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Classification of $\mathscr{L}$-cross-sections of the finite symmetric semigroup up to isomorphism

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Abstract. Let $\mathcal{T}_n$ be the symmetric semigroup of full transformations on a finite set with n elements. In the paper we give a counting formula for the number of $\mathscr{L}$-cross-sections of $\mathcal{T}_n$ and classify all $\mathscr{L}$-cross-sections of $\mathcal{T}_n$ up to isomorphism.

Introduction

Let ρ be an equivalence relation on a semigroup S. A subsemigroup S' of S is called a ρ-cross-section of S provided that S' contains exactly one representative from each equivalence class of ρ. Thus, the restriction ρ to the subsemigroup S' is the identity relation. It is natural to investigate the cross-sections with respect to equivalences related somehow to the semigroup operation: Green's relations, conjugacy and various congruences. In general, a semigroup need not to have a ρ-cross-section. It is possible, for example, that a semigroup S has an $\mathscr{R}$-cross-section, while $\mathscr{L}$-cross-sections of S do not exist at all. Thus, the existence of cross-sections of a given semigroup is an essential and non-obvious fact.

The transformation semigroups are classical objects for investigations in semigroup theory (see [1]). For the full finite symmetric semigroup $\mathcal{T}_n$, all $\mathscr{H}$- and $\mathscr{R}$-cross-sections have been described in [3]. It has been proved

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that there exists a unique $\mathscr{R}$-cross-section up to isomorphism. A pair of non-isomorphic $\mathscr{L}$-cross-sections of $\mathscr{T}_4$ has been constructed in [4]. The author has obtained a description of the $\mathscr{L}$-cross-sections of $\mathscr{T}_n$ in [5] (see Theorem 1).

In the present paper we continue to investigate $\mathscr{L}$-cross-sections of $\mathscr{T}_n$. We give necessary information in Section 1. Section 2 is devoted to some additional definitions. In Section 3 we show how to count all different $\mathscr{L}$-cross-sections of $\mathscr{T}_n$ (Theorem 2). In Section 4 we classify all $\mathscr{L}$-cross-sections up to isomorphism (Theorem 3).

1. Preliminaries

For any nonempty set X, the set of all transformations of X into itself, written on the right, constitutes a semigroup under the composition $x(\alpha\beta) = (x\alpha)\beta$ for all $x \in X$. This semigroup is denoted by $\mathscr{T}(X)$ and called the symmetric semigroup. If $|X| = n$, then the symmetric semigroup $\mathscr{T}(X)$ is also denoted by $\mathscr{T}_n$. We write id_X for the identity transformation on X, and c_x for the constant transformation whose image is the singleton $\{x\}$, $x \in X$. For the image of a transformation $\alpha \in \mathscr{T}_n$ we write $\mathrm{im}\,(\alpha)$. The cardinality $|\,\mathrm{im}\,(\alpha)|$ of the image of α is called the rank of this transformation and is denoted by $\mathrm{rk}\,(\alpha)$. The kernel of α is denoted by $\ker\alpha$. Recall that $\ker\alpha = \{(a,b) \in X \times X \mid a\alpha = b\alpha\}$. If X' is a subset of X, then $\alpha|_{X'}$ is the restriction α to X'. We will assume X is finite. As the nature of elements of X is not important for us, suppose further that $X = \{1, 2, \ldots, n\}$.

We recall that two elements in a semigroup S are called $\mathscr{L}$-equivalent provided that they generate the same principal left ideal in S. Transformations $\alpha, \beta \in \mathscr{T}_n$ are $\mathscr{L}$-equivalent if and only if $\mathrm{im}\,(\alpha) = \mathrm{im}\,(\beta)$ (see e.g. [2]). The last means that an $\mathscr{L}$-cross-section of $\mathscr{T}_n$ contains exactly one transformation with the image M for each nonempty $M \subseteq X$. We will use the last fact frequently. Suppose further that L is an $\mathscr{L}$-cross-section in $\mathscr{T}_n$.

First we isolate two trivial cases:

(i) $L = \{c_1 = \mathrm{id}_X\}$, if $n = 1$;

(ii) $L = \{\mathrm{id}_X, c_1, c_2\}$, if $n = 2$.

For the rest of the paper we may and will assume that $n \geqslant 3$.

In order to present our description of $\mathscr{L}$-cross-sections for an arbitrary finite $\mathscr{T}_n$ [5], we need following definitions.

Let X be a nonempty finite set and let $<$ be a strict total order on X. We define a strict order $\prec$ on the family of all nonempty subsets of X by: $A \prec B$ if for all $a \in A$ and all $b \in B$, $a < b$.

Denote by $\{1,2\}^+$ the free semigroup of words over the alphabet $\{1,2\}$, and by $\{1,2\}^*$ the free monoid over $\{1,2\}$, with 0 as the empty word. Recall, that *a subsequence* of $b \in \{1,2\}^*$ is a word a that can be derived from b by deleting some symbols without changing the order of the remaining symbols. If a is a subsequence of b we will write $a \subseteq b$.

Definition 1. Let X be a finite set (possibly empty) and let $<$ be a strict total order on X. An indexed family $\{A_a\}_{a\in\{1,2\}^*}$ of subsets of X is called a Γ-*family* over $(X,<)$ if for every $a \in \{1,2\}^*$:
(a) $A_0 = X$;
(b) if $|A_a| \leqslant 1$, then $A_{a1} = A_{a2} = \varnothing$;
(c) if $|A_a| > 1$, then A_{a1} and A_{a2} are nonempty with $A_{a1} \prec A_{a2}$ and $A_a = A_{a1} \cup A_{a2}$.

We will say that $\{A_a\}_{a\in\{1,2\}^*}$ is a Γ-family over X if $\{A_a\}_{a\in\{1,2\}^*}$ is a Γ-family over $(X,<)$ for some strict total order $<$ on X (necessarily unique). For simplicity, we will write $\Gamma = \{A_a\}$ instead of $\Gamma = \{A_a\}_{a\in\{1,2\}^*}$.

Recall that a tree is a connected graph without cycles. A full binary tree is defined as a tree in which there is exactly one vertex of degree two (referred to as the root) and each of the remaining vertices is of degree one or three. Vertices of degree one are called leaves. Each vertex except the root has a unique parent, that is, the vertex connected to it on the path to the root. A child of a vertex v is a vertex of which v is the parent. Thus, in a full binary tree each vertex v either is a leaf or has exactly two children that we refer to as the left child of v and the right child of v.

It is easy to see that every Γ-family $\Gamma = \{A_a\}$ over a nonempty set can be represented by a rooted full binary tree $T(\Gamma)$ whose vertices are the nonempty sets from $\{A_a\}$ and a pair $\{A_a, A_b\}$, for $a, b \in \{1,2\}^*$, is an edge if and only if $a = bi$ or $b = ai$, where $i \in \{1,2\}$ (see Fig. 1). For the full binary tree that represents a Γ-family Γ, we will write Γ instead of $T(\Gamma)$, and refer to the tree as a Γ-tree.

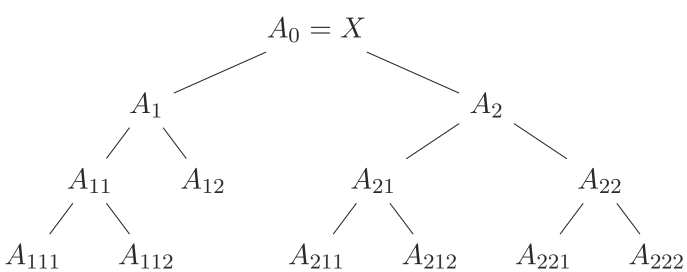

FIGURE 1. A Γ-tree.

Definition 2. A Γ-family $\Gamma = \{A_a\}$ over $(X, <)$ is called an *L-family* over $(X, <)$ if for all $a, b \in \{1,2\}^*$ and all $i, j \in \{1, 2\}$ with $i \neq j$,

$$|A_{aijb}| \leqslant |A_{ajb}|. \tag{1}$$

We will say that $\{A_a\}_{a\in\{1,2\}^*}$ is an *L-family* over X if $\{A_a\}_{a\in\{1,2\}^*}$ is an L-family over $(X, <)$ for some strict total order $<$ on X.

Example 1. Let $\{1, 2, 3, 4, 5\}$ be naturally ordered. Consider the following Γ-family $\{A_a\}$ (see Fig. 2).

$$A_0 = \{1,2,3,4,5\}$$

$$A_1 = \{1,2\} \qquad A_2 = \{3,4,5\}$$

$$A_{11} = \{1\} \quad A_{12} = \{2\} \qquad A_{21} = \{3\} \quad A_{22} = \{4,5\}$$

$$A_{221} = \{4\} \quad A_{222} = \{5\}$$

FIGURE 2. Γ-family $\{A_a\}$.

This Γ-family satisfies condition (2) for all $a, b \in \{1, 2\}^*$ and all $i, j \in \{1, 2\}$ with $i \neq j$, hence $\{A_a\}$ is an L-family by definition.

Figure 3 shows a Γ-family $\{B_a\}$ that does not satisfy condition (2) since $|B_{21}| \geqslant |B_1|$.

$$B_0 = \{1,2,3,4,5\}$$

$$B_1 = \{1\} \qquad B_2 = \{2,3,4,5\}$$

$$B_{21} = \{2,3\} \qquad B_{22} = \{4,5\}$$

$$B_{211} = \{2\} \quad B_{212} = \{3\} \quad B_{221} = \{4\} \quad B_{222} = \{5\}$$

FIGURE 3. Γ-family $\{B_a\}$.

Let Γ be an L-family of subsets of X, $M \subseteq X$ and $M \neq \varnothing$. Our aim now is to construct a map $\alpha_M^{A_a} : A_a \to M$ with $\operatorname{im}(\alpha_M^{A_a}) = M$. We construct this map inductively using partial transformations, whose domains go through vertices of a Γ-tree bottom up. For the domain of a partial transformation f we write $\operatorname{dom}(f)$.

For functions f and g with disjoint domains, we denote by $f \cup g$ the union of f and g (viewed as sets of pairs). In other words, if $h = f \cup g$, then $\operatorname{dom}(h) = \operatorname{dom}(f) \cup \operatorname{dom}(g)$ and for all $x \in \operatorname{dom}(h)$, $xh = xf$ if $x \in \operatorname{dom}(f)$, and $xh = xg$ if $x \in \operatorname{dom}(g)$.

Definition 3. Let $\Gamma = \{A_a\}$ be an L-family over X and let $M \subseteq X$ with $M \neq \varnothing$. Denote by $\langle M \rangle$ the intersection of all $A_c \in \Gamma$ such that $M \subseteq A_c$, and note that $\langle M \rangle = A_b$ for some $b \in \{1,2\}^*$. For every $a \in \{1,2\}^*$, we define the *mapping* $\alpha_M^{A_a}$ inductively as follows:

(a) if $A_a = \varnothing$ then $\alpha_M^{A_a} = \varnothing$ (empty mapping);

(b) if $M = \{m\}$ and $A_a \neq \varnothing$, then $\operatorname{dom}(x\alpha_M^{A_a}) = A_a$ and $x\alpha_M^{A_a} = m$ for every $x \in A_a$;

(c) if $|M| > 1$ and $A_a \neq \varnothing$, then $\alpha_M^{A_a} = \alpha_{M\cap A_{b1}}^{A_{a1}} \cup \alpha_{M\cap A_{b2}}^{A_{a2}}$.

Lemma 1. *Let $\Gamma = \{A_a\}$ be an L-family over X. If $M \subseteq A_a$ or $A_a \neq \varnothing$ and $M \cap A_a = \varnothing$ then* $\operatorname{dom}(x\alpha_M^{A_a}) = A_a$ *and* $\operatorname{im}(x\alpha_M^{A_a}) = M$.

Proof. The proof is by induction on $|M|$. If $M = \{m\}$, then the statement is true by (b) of Definition 3. Let $|M| > 1$ and suppose the statement is true for every M' with $1 \leqslant |M'| < |M|$. Assume $M \subseteq A_a$ or $A_a \neq \varnothing$ and $M \cap A_a = \varnothing$. By (c) of Definition 3, $\alpha_M^{A_a} = \alpha_{M\cap A_{b1}}^{A_{a1}} \cup \alpha_{M\cap A_{b2}}^{A_{a2}}$. Consider two possible cases.

Case 1. $M \subseteq A_a$. Then $A_b \subseteq A_a$ since A_b is the intersection of all A_c such that $M \subseteq A_c$. If $A_b = A_a$ then

$$M \cap A_{b1} = M \cap A_{a1} \subseteq A_{a1},$$
$$M \cap A_{b2} = M \cap A_{a2} \subseteq A_{a2},$$

and $|M \cap A_{b1}|$, $|M \cap A_{b2}| < |M|$ (since $\langle M \rangle = A_b$). Thus, by the inductive hypothesis, the statement is true for $\alpha_{M\cap A_{b1}}^{A_{a1}}$ and for $\alpha_{M\cap A_{b2}}^{A_{a2}}$. Hence it is true for $\alpha_M^{A_a}$.

If $A_b \neq A_a$ then, since $A_a = A_{a1} \cup A_{a2}$ and $A_{a1} \cap A_{a2} = \varnothing$, we get either $A_b \subseteq A_{a1}$ or $A_b \subseteq A_{a2}$. We may assume that $A_b \subseteq A_{a1}$. Then

$$M \cap A_{b1} \subseteq A_{a1},$$
$$(M \cap A_{b2}) \cap A_{a2} = \varnothing.$$

Note that $A_{a2} \neq \varnothing$ (since $M \subseteq A_a$ and $|M| > 1$) and $|M \cap A_{b1}|$, $|M \cap A_{b2}| < |M|$ (since $\langle M \rangle = A_b$). Again, the statement follows by the inductive hypothesis from $\alpha_M^{A_a} = \alpha_{M\cap A_{b1}}^{A_{a1}} \cup \alpha_{M\cap A_{b2}}^{A_{a2}}$.

Case 2. $A_a \neq \varnothing$ and $M \cap A_a = \varnothing$. Then $(M \cap A_{b1}) \cap A_{a1} = \varnothing$ and $(M \cap A_{b2}) \cap A_{a2} = \varnothing$. As before, we get the statement by the inductive hypothesis from $\alpha_M^{A_a} = \alpha_{M\cap A_{b1}}^{A_{a1}} \cup \alpha_{M\cap A_{b2}}^{A_{a2}}$. $\square$

Denote by L_X^Γ the set of all transformations of the form α_M^X, where $M \subseteq X$, $M \neq \varnothing$. We will denote the elements α_M^X also by α_M.

Example 2. Let $\{1,2,3,4,5\}$ be naturally ordered. We will construct the transformation $\alpha = \alpha_M$ with $M = \{1,2,4,5\}$ for the L-family $\{A_a\}$ from Example 1. Clearly, $\langle M\rangle = A_0$, so by definition of α_M

$$\alpha = \alpha_M^{A_0} = \alpha_{M\cap A_1}^{A_1} \cup \alpha_{M\cap A_2}^{A_2} = \alpha_{\{1,2\}}^{A_1} \cup \alpha_{\{4,5\}}^{A_2}.$$

Since $\langle\{1,2\}\rangle = A_1$, $\langle\{4,5\}\rangle = A_{22}$, thus

$$\alpha_{\{1,2\}}^{A_1} = \alpha_{\{1,2\}\cap A_{11}}^{A_{11}} \cup \alpha_{\{1,2\}\cap A_{12}}^{A_{12}} = \alpha_{\{1\}}^{A_{11}} \cup \alpha_{\{2\}}^{A_{12}},$$

$$\alpha_{\{4,5\}}^{A_2} = \alpha_{\{4,5\}\cap A_{221}}^{A_{21}} \cup \alpha_{\{4,5\}\cap A_{222}}^{A_{22}} = \alpha_{\{4\}}^{A_{21}} \cup \alpha_{\{5\}}^{A_{22}}.$$

Thus, since $A_{11} = \{1\}$, $A_{12} = \{2\}$, $A_{21} = \{3\}$, and $A_{22} = \{4,5\}$, we have

$$\alpha = \alpha_{\{1\}}^{A_{11}} \cup \alpha_{\{2\}}^{A_{12}} \cup \alpha_{\{4\}}^{A_{21}} \cup \alpha_{\{5\}}^{A_{22}} = \begin{pmatrix} 12345 \\ 12455 \end{pmatrix}.$$

The other transformations from L_X^Γ can be obtained in the same way (see [5, Example 3]).

The following theorem describes the $\mathscr{L}$-cross-sections of $\mathscr{T}_n$:

Theorem 1 ([5, Theorem 1])**.** *For each L-family Γ of X, the set L_X^Γ is an $\mathscr{L}$-cross-section of the symmetric semigroup $\mathscr{T}_n$. Conversely, every $\mathscr{L}$-cross-section of the symmetric semigroup $\mathscr{T}_n$ is given by L_X^Γ for a suitable L-family Γ on X.*

2. Alternative definition of L-family

Since the definition of an L-family may seem difficult to use and understand, we try to find a way to make it easy and more visual. We state a new definition in Proposition 1. But first we need some preparation.

Definition 4. Let Γ_1, Γ_2 be the full binary trees that represent Γ-families $\{A_a\}$ over X_1 and $\{B_a\}$ over X_2 respectively. We say that Γ_1 *is less than or equal to* Γ_2, written $\Gamma_1 \leqslant \Gamma_2$, if $|A_a| \leqslant |B_a|$ for all $a \in \{1,2\}^*$.

Let $\Gamma = \{A_a\}_{a\in\{1,2\}^*}$ be a Γ-family over X. For every $a \in \{1,2\}^*$, denote by $\Gamma(a)$ the family $\{B_b\}_{b\in\{1,2\}^*}$ of subsets of A_a such that $B_b = A_{ab}$ for each $b \in \{1,2\}^*$. It is clear that $\Gamma(a)$ is a Γ-family over the set A_a and that, if $A_a \neq \varnothing$, then $\Gamma(a)$ is represented by the subtree $\Gamma(a)$ of the full binary tree Γ with the root A_a.

Definition 5. Let Γ be a Γ-tree. For all $a \in \{1,2\}^*$ and $i \in \{1,2\}$, we call the tree $\Gamma(a)$ the *parent tree* of the subtree $\Gamma(ai)$. We will say that Γ is *monotone* if for all $a \in \{1,2\}^*$ and $i \in \{1,2\}$, $\Gamma(ai) \leqslant \Gamma(a)$.

Proposition 1. *A Γ-tree Γ with root X represents an L-family over X if and only if Γ is monotone.*

Proof. Necessity. Suppose that a Γ-tree Γ with root X represents an L-family over X and let $a \in \{1,2\}^*$. We aim to prove that $\Gamma(a1) \leqslant \Gamma(a)$. If $|A_{a1}| = 1$, then it is clear that $\Gamma(a1) \leqslant \Gamma(a)$. Let $|A_{a1}| > 1$. To prove $\Gamma(a1) \leqslant \Gamma(a)$ we show first $\Gamma(a12) \leqslant \Gamma(a2)$ and then $\Gamma(a11) \leqslant \Gamma(a1)$.

Let $\Gamma(a12)$ represent $\{B_b\}$ and let $\Gamma(a2)$ represent $\{C_b\}$. Then, for every $b \in \{1,2\}^*$,

$$|B_b| = |A_{a12b}| \leqslant |A_{a2b}| = |C_b|,$$

where $\leqslant$ follows by (2). Thus, $\Gamma(a12) \leqslant \Gamma(a2)$.

To prove that $\Gamma(a11) \leqslant \Gamma(a1)$, denote by $\{B_b\}$ and $\{C_b\}$ the L-families that are represented by $\Gamma(a11)$ and $\Gamma(a1)$, respectively. Denote by $\overline{k}$, $k \geqslant 0$, the empty word 0 if $k = 0$ and $\underbrace{11\ldots1}_{k} \in \{1,2\}^*$ if $k \geqslant 1$. Then, for every $b \in \{1,2\}^*$, if $b = \overline{k}$, $k \geqslant 0$, then

$$|B_b| = |A_{a11\overline{k}}| \leqslant |A_{a1\overline{k}}| = |C_b|,$$

since $A_{a11\overline{k}} \subset A_{a1\overline{k}}$; and if $b = \overline{k}2c$ ($k \geqslant 0,\ c \in \{1,2\}^*$), then

$$|B_b| = |A_{a11\overline{k}2c}| \leqslant |A_{a1\overline{k}2c}| = |C_b|,$$

where $\leqslant$ follows by (2).

Now, since $|A_{a1}| < |A_a|$, $\Gamma(a11) \leqslant \Gamma(a1)$ and $\Gamma(a12) \leqslant \Gamma(a2)$, we get $\Gamma(a1) \leqslant \Gamma(a)$. In dual way, one can show that $\Gamma(a2) \leqslant \Gamma(a)$. So any subtree of Γ is less than or equal to the parent tree of this subtree, thus Γ is monotone.

Sufficiency. Let $a \in \{1,2\}^*$ and $i,j \in \{1,2\}$ with $i \neq j$. Let the subtrees $\Gamma(ai)$ and $\Gamma(a)$ of Γ represent $\{B_b\}_{b\in\{1,2\}^*}$ and $\{C_b\}_{b\in\{1,2\}^*}$, respectively. Since $\Gamma(ai) \leqslant \Gamma(a)$,

$$|A_{aijb}| = |B_{jb}| \leqslant |C_{jb}| = |A_{ajb}|.$$

Hence (2) holds, that is, Γ is an L-family. $\square$

Definition 6. For $n \in \mathbb{N}$ we will write Γ^n to mean an L-family over a set with n elements. Let $\Gamma^n = \{A_a\}$ be an L-family with $n \geqslant 2$. Let $s, t \in \{1, 2, \ldots, n\}$ with $s + t \leqslant n$. We denote by $Q_{s,t}$ the set of all pairs (Γ^s, Γ^t) of L-families Γ^s and Γ^t such that:

(a) $\Gamma^s = \Gamma^n(a)$ and $\Gamma^t = \Gamma^n(b)$ for $a, b \in \{1, 2\}^*$ such that $A_a \cap A_b = \varnothing$;

(b) if $s > 1$ then $\Gamma^s(2) \leqslant \Gamma^t$, and if $t > 1$ then $\Gamma^t(1) \leqslant \Gamma^s$.

Example 3. Figure 4 shows a pair of L-families (Γ^4, Γ^5) that does not belong to $Q_{4,5}$. To simplify the picture we denote the nodes of the trees by their cardinalities.

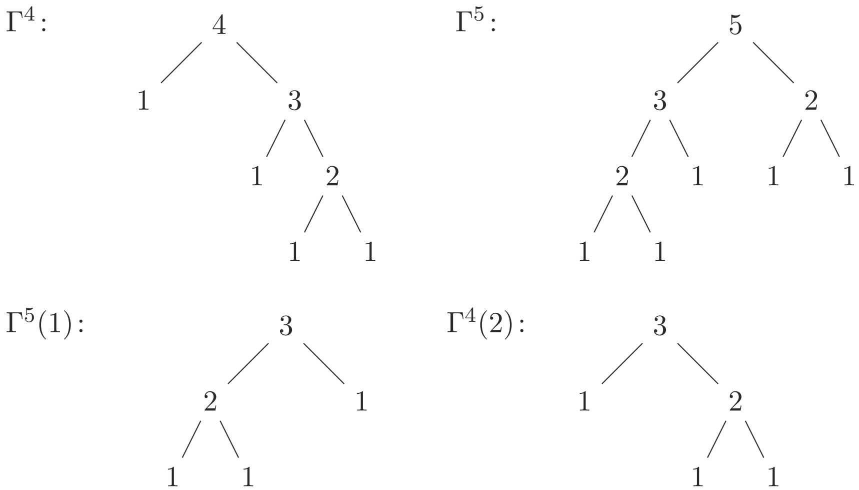

FIGURE 4. Γ^4 and Γ^5 such that $(\Gamma^4, \Gamma^5) \notin Q_{4,5}$.

As the picture shows, Γ^4 and $\Gamma^5(1)$ do not satisfy the condition $\Gamma^5(1) \leqslant \Gamma^4$ ($2 > 1$ in the first position). However, Γ^5 and $\Gamma^4(2)$ satisfy the condition $\Gamma^4(2) \leqslant \Gamma^5$.

Fix a total order $<$ on an n-element set X and denote by Q_n the number of L-families over X.

Proposition 2. *The number Q_n of all distinct L-families Γ on the totally ordered set $(X, <)$, with $|X| = n$, is given by the formula:*

$$Q_1 = 1, \quad Q_n = \sum_{\substack{s,t \\ s+t=n}} |Q_{s,t}| \quad \text{if} \quad n \geqslant 2.$$

Proof. Obviously, $Q_1 = 1$. Let $n \geqslant 2$. Let Γ^n be an L-family over $(X, <)$ and let $\Gamma^s = \Gamma^n(1)$ and $\Gamma^t = \Gamma^n(2)$. It is clear that $s + t = n$. Using

Proposition 1, we get $\Gamma^s \leqslant \Gamma^n$, $\Gamma^t \leqslant \Gamma^n$, whence $\Gamma^s(2) \leqslant \Gamma^t$, $\Gamma^t(1) \leqslant \Gamma^s$ and thus $(\Gamma^s, \Gamma^t) \in Q_{s,t}$. It is then clear that the mapping $\Gamma^n \to (\Gamma^{n(1)}, \Gamma^{n(2)})$ is a bijection from the set of L-families over $(X, <)$ onto the union of the sets $Q_{s,t}$ with $s + t = n$.

Second variant of proof, using the first definition of an L-family (condition (2)): Since Γ^n is an L-family, if $|A_1| > 1$ then $|A_{12b}| \leqslant |A_{2b}|$ for all $b \in \{1,2\}^*$, therefore $\Gamma^s(2) \leqslant \Gamma^t$. Analogously we obtain $\Gamma^t(1) \leqslant \Gamma^s$.

Thus, $\Gamma^s, \Gamma^t \in Q_{s,t}$ and $Q_n = \sum_{s+t=n} |Q_{s,t}|$, for $n \geqslant 2$. □

We give the initial values of Q_n, $n \in \mathbb{N}$ below. To calculate them we have used a computer programm.

n	1	2	3	4	5	6	7	8	9	10
Q_n	1	1	2	3	6	10	18	32	58	101

3. The number of $\mathscr{L}$-cross-sections of $\mathscr{T}_n$

Suppose $a \in \{1,2\}^*$ is an arbitrary word. The word obtained from a by replacing each 1 by 2 and each 2 by 1, is denoted by $\bar{a}$.

Definition 7. Let $\Gamma_1 = \{A_a\}$, $\Gamma_2 = \{B_a\}$ be L-families over X_1 and X_2, respectively. We say that Γ_1 and Γ_2 are *similar* if

$$\forall a \in \{1,2\}^* \ |A_a| = |B_a| \ \text{ or } \ \forall a \in \{1,2\}^* \ |A_a| = |B_{\bar{a}}|.$$

The similarity of L-families Γ_1 and Γ_2 is denoted by $\Gamma_1 \sim \Gamma_2$.

The relation of similarity is clearly an equivalence and partitions the set of all L-families over the n-element set into disjoint equivalence classes.

Lemma 2. *Let $<_1$, $<_2$ be strict total orders on X, $\Gamma_1 = \{A_a\}$, $\Gamma_2 = \{B_a\}$ be arbitrary L-families over $(X, <_1)$ and $(X, <_2)$, respectively. If $L_1 = L^{\Gamma_1}_{<_1}$, $L_2 = L^{\Gamma_2}_{<_2}$ are corresponding $\mathscr{L}$-cross-sections of $\mathscr{T}_n$, then $L_1 = L_2$ if and only if one of the following conditions is satisfied:*

(i) $\Gamma_1 = \Gamma_2$ *(i. e. $\Gamma_1 \sim \Gamma_2$ and $<_1 = <_2$);*

(ii) $\Gamma_1 \sim \Gamma_2$ *and* $<_2 = <_1^{-1}$.

Proof. Sufficiency. Obviously (i) implies $L_1 = L_2$. Suppose (ii) holds. Then $A_a = B_{\bar{a}}$ for all $a \in \{1,2\}^*$. To prove that $L_1 = L_2$, it suffices to show that $\alpha_M^{A_a} = \alpha_M^{B_{\bar{a}}}$ for all $a \in \{1,2\}^*$ and $M \subseteq X$ with $M \neq \varnothing$. We proceed by induction on $|M|$. Let $M = \{m\}$. If $A_a = \varnothing$, then $B_{\bar{a}} = A_a = \varnothing$, and so $\alpha_M^{A_a} = \varnothing = \alpha_M^{B_{\overline{a}}}$. If $A_a \neq \varnothing$ then $\mathrm{dom}\,(\alpha_M^{A_a}) = A_a = B_{\overline{a}} = \mathrm{dom}\,(\alpha_M^{B_{\overline{a}}})$ and for all x in the common domain, $x\alpha_M^{A_a} = m = x\alpha_M^{B_{\overline{a}}}$, which implies $\alpha_M^{A_a} = \alpha_M^{B_{\overline{a}}}$.

Let $|M| > 1$ and suppose that the statement is true for all $M_1 \subseteq X$ with $|M_1| < M$. Again, if $A_a = \varnothing$, then $B_{\overline{a}} = A_a = \varnothing$, and so $\alpha_M^{A_a} = \varnothing = \alpha_M^{B_{\overline{a}}}$. Suppose that $A_a \neq \varnothing$ and let $\langle M \rangle = A_b$, $b \in \{1,2\}^*$. Then $B_{\overline{b}} = A_b = \langle M \rangle$, and so

$$\begin{aligned}
\alpha_M^{A_a} &= \alpha_{M\cap A_{b1}}^{A_{a1}} \cup \alpha_{M\cap A_{b2}}^{A_{a2}},\\
\alpha_M^{B_{\bar{a}}} &= \alpha_{M\cap A_{\bar{b}1}}^{B_{\bar{a}1}} \cup \alpha_{M\cap A_{\bar{b}2}}^{B_{\bar{a}2}} = \alpha_{M\cap A_{\overline{b2}}}^{B_{\overline{a2}}} \cup \alpha_{M\cap A_{\overline{b1}}}^{B_{\overline{a1}}}.
\end{aligned}$$

By the inductive hypothesis, $\alpha_{M\cap A_{b1}}^{A_{a1}} = \alpha_{M\cap A_{\overline{b1}}}^{B_{\overline{a1}}}$ and $\alpha_{M\cap A_{b2}}^{A_{a2}} = \alpha_{M\cap A_{\overline{b2}}}^{B_{\overline{a2}}}$. Thus $\alpha_M^{A_a} = \alpha_M^{B_{\bar{a}}}$.

Necessity. Let $L_1 = L_2$. According to [5, Corollary 4], Γ_1 and $\bigcup_{\alpha\in L_1} X/\ker\alpha$ coincide as unindexed families of sets. The same result is true for Γ_2 and L_2. Since $L_1 = L_2$, it follows that Γ_1 and Γ_2 are the same as unindexed families of sets.

If $<_1 = <_2$, then Γ_1 and Γ_2 coincide as L-families, so (i) holds. Suppose $<_2 = <_1^{-1}$. Then $A_a = B_{\overline{a}}$ for all $a \in \{1,2\}^*$, which implies $\Gamma_1 \sim \Gamma_2$, so (ii) holds.

To complete the proof we show that in all other cases one gets a contradiction. Let $<_1 \neq <_2 \neq {<_1}^{-1}$. Since Γ_1 and Γ_2 are the same as unindexed families of sets, we have either $A_1 = B_1$ and $A_2 = B_2$ or $A_1 = B_2$ and $A_2 = B_1$. First suppose that $A_i = B_i$, $i \in \{1,2\}$. Let $x, y \in X$ such that $x <_1 y$, $y <_2 x$. Then $\begin{pmatrix} A_1 & A_2 \\ x & y \end{pmatrix} \in L_1$, $\begin{pmatrix} B_1 & B_2 \\ y & x \end{pmatrix} = \begin{pmatrix} A_1 & A_2 \\ y & x \end{pmatrix} \in L_2$ and we get a contradiction with $L_1 = L_2$.

Suppose now that $A_1 = B_2$, $A_2 = B_1$. Let $x, y \in X$ such that $x <_1 y$ and $x <_2 y$. In this case we have $\begin{pmatrix} A_1 & A_2 \\ x & y \end{pmatrix} \in L_1$, $\begin{pmatrix} B_1 & B_2 \\ x & y \end{pmatrix} = \begin{pmatrix} A_2 & A_1 \\ x & y \end{pmatrix} \in L_2$. The last is impossible since $L_1 = L_2$. □

Theorem 2. *The number of different $\mathscr{L}$–cross-sections in the semigroup $\mathcal{T}_n$, $n \geqslant 2$, equals $Q_n \cdot \frac{n!}{2}$.*

Proof. Let A_Γ and A_L be the sets of L-families over X and $\mathscr{L}$-cross-sections in $\mathcal{T}_n$, respectively. Since there are $n!$ strict orders on X and Q_n L-families for each strict order $<$, $|A_\Gamma| = Q_n \cdot n!$. Define a mapping $\omega : A_\Gamma \to A_L$ by $\Gamma\omega = L^\Gamma$. By Theorem 1, ω is onto. Suppose that $\Gamma_1\omega = \Gamma_2\omega$ with $\Gamma_1 \neq \Gamma_2$. Let $\Gamma_1 = \{A_a\}$ and $\Gamma_2 = \{B_a\}$. By Lemma 2, we have $B_a = A_{\bar{a}}$ for every $a \in \{1,2\}$. Thus ω is two-to-one, and so $|A_L| = \frac{|A_\Gamma|}{2} = Q_n \cdot \frac{n!}{2}$. $\square$

4. The classification of $\mathscr{L}$-cross-sections of $\mathscr{T}_n$ up to isomorphism

It is well known that not all $\mathscr{L}$-cross-sections in semigroup $\mathscr{T}(X)$ are isomorphic to each other (see [4]). We now investigate when two L-families correspond to isomorphic $\mathscr{L}$-cross-sections. Throughout this section let L_1 and L_2 be two $\mathscr{L}$-cross-sections of $\mathscr{T}(X_n)$; $\Gamma_1 = \{A_a\}$, $\Gamma_2 = \{B_a\}$ be the L-families associated with L_1 and L_2, i. e. $L_1 = L_X^{\Gamma_1}$ and $L_2 = L_X^{\Gamma_2}$.

Note that if $|X| \leqslant 3$ all the possible $\mathscr{L}$-cross-sections are isomorphic and all the possible L-families are similar. The following is true for an arbitrary finite set X.

Lemma 3. *If $\Gamma_1 \sim \Gamma_2$, then $L_1 \cong L_2$.*

Proof. If $|A_a| = |B_a|$, for all $a \in \{1,2\}^*$, then set

$$\theta : \Gamma_1 \to \Gamma_2 : A_a \mapsto B_a$$

and if $|A_a| = |B_{\bar{a}}|$ for all $a \in \{1,2\}^*$, then set

$$\theta : \Gamma_1 \to \Gamma_2 : A_a \mapsto B_{\bar{a}}.$$

Without loss of generality we can assume that $|A_a| = |B_a|$, for all $a \in \{1,2\}^*$. Let $x, y \in X$ be arbitrary elements and $A_a = \{x\}$, $A_a \in \Gamma_1$, $a \in \{1,2\}^*$. Set

$$\psi : X \to X : x \mapsto y \Leftrightarrow A_a\theta = \{y\}.$$

It is clear that this mapping is a bijection, and for all $a \in \{1,2\}^*$, we have $A_a\psi = A_a\theta$, where $A_a\psi = \{x\psi \mid x \in A_a\}$. Let

$$\tau : L_1 \to L_2 : \varphi \mapsto \varphi' = \psi^{-1}\varphi\psi.$$

Now we verify that $\varphi' \in L_X^{\Gamma_2}$. To be more precise, we show that $\varphi' = \alpha_{(\operatorname{im}\varphi)\psi}$ for $\varphi \in L_1$ with $\varphi\tau = \varphi'$. Let $a \in \{1,2\}^*$ be an arbitrary element such that $A_a \neq \varnothing$. Consider the image of B_a under the map φ'. Since ψ is a bijection, we have

$$\langle B_a\varphi' \rangle = \langle (A_a\psi)(\psi^{-1}\varphi\psi) \rangle = \langle A_a(\varphi\psi) \rangle = \langle (A_a\varphi)\psi \rangle. \tag{2}$$

We denote by M the image of φ, so $\varphi = \alpha_M$. Let $M = \{m_1, m_2, \dots, m_k\}$ for $m_1, m_2, \dots, m_k \in X$. By definition of α_M we have

$$\alpha_M = \alpha_{\{m_1\}}^{A_{b_1}} \cup \alpha_{\{m_2\}}^{A_{b_2}} \cup \ldots \cup \alpha_{\{m_k\}}^{A_{b_k}}$$

for suitable $b_1, b_2, \dots, b_k \in \{1,2\}^*$. In virtue of arbitrariness of $a \in \{1,2\}^*$ in (2) we obtain $\langle B_{b_i}\varphi'\rangle = \langle (A_{b_i}\varphi)\psi\rangle = \langle \{m_i\psi\}\rangle$, $1 \leqslant i \leqslant k$. Since B_{bi}, $1 \leqslant i \leqslant k$, are pairwise disjoint and $|B_{b_1} \cup B_{b_2} \cup \ldots B_{b_k}| = |A_{b_1} \cup A_{b_2} \cup \ldots A_{b_k}| = |X|$, we get $B_{b_1} \cup B_{b_2} \cup \ldots B_{b_k} = X$, consequently $\operatorname{im}(\varphi') = (\operatorname{im}\varphi)\psi$.

Now to prove $\varphi' = \alpha_{(\operatorname{im}\varphi)\psi}$ it suffices to show $\varphi'|_{B_a} = \alpha^{B_a}_{(A_a\varphi)\psi}$ for all $a \in \{1,2\}^*$. We proceed by induction on $|(A_a\varphi)\psi|$. If $B_a = \varnothing$, then $\varphi'|_{B_a} = \varnothing = \alpha^{B_a}_{(A_a\varphi)\psi}$. If $|B_a| = |A_a| \neq 0$ and $(A_a\varphi)\psi = \{m\}$ then $\operatorname{dom}(\alpha^{B_a}_{(A_a\varphi)\psi}) = B_a = \operatorname{dom}(\varphi'|_{B_a})$ and by (2)

$$\langle \operatorname{im}(\varphi'|_{B_a})\rangle = \langle (A_a\varphi)\psi\rangle = \langle \{m\}\rangle,$$

thus, for all x in the common domain, $x\varphi'|_{B_a} = m = x\alpha^{B_a}_{(A_a\varphi)\psi}$, which implies $\varphi'|_{B_a} = \alpha^{B_a}_{(A_a\varphi)\psi}$.

Let $|(A_a\varphi)\psi| > 1$ and suppose the statement is true for all $M_1 \subseteq X$ with $M_1 \neq \varnothing$ and $|M_1| < |(A_a\varphi)\psi|$. Again, if $B_a = \varnothing$, then $\varphi'|_{B_a} = \varnothing = \alpha^{B_a}_{(A_a\varphi)\psi}$. Suppose $B_a \neq \varnothing$, then, clearly, $\varphi'|_{B_a} = \varphi'|_{B_{a1}} \cup \varphi'|_{B_{a2}}$. By the inductive hypothesis $\varphi'|_{B_{a1}} = \alpha^{B_{a1}}_{(A_{a1}\varphi)\psi}$ and $\varphi'|_{B_{a2}} = \alpha^{B_{a2}}_{(A_{a2}\varphi)\psi}$. Thus

$$\varphi'|_{B_a} = \alpha^{B_{a1}}_{(A_{a1}\varphi)\psi} \cup \alpha^{B_{a2}}_{(A_{a2}\varphi)\psi} = \alpha^{B_a}_{(A_a\varphi)\psi} \quad \text{for all } a \in \{1,2\}^*.$$

Hence,

$$\varphi' = \alpha^{B_1}_{(A_1\varphi)\psi} \cup \alpha^{B_2}_{(A_2\varphi)\psi} = \alpha^{B_1 \cup B_2}_{(A_1\varphi \cup A_2\varphi)\psi} = \alpha_{(\operatorname{im}\varphi)\psi} \in L_X^{\Gamma_2}.$$

Since $\alpha_M\tau = \alpha_{M\psi}$, $M \subseteq X$ and ψ is bijective, we get τ is bijective too.

Finally, for all $\beta, \gamma \in L_1$, we have

$$(\beta)\tau(\gamma)\tau = \psi^{-1}(\beta\gamma)\psi = (\beta\gamma)\tau. \qquad \square$$

To prove the converse we first need some preparations.

Let $\tau : L_1 \to L_2$ be an isomorphism. In both L_1 and L_2, the set $\{c_x \mid x \in X\}$ of constant transformations is the minimum ideal. Thus, τ maps $\{c_x \mid x \in X\}$ onto $\{c_x \mid x \in X\}$. For $x \in X$, denote by x' the element of X such that $c_x\tau = c_{x'}$.

If $A_a \in \Gamma_1$ and $x \in A_a$ is an arbitrary fixed element, then denote by $\varphi(A_a, x)$ the transformation in L_1 with the image $(X \setminus A_a) \cup \{x\}$.

Whenever we say that $L_1 \cong L_2$, we will assume that τ is an isomorphism from L_1 to L_2.

It is clear, that if $\Gamma_1 \sim \Gamma_2$, then $|A_a\theta| = |B_a|$ ($|A_a\bar{\theta}| = |B_a|$). Obviously, if $L_1 \cong L_2$, then $|\Gamma_1| = |\Gamma_2|$. We show in following Lemma, that if $L_1 \cong L_2$, then for every set in Γ_1 there exists a unique set in Γ_2 with the same cardinality.

Lemma 4. *Let $L_1 \cong L_2$. For every $A_a \in \Gamma_1$, $x \in A_a$, the following statements hold true:*

(i) $\varphi(A_a, x)|_{X\setminus A_a} = \mathrm{id}_{X\setminus A_a}$, $\varphi(A_a, x)|_{A_a} = c_x$.

(ii) *there exists $B_{a'} \in \Gamma_2$ such that $|A_a| = |B_{a'}|$ and $\varphi(A_a, x)\tau = \varphi(B_{a'}, x')$, where $c_x\tau = c_{x'}$.*

Proof. (i) For every $A_a \in \Gamma_1$, $x \in A_a$, consider the elements $\varphi(A_a, x) \in L_1$ such that

$$\mathrm{im}\,(\varphi(A_a, x)) = (X \setminus A_a) \cup \{x\}.$$

If $A_a = X$ we get $\varphi(A_a, x) = c_x$, and $\varphi(A_a, x) = \mathrm{id}_X$ if $|A_a| = 1$.

Suppose $A_a \neq X$, $|A_a| > 1$. In this case denote subsets of Γ_1 as follows: put $X = X_1 \uplus X_1'$, if $A_a \subseteq X_1'$; $X_1' = X_2 \uplus X_2'$ if $A_a \subseteq X_2'$; $\ldots$, etc., until we get, for a natural p, that $X_{p-1}' = X_p \uplus X_p'$ and $A_a = X_p'$, where $C = D \uplus E$ means that $C = D \cup E$ and $D \cap E = \varnothing$.

In the proof of [5, Lemma 4, (ii)] it was shown that

$$\sigma_p = \begin{pmatrix} X_1 & X_2 & \ldots & X_p & X_p' \\ x_1 & x_2 & \ldots & x_p & x_p' \end{pmatrix} \in L_1, \tag{3}$$

where $x_p' \in X_p'$, $x_j \in X_j$, $1 \leqslant j \leqslant p$. Since $X \setminus A_a = X_1 \cup X_2 \cup \ldots \cup X_p$, and $x \in A_a = X_p'$ with $X_p' \cap X_i = \varnothing$ for all $1 \leqslant i \leqslant p$, we get

$$\mathrm{im}\,(\varphi(A_a, x)\sigma_p) = \mathrm{im}\,(\sigma_p).$$

From $\varphi(A_a, x)\sigma_p$, $\sigma_p \in L_1$, we obtain $\varphi(A_a, x)\sigma_p = \sigma_p$. The last equality is true for every σ_p as in (3), which is only possible if $\varphi(A_a, x)|_{X\setminus A_a} = \mathrm{id}_{X\setminus A_a}$, $\varphi(A_a, x)|_{A_a} = c_x$.

(ii) Let $x, t \in A_a$, $z \in X \setminus A_a$. On the one hand,

$$(c_z\varphi(A_a, x))\tau = c_z\tau = c_{z'} \quad \text{and} \quad (c_z\tau)(\varphi(A_a, x)\tau) = c_{z'}(\varphi(A_a, x)\tau), \tag{4}$$

so $c_{z'}(\varphi(A_a, x)\tau) = c_{z'}$. On the other hand,

$$(c_t\varphi(A_a, x))\tau = c_x\tau = c_{x'} \quad \text{and} \quad (c_t\tau)(\varphi(A_a, x)\tau) = c_{t'}(\varphi(A_a, x)\tau), \tag{5}$$

so $c_{t'}(\varphi(A_a,x)\tau)=c_{x'}$. Consider $x'(\varphi(A_a,x)\tau)^{-1}\in\ker\varphi(A_a,x)\tau$. For all $t\in A_a$, $z\in X\setminus A_a$ we have $c_{z'}\neq c_{x'}$ (since $x\neq z$), $c_{z'}(\varphi(A_a,x)\tau)=c_{z'}$, $c_{t'}(\varphi(A_a,x)\tau)=c_{x'}$. It follows that

$$x'(\varphi(A_a,x)\tau)^{-1}=\{t'\mid t\in A_a\},$$

and so $\{t'\mid t\in A_a\}\in X/\ker\varphi(A_a,x)\tau$. By [5, Corollary 4], Γ_2 and $\cup_{\alpha\in L_2}X/\ker\alpha$ are the same as unindexed families of sets, thus there exists $B_{a'}\in\Gamma_2$, for some $a'\in\{1,2\}^*$, with $B_{a'}=\{t'\mid t\in A_a\}$. Due to bijectivity of τ, we have $|A_a|=|B_{a'}|$. Furthermore, by (4) and (5), $(\varphi(A_a,x)\tau)|_{X\setminus B_{a'}}=\mathrm{id}_{X\setminus B_{a'}}$ and $(\varphi(A_a,x)\tau)|_{B_{a'}}=c_{x'}$, $x'\in B_{a'}$. Hence, $\varphi(A_a,x)\tau=\varphi(B_{a'},x')$ and $c_{x'}=c_x\tau$. □

Denote the set of all nonempty subsets of X by $U(X)$.

Lemma 5. *Let $L_1\cong L_2$, $\psi:U(X)\to U(X):M\mapsto M'\Leftrightarrow\ \alpha_M\tau=\alpha_{M'}$. The following statements hold true:*
(i) *for all $A_a\in\Gamma_1$, $A_a\psi\in\Gamma_2$;*
(ii) *for all $A_a\in\Gamma_1$ and $\beta\in L_1$, $(A_a\beta)\psi=A_a\psi(\beta\tau)$.*

Proof. (i) It is clear that if $|A_a|=1$, then $A_a\psi\in\Gamma_2$. Let $|A_a|>1$ and $\alpha=\alpha_{A_a}\in L_1$. Let $x\in A_a$ be an arbitrary fixed element, and $\varphi(A_a,x)\tau=\varphi(B_{a'},x')$, $B_{a'}\in\Gamma_2$, $x'\in B_{a'}$, $|A_a|=|B_{a'}|$. Since $\alpha\varphi(A_a,x)=c_x$, we have $(\alpha\tau)\varphi(B_{a'},x')=c_{x'}$, therefore $\mathrm{im}\,(\alpha\tau)\subseteq B_{a'}$. Suppose that $\mathrm{rk}\,(\alpha)>\mathrm{rk}\,(\alpha\tau)$ and denote by β' the transformation from L_2 with $\mathrm{im}\,(\beta')=B_{a'}$.

Let $\delta\in L_1$ such that $\mathrm{im}\,(\delta)\subseteq A_a$. Just as in [5, Lemma 4,(iv)] it can be shown, that there exists $\gamma\in L_1$ with $\mathrm{im}\,(\gamma|_{A_a})=\mathrm{im}\,(\delta)$. We denote this transformation by γ^δ. Thus, for all $\delta\in L_1$ with $\mathrm{im}\,(\delta)\subseteq A_a$, there exists $\gamma^\delta\in L_1$ such that $\delta=\alpha\gamma^\delta$.

Let $\beta=\beta'\tau^{-1}$. Since $\beta'\varphi(B_{a'},x')=c'_x$, it follows that $(\beta'\varphi(B_{a'},x'))\tau^{-1}=\beta\varphi(A_a,x)=c_x$, hence $\mathrm{im}\,(\beta)\subseteq A_a$. Thus, $\beta=\alpha\delta^\beta$, whence $\beta'=(\alpha\tau)(\delta^\beta\tau)$. But

$$\mathrm{rk}\,(\beta')=\mathrm{rk}\,((\alpha\tau)(\delta^\beta\tau))\leqslant\mathrm{rk}\,(\alpha\tau)<\mathrm{rk}\,(\alpha).$$

The latter contradiction proves that $\mathrm{rk}\,(\alpha)=\mathrm{rk}\,(\alpha\tau)$. Hence $|\,\mathrm{im}\,(\alpha\tau)|=|A_a|=|B_{a'}|$ and $\mathrm{im}\,(\alpha\tau)\subseteq B_{a'}$, which implies $\mathrm{im}\,(\alpha\tau)=B_{a'}$. Thus, $\alpha_{A_a}\tau=\alpha_{B_{a'}}$, hence $A_a\psi=B_{a'}\in\Gamma_2$.

(ii) Suppose that $A_a\in\Gamma_1$ and $\beta\in L_1$. Let $\alpha_{A_a}\tau=\alpha_{B_{a'}}$, $B_{a'}\in\Gamma_2$. Denote $\beta\tau$ by β'. Then

$$(\alpha_{A_a\beta})\tau=(\alpha_{A_a}\beta)\tau=(\alpha_{B_{a'}})\beta'=\alpha_{A_a\psi}\beta'=\alpha_{(A_a\psi)\beta'},$$

which implies $(A_a\beta)\psi=A_a\psi(\beta\tau)$ by the definition of ψ. □

Corollary 1. *Let* $L_1 \cong L_2$, ψ *be the function from* Lemma 5*. Then, for all* $A_a, A_b \in \Gamma_1$*:*

(i) $|A_a| = |A_a\psi|$;

(ii) *if* $A_a \subseteq A_b$, *then* $A_a\psi \subseteq A_b\psi$;

(iii) *if* $A_a \cap A_b = \varnothing$, *then* $A_a\psi \cap A_b\psi = \varnothing$.

Proof. (i) Let $A_a \in \Gamma_1$, $x \in A_a$, and $(\varphi(A_a, x))\tau = \varphi(B_{a'}, x')$ for $B_{a'} \in \Gamma_2$, $x' \in B_{a'}$, with $|A_a| = |B_{a'}|$ (see Lemma 4, (ii)). The proof of Lemma 5, (i) implies that $A_a\psi = B_{a'}$. So $|A_a| = |A_a\psi|$.

(ii) Let $A_a \subseteq A_b$ and $z \in A_b$ be an arbitrary fixed element. Suppose $(\varphi(A_b, z))\tau = \varphi(B_{b'}, z')$ with $B_{b'} \in \Gamma_2$, $z' \in B_{b'}$. By Lemma 4, (ii), $c_z\tau = c_{z'}$, consequently $\{z\}\psi = \{z'\}$. On the one hand,

$$(A_a\varphi(A_b, z))\psi = \{z\}\psi = \{z'\}.$$

On the other hand, by Lemma 5, (ii),

$$(A_a\varphi(A_b, z))\psi = (A_a\psi)(\varphi(A_b, z)\tau) = (A_a\psi)\varphi(B_{b'}, z').$$

So $(A_a\psi)\varphi(B_{b'}, z') = \{z'\}$, which is implies $A_a\psi \subseteq B_{b'} = A_b\psi$.

(iii) Let $A_a \cap A_b = \varnothing$. Fix $z \in A_b$ and let $(\varphi(A_b, z))\tau = \varphi(B_{b'}, z')$ with $B_{b'} \in \Gamma_2$, $z' \in B_{b'}$. Suppose $y \in A_a$ is an arbitrary element, and $c_y\tau = c_{y'}$. By definition of ψ then we have $\{y\}\psi = \{y'\}$. On the one hand,

$$(y\varphi(A_b, z))\psi = \{y\}\psi = \{y'\}, \text{ where } y' \neq z'.$$

On the other hand, by Lemma 5, (ii),

$$(\{y\}\varphi(A_b, z))\psi = (\{y\}\psi)(\varphi(A_b, z)\tau) = \{y'\}\varphi(B_{b'}, z').$$

So $y'\varphi(B_{b'}, z') = y'$, $y' \neq z'$ for all $y \in A_a$. Thus

$$\{y' \mid c_{y'} = c_y\tau,\ y \in A_a\} \cap A_b\psi = \{y' \mid c_{y'} = c_y\tau,\ y \in A_a\} \cap B_{b'} = \varnothing.$$

By (ii) of this Corollary $\{y'\} = \{y\}\psi \subseteq A_a\psi$. Since τ is a bijection, we have

$$|A_a| = |\{y' \mid c_{y'} = c_y\tau,\ y \in A_a\}|.$$

By (i), we get $|A_a| = |A_a\psi|$, thus $A_a\psi = \{y' \mid c_{y'} = c_y\tau,\ y \in A_a\}$. Hence $A_a\psi \cap A_b\psi = \varnothing$. $\square$

Now we are ready to prove

Lemma 6. *If* $L_1 \cong L_2$, *then* $\Gamma_1 \sim \Gamma_2$.

Proof. The result is clearly true if $|X| = 1$. Suppose that $|X| \geqslant 2$. Consider the restriction of the function ψ from Lemma 5 to Γ_1 (which we will also call ψ). By (i) of Lemma 5, $\psi : \Gamma_1 \to \Gamma_2$. It easily follows from Corollary 1 that either $A_1\psi = B_1$ and $A_2\psi = B_2$ or $A_1\psi = B_2$ and $A_2\psi = B_1$. Suppose that $A_1\psi = B_1$ and $A_2\psi = B_2$. We will prove by induction on $|a|$ that for all $a \in \{1,2\}^*$, $A_a\psi = B_a$. We already know that this is true if $|a| = 1$. Let $k \geqslant 1$ and suppose that $A_a\psi = B_a$ for every $a \in \{1,2\}^*$ with $|a| \leqslant k$.

Note, that for all $\mathscr{T}_n$, $n \in \mathbb{N}$ if $|A_1| = 1$ or $|A_2| = 1$, $A_1, A_2 \in \Gamma_1$, then the structure of Γ_1 is uniquely determined in virtue of (2). Thus, if $L_1 = L_X^{\Gamma_1} \cong L_2$, then we get immediately that $\Gamma_1 \sim \Gamma_2$.

Assume further that $|A_1|$, $|A_2| > 1$. We will prove by induction on $|a|$ that for all $a \in \{1,2\}^*$ $A_a\psi = B_a$ or $A_a\psi = B_{\bar{a}}$.

Suppose, that condition $A_a\psi = B_a$ or $A_a\psi = B_{\bar{a}}$ holds for all $A_a \in \Gamma_1$, $|a| \leqslant k$, $k \in \mathbb{N}$. Without loss of generality set $A_a\psi = B_a$, for all $A_a \in \Gamma_1$, $B_a \in \Gamma_2$ if $|a| \leqslant k$, $k \in \mathbb{N}$.

Let $bi \in \{1,2\}^*$, $|bi| = k$ and $A_{bi} \in \Gamma_1$. As has been shown in [5, Lemma 4, (iv)], there exists a transformation $\gamma \in L_1$ such that $A_b\gamma = A_{bi}$, i.e., $\gamma|_{A_b} = \alpha^{A_b}_{A_{bi}}$. According to Definition 3,

$$\alpha^{A_b}_{A_{bi}} = \alpha^{A_{b1}}_{A_{bi}\cap A_{bi1}} \cup \alpha^{A_{b2}}_{A_{bi}\cap A_{bi2}} = \alpha^{A_{b1}}_{A_{bi1}} \cup \alpha^{A_{b2}}_{A_{bi2}},$$

so $A_{bj}\gamma = A_{bij}$, $j \in \{1,2\}$. Moreover, by the induction hypothesis, the following conditions hold:

$$(\alpha_{A_b}\gamma)\tau = (\alpha_{A_{bi}})\tau = \alpha_{A_{bi}\psi} = \alpha_{B_{bi}},$$
$$(\alpha_{A_b}\gamma)\tau = (\alpha_{A_b}\tau)(\gamma\tau) = (\alpha_{A_b\psi})(\gamma\tau) = \alpha_{B_b}(\gamma\tau).$$

Consequently, $\alpha_{B_b}(\gamma\tau) = \alpha_{B_{bi}}$, and so $B_b(\gamma\tau) = B_{bi}$. Since $\gamma\tau \in L_2$, we have $(\gamma\tau)|_{B_b} = \alpha^{B_b}_{B_{bi}}$. According to Definition 3,

$$\alpha^{B_b}_{B_{bi}} = \alpha^{B_{b1}}_{B_{bi}\cap B_{bi1}} \cup \alpha^{B_{b2}}_{B_{bi}\cap B_{bi2}} = \alpha^{B_{b1}}_{B_{bi1}} \cup \alpha^{B_{b2}}_{B_{bi2}},$$

so $B_{bj}(\gamma\tau) = B_{bij}$, $j \in \{1,2\}$.

Now, on the one hand, we have $A_{bj}\psi(\gamma\tau) = (A_{bj}\gamma)\psi = A_{bij}\psi$, $j \in \{1,2\}$, by (ii) of Lemma 5. On the other hand, using the induction hypothesis, we get $A_{bj}\psi(\gamma\tau) = B_{bj}(\gamma\tau) = B_{bij}$, $j \in \{1,2\}$. Thus, $A_{bij}\psi = B_{bij}$, $j \in \{1,2\}$. Since A_{bj} is an arbitrary element with $|bj| = k$, we get $A_c\psi = B_c$ for all $c \in \{1,2\}^*$, $|c| = k+1$, $A_c \in \Gamma_1$. So for all $a \in \{1,2\}^*$ $A_a\psi = B_a$.

In a dual way, we can prove that if $A_1\psi = B_2$ and $A_2\psi = B_1$, then $A_a\psi = B_{\bar{a}}$ for every $a \in \{1,2\}^*$. Since $|A_a| = |A_a\psi|$ for every $a \in \{1,2\}^*$ (by Corollary 1), it follows that $\Gamma_1 \sim \Gamma_2$. □

Now Lemmas 3 and 6 yield

Theorem 3. *Two $\mathscr{L}$-cross-sections of $\mathcal{T}_n$ are isomorphic if and only if the L-families associated with them are similar.*

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