account the assumption that R is a prime Goldie ring, for any $0 \neq x \in R$ any two-sided ideal RxR is essential, thus there exists a regular element $a = \sum_{i=1}^n r_i x s_i \in RxR$ (Using Goldie theorem). Let $V = \{r_1, \dots, r_n\}$ and $y \in (0 : Vx)$, then $ya = \sum y r_i x s_i = 0$. Since a is regular, it follows that $y = 0$, hence $0 \in \text{spec}(R)$ indeed.

Clearly, it is necessary to demonstrate how to calculate prime left ideals in an easy example. For this purpose we use the following example.

Example 1. Consider the matrix ring $R = M_2(k)$ over a (commutative) field k .

It is well known that $\text{Spec}(R) = \{0\}$. Let L be a nonzero left R -ideal and $0 \neq r \in L$. Since all nonzero left ideals of the ring R are maximal, $L = Rr$. Multiplying r by the matrix units e_{11} and e_{12} resp., it easily follows that we may assume r to be of the form $r = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$, for some nonzero string $(a \ b) \in k^2$. One thus finds $L = R \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = [a, b]_k$.

Moreover, $[a, b]_k = [a', b']_k$ if and only if there exists such $c \in k$ that $a = ca'$ and $b = cb'$. Then $\text{spec}(R) = \{[a, b]_k \mid a, b \in k\}$ may be identified with the projective line P_k^1 (with "generic point" $(0) = [0, 0]_k$). (See [6])

As in [8] we introduce a preorder \leq on the set of all left ideals by putting $K \leq L$ for a pair of left R -ideals L and K if and only if there exists a finite subset V of the ring R such that $(K : V) \subseteq L$.

Let us try to establish a preorder on the modules. Let R be a regular module over itself with generator 1. Then $M = R \cdot 1$ is a cyclic module.

Theorem 1. *Every cyclic module is isomorphic to the quotient module of a regular module by the annihilator of a generator $R \cdot m = R/\text{Ann}(m)$, where $\text{Ann}(m)$ is the left annihilator of a generator m .*

Consider some submodules of a cyclic module M which is presented as $Rm = R/\text{Ann}(m)$ for the generator m . Let L, K be some submodules. We can represent $L = \mathfrak{A}/\text{Ann}(m)$ and $K = \mathfrak{B}/\text{Ann}(m)$ for some left ideals \mathfrak{A} and \mathfrak{B} of a ring R . Then we define $L = \mathfrak{A}/\text{Ann}(m) \leq K = \mathfrak{B}/\text{Ann}(m)$ if and only if $\mathfrak{A} \leq \mathfrak{B}$ as the Rosenberg ideals. All properties are carried out. Thus the spectrum of a cyclic module is the set of all ideals that are in the spectrum of ring R .

It is well known that any module is the sum of its cyclic submodules. Then the *cyclic spectrum* of a arbitrary module M is defined as the union of all spectra of its cyclic submodules. The cyclic spectrum of module M is denoted by $\text{Cspec}(M)$. Then we can define $L \leq K \iff \text{Cspec}(L) \subseteq \text{Cspec}(K)$ for all submodules of the module M and obtain a preorder on the family of such submodules.

Example 2. Let $M = \{ \begin{pmatrix} a \\ b \end{pmatrix} \mid a, b \in k \}$ be module of columns with height 2 over ring $R = M_2(k)$, where k is commutative field.

This module is cyclic with generator $e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, that is, $M = R \times \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then $\text{Ann}(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = \{ \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} \mid b, d \in k \}$, thus $M/\text{Ann}(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) \cong \{ \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \mid a, c \in k \}$. The maximal submodule is $\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \}$, hence cyclic spectrum consists of one point.

Lemma 1. *Let L and K be left cyclic R -modules. Then $L \leq K$ if and only if there exists a cyclic left R -module X , a monomorphism $X \hookrightarrow L^n$ and an epimorphism $X \twoheadrightarrow K$. In other words, there exists a diagram $(L)^n \hookleftarrow X \twoheadrightarrow K$.*

Proof. Recall the definition of preorder for submodules of a cyclic module. Let L, K be some submodules. We can represent $L = \mathfrak{A}/\text{Ann}(m)$ and $K = \mathfrak{B}/\text{Ann}(m)$ for some left ideals \mathfrak{A} and \mathfrak{B} of the ring R . Then we define $L = \mathfrak{A}/\text{Ann}(m) \leq K = \mathfrak{B}/\text{Ann}(m)$ iff $\mathfrak{A} \leq \mathfrak{B}$ as Rosenberg ideals. Thus consider two cyclic modules L and K . They are fully represented by their ideals \mathfrak{A} and \mathfrak{B} . Than if $\mathfrak{A} \leq \mathfrak{B}$ by the definition, than there exists a finite subset $V \subseteq R$, such that $(\mathfrak{A} : V) \leq \mathfrak{B}$. Put $V = \{v_1, \dots, v_n\}$ and let $X = R\vec{v}$ be a cyclic module, where $\vec{v} = \{v_1, \dots, v_n\} \in (L)^n$. Than we have

$$(0 : \vec{v}) = \cap_{i=1}^n (\mathfrak{A} : v_i) = (\mathfrak{A} : V) \subseteq \mathfrak{B},$$

which implies that there exists a surjection $X \twoheadrightarrow K$.

On the other hand, assume that there exists a diagram $(L)^n \hookleftarrow^\alpha X \twoheadrightarrow^\beta K$. Thus we can find such element $x \in X$, that $\beta(x) = \vec{1}$. Put

$\alpha(x) = (\vec{v}_1, \dots, \vec{v}_n) \in (L)^n$, where $(\vec{v}_1, \dots, \vec{v}_n) \in \mathfrak{A}$ for some $v_i \in R$. Put $V = \{v_1, \dots, v_n\}$ and then we have

$$(\mathfrak{A} : V) = \bigcap_{i=1}^n (\mathfrak{A} : v_i) = (0 : \vec{v}) = (0 : x) \subseteq \mathfrak{B},$$

so $\mathfrak{A} \leq \mathfrak{B}$ and $L \leq K$. □

Usually from the preorder \leq we obtain an *equivalence relation* \sim as follows: $K \sim L$ iff $K \leq L$ and $L \leq K$. The equivalence class of the submodule L will be denoted by $[L]$.

Lemma 2 (11). *If \mathfrak{P} is a strongly-prime module, then for any element $x \in M$ the following properties are equivalent:*

- (1) $x \notin \mathfrak{P}$;
- (2) $(\mathfrak{P} : x) \leq \mathfrak{P}$;
- (3) $(\mathfrak{P} : x) \in [\mathfrak{P}]$.

Lemma 3. *Let M be cyclic module. If $\mathfrak{P} \in \text{Cspec}(M)$ and if L and K are submodules such that $L \cap K \leq \mathfrak{P}$, then either $L \leq \mathfrak{P}$ or $K \leq \mathfrak{P}$.*

Proof. Let $L \not\leq \mathfrak{P}$ and $K \not\leq \mathfrak{P}$ and let $L \cap K \leq \mathfrak{P}$. Thus, by the definition, there exist ideals \mathfrak{A} , \mathfrak{B} and \mathfrak{p} of the ring R , such that $L = \mathfrak{A}/\text{Ann}(m)$, $K = \mathfrak{B}/\text{Ann}(m)$ and $P = \mathfrak{p}/\text{Ann}(m)$. Then there exists a finite subset V of the ring R , such that $(\mathfrak{A} \cap \mathfrak{B} : V) \subseteq \mathfrak{p}$. Since $\mathfrak{A} \not\leq \mathfrak{p}$, this implies that $(\mathfrak{A} : F) \not\subseteq \mathfrak{p}$ for some finite subset F of the ring R . Thus, if we take $F = V$, we obtain the fact, that $(\mathfrak{A} : V) \not\subseteq \mathfrak{p}$. Now, if $x \in (\mathfrak{A} : V) - \mathfrak{p}$, then there exists a finite set $W \subseteq R$ with the property that $(\mathfrak{p} : Wx) \subseteq \mathfrak{p}$. Since $K \not\leq \mathfrak{p}$, we have $\mathfrak{b} \not\leq \mathfrak{p}$, get fact that $(\mathfrak{B} : F) \not\subseteq \mathfrak{p}$ for any finite set $F \subseteq R$. In particular, this holds for $F = WxV$, thus we can find an element $y \in (\mathfrak{B} : WxV) - \mathfrak{p}$. Finally, $x \in (\mathfrak{A} : V)$ implies that $yWxV \subseteq \mathfrak{B}$, and y belongs to the set $(\mathfrak{B} : WxV)$. Certainly, $yWxV \subseteq \mathfrak{B}$, then $yWxV \subseteq \mathfrak{A} \cap \mathfrak{B}$ and $yWx \subseteq (\mathfrak{A} \cap \mathfrak{B} : V) \subseteq \mathfrak{p}$. Thus, $y \in (\mathfrak{p} : Wx) \subseteq \mathfrak{p}$, that contradicts to the fact, that $y \notin \mathfrak{p}$. □

Similarly

Lemma 4. *If $\mathfrak{P} \in \text{Cspec}(R)$ and if L and K are submodules such that $LK \leq \mathfrak{P}$, then either $L \leq \mathfrak{P}$ or $K \leq \mathfrak{P}$.*

Recall the operation of multiplication of the submodules of cyclic module R/\mathfrak{c} . Any submodule of cyclic module can be viewed as the quotient-module of some left ideal by some other left ideal. Let we have two such submodules $L \cong \mathfrak{a}/\mathfrak{c}$ and $K \cong \mathfrak{b}/\mathfrak{c}$. Then $L \cdot K = \mathfrak{a}/\mathfrak{c} \cdot \mathfrak{b}/\mathfrak{c} = \mathfrak{ab}/\mathfrak{c}$.

Lemma 5. *Let \mathfrak{P} and \mathfrak{Q} be strongly-prime submodules of the cyclic module M . Then the following holds:*

- (1) *If $\mathfrak{P} \sim \mathfrak{Q}$, then $\mathfrak{P} \cap \mathfrak{Q}$ is a strongly-prime module and $\mathfrak{P} \sim \mathfrak{P} \cap \mathfrak{Q}$;*
- (2) *If $\mathfrak{P} \cap \mathfrak{Q}$ is a strongly-prime module, then either $\mathfrak{P} \subseteq \mathfrak{Q}$ or $\mathfrak{P} \supseteq \mathfrak{Q}$ or $\mathfrak{P} \sim \mathfrak{Q}$.*

Proof. Let \mathfrak{P} and \mathfrak{Q} be strongly-prime submodules of a cyclic module M . Thus, for every submodule of a cyclic module there exist ideals $\mathfrak{P} = \mathfrak{p}/\text{Ann}(m)$ and $\mathfrak{Q} = \mathfrak{q}/\text{Ann}(m)$, where $\mathfrak{P} \leq \mathfrak{Q}$ if and only if $\mathfrak{p} \leq \mathfrak{q}$ as Rosenberg ideals. Similarly, we can formulate the definition of equivalence relation. Thus let $\mathfrak{p} \sim \mathfrak{q}$ and $x \notin \mathfrak{p} \cap \mathfrak{q}$. Let $x \notin \mathfrak{p}$, thus there exists a finite subset $V \subseteq R$, such that $(\mathfrak{p} : Vx) \subseteq \mathfrak{p}$. If $x \notin \mathfrak{q}$, then $(\mathfrak{q} : Wx) \subseteq \mathfrak{q}$ for some finite subset W of the ring R . Let $U = V \cup W$, then $(\mathfrak{p} \cap \mathfrak{q} : Ux) \subseteq \mathfrak{p} \cap \mathfrak{q}$. If $x \in \mathfrak{q}$, then $(\mathfrak{q} : Vx) = R$, hence $(\mathfrak{p} \cap \mathfrak{q} : Vx) \subseteq \mathfrak{p}$. Since $\mathfrak{p} \sim \mathfrak{q}$ by the assumption, $\mathfrak{p} \leq \mathfrak{q}$, and thus $(\mathfrak{p} : U) \subseteq \mathfrak{q}$ for some finite subset $U \subseteq R$, and since we may assume that $1 \in U$, we obtain

$$(\mathfrak{p} \cap \mathfrak{q} : UVx) = ((\mathfrak{p} \cap \mathfrak{q} : Vx) : U) \subseteq (\mathfrak{p} : U) \subseteq \mathfrak{q}.$$

Moreover, since $V \subseteq UV$, we also have

$$(\mathfrak{p} \cap \mathfrak{q} : UVx) \subseteq (\mathfrak{p} \cap \mathfrak{q} : Vx) \subseteq \mathfrak{p},$$

hence $(\mathfrak{p} \cap \mathfrak{q} : UVx) \subseteq \mathfrak{p} \cap \mathfrak{q}$, thus $\mathfrak{p} \cap \mathfrak{q}$ is a strongly prime ideal. Clearly $\mathfrak{p} \cap \mathfrak{q} \leq \mathfrak{p}$. On the other hand, since $\mathfrak{p} \leq \mathfrak{q}$, there exists a finite subset $V \subseteq R$, with $(\mathfrak{p} : V) \subseteq \mathfrak{q}$. We may obviously assume that $1 \in V$, thus we have $(\mathfrak{p} : V) \subseteq \mathfrak{p}$. Hence $(\mathfrak{p} : V) \subseteq \mathfrak{p} \cap \mathfrak{q}$, so $\mathfrak{p} \leq \mathfrak{p} \cap \mathfrak{q}$ and $\mathfrak{p} \sim \mathfrak{p} \cap \mathfrak{q}$.

Let us now assume that $\mathfrak{p} \cap \mathfrak{q}$ is a strongly-prime ideal while $\mathfrak{p} \not\subseteq \mathfrak{q}$ and $\mathfrak{p} \not\supseteq \mathfrak{q}$. Since such a $\mathfrak{p} \not\subseteq \mathfrak{q}$ there exists an element $x \in \mathfrak{p} - \mathfrak{q}$. Thus $x \notin \mathfrak{p} \cap \mathfrak{q}$ and we may find a finite subset $V \subseteq R$ such that $(\mathfrak{p} \cap \mathfrak{q} : Vx) \subseteq \mathfrak{p} \cap \mathfrak{q}$. Since $(\mathfrak{p} : Vx) = R$, this yields $(\mathfrak{q} : Vx) \subseteq \mathfrak{p} \cap \mathfrak{q} \subseteq \mathfrak{p}$, hence $\mathfrak{p} \leq \mathfrak{q}$. By symmetry $\mathfrak{p} \geq \mathfrak{q}$, and thus $\mathfrak{p} \sim \mathfrak{q}$. □

We easily obtain the following corollary:

Corollary 1. *Let $\mathfrak{P}_1, \dots, \mathfrak{P}_n$ be a finite family of strongly-prime modules, such that $\mathfrak{P}_1 \sim \dots \sim \mathfrak{P}_n$, then $\bigcap_{i=1}^n \mathfrak{P}_i$ is a strongly-prime module and $\mathfrak{P}_1 \sim \bigcap_{i=1}^n \mathfrak{P}_i$.*

For any left module M , its submodule N is called *strongly two-sided*, if left annihilator of every element of N is two-sided ideal. Clearly, new

submodule is two-sided. Thus the set of such submodules is not empty, because the zero submodule is strongly two-sided submodule. The sum of all strongly two-sided submodules is called the *bound* of the submodule N . In other words, the *bound of the module* is the largest submodule among those that have two-sided left annihilators for all their elements. In the case when $M = N$ we are talking about the concept of a *bound of the module*. As follows, the *bound* of the module M is the largest strongly two-sided submodule of the module M . Denote the bound of a submodule N by $b(N)$, the bound of the module M by $b(M)$.

Lemma 6. *For every strongly-prime left submodule \mathfrak{P} of the module M we have $b(\mathfrak{p}) \in \text{Cspec}(M)$.*

Proof. Let $x, y \in M$ by elements, such that $xRy \subseteq b(\mathfrak{P})$. Assume that $y \notin b(\mathfrak{P})$. Then there exists such an element $s \in R$ with $ys \notin \mathfrak{P}$. For every $r \in R$, $(xr)R(ys) \subseteq (xRy)s \subseteq b(\mathfrak{P})s \subseteq b(\mathfrak{P}) \subseteq \mathfrak{P}$. Hence $rx \in \mathfrak{P}$. Thus $xR \subseteq b(\mathfrak{P})$, which proves the assertion. \square

Lemma 7. *If $L \leq K$ are left R -modules, then $b(L) \subseteq b(K)$. Conversely, if R is a left noetherian fully-bounded ring, and if $b(L) \subseteq b(K)$, then $L \leq K$.*

Proof. Since $L \leq K$, there exists a representation $L = \mathfrak{A}/\text{Ann}(m)$ and $K = \mathfrak{B}/\text{Ann}(m)$ for some left ideals \mathfrak{A} and \mathfrak{B} of the ring R . Then $\mathfrak{A} \leq \mathfrak{B}$. Thus there exist a finite subset $V \subseteq R$, that $(\mathfrak{A} : V) \subseteq \mathfrak{B}$. Then for every elements $r \in b(L)$ and $s \in R$, we have $rs \in \mathfrak{A}$, therefor $r \in (\mathfrak{A} : s)$. Thus $r \in (\mathfrak{A} : V) = \bigcap_{s \in V} (\mathfrak{A} : s)$. Since the former is contained in \mathfrak{B} , we have $b(L) \subseteq K$, hence $b(L) \subseteq b(K)$.

On the other hand, if R is a left noetherian fully-bounded ring, then there exists a finite subset $V = \{v_1, \dots, v_n\} \subseteq R$ such that $b(L) = \bigcap_{i=1}^n (\mathfrak{A} : v_i) = (\mathfrak{A} : V)$. Hence $(\mathfrak{A} : V) = b(\mathfrak{A}) \subseteq b(\mathfrak{B}) \subseteq \mathfrak{B}$, and $\mathfrak{A} \leq \mathfrak{B}$, therefore $L \leq K$. \square

Corollary 2. *Let L and K be left modules such that $L \sim K$, then $b(L) = b(K)$. Moreover if R is a left noetherian fully-bounded ring, then the converse is also true.*

3. Functoriality of cyclic spectrum of module

The cyclic spectrum construction can be regarded as a contravariant functor from the category of modules to the category of sets,

$$\text{CSpec: Mod} \rightarrow \text{Set}.$$

A contravariant functor CSpec is a rule assigning to each module M over an associative ring R the set $\text{CSpec}(M)$, the cyclic spectrum, i.e. the set of submodules that are related in that spectrum, and to each module homomorphism $f: M_1 \rightarrow M_2$ the map of sets

$$\begin{aligned} \text{Cspec}(M_1) &\rightarrow \text{Cspec}(M_2), \\ P &\mapsto f^{-1}(P). \end{aligned}$$

Consider the endomorphism ring $E = \text{End}(M)$, and also consider the center of that ring, denoted by $C = \{c \in E \mid cr = rc, \forall r \in E\}$. Consider the construction of partial algebra over the ring C . It is the set Q with a reflexive, symmetric binary relation $\perp \subseteq Q \times Q$ (called commeasureability), partial addition and multiplication operations "+" and ".", that are functions $I \rightarrow Q$, a scalar multiplication operation $E \times Q \rightarrow Q$, and elements $0, 1 \in C$, such that the following axioms are satisfied:

- (1) for all $q \in Q$, $a \perp 0$ and $a \perp 1$;
- (2) the relation \perp is preserved by the partial binary operations: for all $q_1, q_2, q_3 \in Q$, with $q_i \perp q_j$ ($1 \leq i, j \leq 3$) and for all $\lambda \in C$, one has $(q_1 + q_2) \perp q_3$, $(q_1 \cdot q_2) \perp q_3$ and $(\lambda q_1) \perp q_2$;
- (3) if $q_i \perp q_j$ for $1 \leq i, j \leq 3$, then the values of all polynomials in q_1, q_2 and q_3 form a commutative algebra.

Commeasureability subalgebra of a partial C -algebra Q is a subset $Z \subseteq Q$ consisting of pairwise commeasureable elements that is closed under C -scalar multiplication and the partial binary operations of Q .

Given functors $K: \mathcal{A} \rightarrow \mathcal{B}$ and $S: \mathcal{A} \rightarrow \mathcal{C}$, we recall that the (right) Kan extension of S along K is a functor $L: \mathcal{B} \rightarrow \mathcal{C}$ with a natural transformation $\varepsilon: LK \rightarrow S$ such that for any other functor $F: \mathcal{B} \rightarrow \mathcal{C}$ with a natural transformation $\eta: FK \rightarrow S$ there is a unique natural transformation $\delta: F \rightarrow L$, such that $\eta = \varepsilon \circ (\delta K)$.

Theorem 2. *The functor $\text{Cspec}: \text{Mod}^{\text{op}} \rightarrow \text{Set}$, with the identity natural transformation $\text{Cspec}|_{\text{Comm Mod}^{\text{op}}} \rightarrow \text{CSpec}$ is the Kan extension of the functor $\text{Cspec}: \text{Comm Mod}^{\text{op}} \rightarrow \text{Set}$ along the embedding $\text{Comm Mod}^{\text{op}} \subseteq \text{Mod}^{\text{op}}$.*

Proof. Let $F: \text{Mod} \rightarrow \text{Set}$ be a contravariant functor with a fixed natural transformation $\eta: F|_{\text{Comm Mod}} \rightarrow \text{Spec}$. Consider functor $\text{C-Spec}: \text{Comm Mod} \rightarrow \text{CSpec}$. We need to show that there

is a unique natural transformation $\delta: F \rightarrow \text{CSpec}$, that induces $\eta: F|_{\text{CommMod}} \rightarrow \text{CSpec}$ upon a restriction to $\text{CommMod} \subseteq \text{Mod}$. To construct it, fix ring R and module M over it. For every submodule $N \subseteq M$ over ring R the inclusion $N \subseteq M$ given a morphisms of sets $F(M) \rightarrow F(N)$, and η provides a morphisms $\eta_N: F(N) \rightarrow \text{CSpec}(N)$; these compose to give morphisms $F(M) \rightarrow \text{CSpec}(N)$. By naturality of the morphisms involved, these maps of $F(M)$ collectively form a cone over the diagram obtained for submodules of module. By the universal property of limit, there exists a unique arrow making corresponding diagram commutative for all $N \subseteq M$.

Defined morphisms δ_M form the components of a natural transformation $\delta: F \rightarrow \text{CSpec}$. By construction, δ induces η when restricted to CommMod . Uniqueness of δ is guaranteed by the uniqueness of the indicated arrow used to define δ_M above. \square

4. Localisations

Recall some definitions. By a *torsion-theoretic spectrum* we mean the space of all prime torsion theories (or prime Gabriel filters of a main ring) in the category of left R -modules with Zarisky topology. Recall that *prime torsion theory* $\pi \in R - \text{tors}$ is a torsion theory, for which $\pi = \chi(R/I)$ for some critical ideal I of the ring R , where $R - \text{tors}$ is class of all torsion theories of the category $R\text{-mod}$ and $\chi(R/I)$ is the torsion theory, cogenerated by module $E(R/I)$. If τ is torsion theory of the category $R\text{-mod}$, then left R -module M is called *torsion free module* if and only if there exist R from M into some member of τ . Class of all torsion free modules for some τ is denote by \mathfrak{F}_τ . Further information about the prime torsion theories can be fund in [5].

Remark 1. The class of all torsion theories $R\text{-tors}$ can be partially ordered by setting $\tau \leq \tau'$ if and only if $\mathfrak{F}_\tau \subseteq \mathfrak{F}_{\tau'}$, namely, the class of all torsion modules of one torsion theory is contained in the class of all torsion modules of other torsion theory.

Introduce the notion of torsion-theoretic spectrum of a module M . Use the concepts of torsion-theoretic spectrum of a ring R introduced above. Introduce the concept of support of module M : $\text{supp}(M) = \{\sigma | \sigma(M) \neq 0\}$. *Torsion-theoretic spectrum of module* M , $R\text{-Sp}(M)$ is defined as $R\text{-sp}(R) \cap \text{supp}(M)$.

If M is a left R -module, denote by $\xi(M)$ the smallest torsion theory such that M will be a torsion module, by $\chi(M)$ the largest torsion theory,

that M will be a torsion-free module. Clearly, $\mathcal{T}_{\chi(M)}$ consists of R -modules N such that $\text{Hom}_R(N, E(M)) = 0$, where $E(M)$ is the injective hull of a module M .

Lemma 8. *If σ is a torsion theory and \mathfrak{P} is a left Rosenberg point of a cyclic module M , then M/\mathfrak{P} is either a σ -torsion module or a σ -torsion free module.*

Proof. Assume that $M/\mathfrak{P} \notin \mathcal{F}_\sigma$. If \mathfrak{P} is a left Rosenberg point, then there exists ideal \mathfrak{p} of a ring R such that $\mathfrak{P} = \mathfrak{p}/\text{Ann}(m)$. Pick an element $0 = \bar{x} \in \sigma(R/\mathfrak{p})$. Thus, there exists a finite subset V of the ring R with $(\mathfrak{p} : Vx) \subseteq \mathfrak{p}$. Obviously, $V\bar{x} \subseteq \sigma(R/\mathfrak{p})$, hence, for every element $v \in V$ there exists left ideal $L_v \in \mathcal{L}(\sigma)$ such that $L_v vx \subseteq \mathfrak{p}$. Let $L = \bigcap_{v \in V} L_v$, then $L \in \mathcal{L}(\sigma)$ and $LVx \subseteq \mathfrak{p}$. Hence $L \subseteq (\mathfrak{p} : Vx) \subseteq \mathfrak{p}$ and $\mathfrak{p} \in \mathcal{L}(\sigma)$, and therefore M/\mathfrak{P} is σ -torsion module. \square

Proposition 1. *If M is a fully bounded left noetherian module and $\mathfrak{P} \in \text{Cspec}(M)$, then the torsion theory $\tau_{\mathfrak{P}} = \chi(M/\mathfrak{P})$ cogenerated by module M/\mathfrak{P} is prime.*

Proof. Obviously, $\mathfrak{P} \notin \mathcal{L}(\tau_{\mathfrak{P}})$, therefore M/\mathfrak{P} is a $\tau_{\mathfrak{P}}$ -torsion free module. Thus, since $\chi(M/\mathfrak{P})$ is the largest torsion theory for which M/\mathfrak{P} is torsion free module. We have $\chi(M/\mathfrak{P}) \leq \tau_{\mathfrak{P}}$. Conversely, assume that $\mathcal{L}(\chi(M/\mathfrak{P})) \not\subseteq \mathcal{L}(\tau_{\mathfrak{P}})$. Take $L \in \mathcal{L}(\chi(M/\mathfrak{P})) - \mathcal{L}(\tau_{\mathfrak{P}})$, then $L \leq \mathfrak{P}$. Thus, by the definition, $\mathfrak{A} \leq \mathfrak{p}$ for some ideals \mathfrak{A} and \mathfrak{p} of the ring R . Thus there exists a finite subset $U \subseteq R$ such that $\bigcap_{u \in U} (\mathfrak{A} : u) = (\mathfrak{A} : U) \subseteq \mathfrak{p}$. Hence $\mathfrak{p} \in \mathcal{L}(\chi(M/\mathfrak{P}))$, contradicting the definition of $\chi(M/\mathfrak{P})$. \square

The previous statements imply the following result.

Theorem 3. *The mapping $\Phi: \text{Cspec}(M) \rightarrow \text{M-sp}$, where $\Phi(\mathfrak{P}) = \chi(M/\mathfrak{P})$ is continuous and surjective.*

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CONTACT INFORMATION

M. Maloid-Glebova Ivan Franko National University of L'viv
E-Mail(s): martamaloid@gmail.com

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