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Representation of Steinitz's lattice in lattices of substructures of relational structures

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Abstract. General conditions under which certain relational structure contains a lattice of substructures isomorphic to Steinitz's lattice are formulated. Under some natural restrictions we consider relational structures with the lattice containing a sublattice isomorphic to the lattice of positive integers with respect to divisibility. We apply to this sublattice a construction that could be called "lattice completion". This construction can be used for different types of relational structures, in particular for universal algebras, graphs, metric spaces etc. Some examples are considered.

1. Introduction

Steinitz's lattice was introduced at the beginning of the XX century by German mathematician A.Steinitz for describing the structure of subfields of algebraically closed field of prime characteristic [1]. It can be determined as the lattice of supernatural numbers with a relation of the divisibility. Steinitz's lattice is complete, i.e. for an arbitrary subset of its elements exists the exact lower and the exact upper bounds. It contains a various sublattices including Boolean algebras. So, an existence of such lattice in the lattice of substructures of mathematical structure shows its inner richness that may be a basis for the use of the structure as an universal object for corresponding class of structures of the same type.

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In the paper we formulate the general conditions in which certain relational structure has some lattice of substructures, which is isomorphic to Steinitz's lattice. If these conditions are satisfied, then the isomorphic representation of Steinitz's lattice in a lattice of substructures of this relational structure is obtained. Under certain natural restrictions it is enough to view structures with the lattice containing sublattice, that is isomorphic to the lattice of positive integers with divisibility. We apply to this sublattice a construction that could be called "lattice completion". This construction can be used to different types of relational structures, in particular — universal algebra, graphs, metric spaces etc. For example, Steinitz's lattice is isomorphic to:

(i) the lattice of all subfields of algebraic closure of finite field (see, e.g., [2]);
(ii) the lattice of Glimm's subalgebras of limited C^*-algebra (see, e.g., [3], [4]);
(iii) lattices of so-called homogeneously symmetric and homogeneously alternating subgroups of symmetric groups of permutations of natural numbers (see, e.g., [5], [6]).

Furthermore, some elements of Steinitz's lattice of substructures in certain relational structures were studied in many works of various authors (see, e.g., [7]–[10]).

The paper is organized as follows. In Section 2 we give the definition of Steinitz's lattice and its basic properties. In Section 3 we present basic information on the theory of relational structures. In Section 4 basic construction of "lattice completion" of relational structures is described. It is shown how to build the certain isomorphism between Steinitz's lattice and a lattice that occurs as a result of "lattice completion". In Section 5 we give examples of an application of the construction of "lattice completion" in the theory of infinite groups and semigroups transformations. This makes it possible to introduce new objects namely, groups and semigroups of periodically defined transformations of natural numbers. Last section describes lattices of subspaces of Besicovitch's space which are constructed by the use of "lattice completion", and therefore are isomorphic to Steinitz's lattice.

Results of the article were earlier partially announced in the article [10] of third author. All symbols are commonly used in the paper. For determination of indefinite terms we refer readers to [11]–[13].

2. Steinitz's lattice

2.1. Let $\mathbb{N}$ be a set of natural numbers and let $\mathbb{P}$ be its subset of primes.

Definition 1. Supernatural number (or Steinitz's number) is called a formal product

$$\prod_{p\in\mathbb{P}} p^{k_p}, \quad k_p \in \mathbb{N} \cup \{0, \infty\}. \tag{1}$$

Denote by the symbol $\mathbb{SN}$ the set of all supernatural numbers. Every natural number is a supernatural number, so that $\mathbb{N} \subset \mathbb{SN}$. Numbers from $\mathbb{SN} \setminus \mathbb{N}$ will be called infinite supernatural numbers. Divisible relation $|$ on $\mathbb{N}$ in natural way is extended to $\mathbb{SN}$. Namely, for arbitrary supernatural numbers

$$u = \prod_{p\in\mathbb{P}} p^{k_p}, \quad v = \prod_{p\in\mathbb{P}} p^{l_p}, \quad k_p, l_p \in \mathbb{N} \cup \{0, \infty\}, \tag{2}$$

we get $u \mid v$ if and only if for all $p \in \mathbb{P}$ inequalities $k_p \leqslant l_p$ hold (it is assumed that ∞ is more than zero and all natural numbers). Main properties of set $\mathbb{SN}$, ordered by the divisible relation $|$, are characterized by the following lemma.

Lemma 1. *The ordered set* $(\mathbb{SN}, |)$ *is a lattice. The lattice* $(\mathbb{SN}, |)$ *is a complete one and contains the largest and the smallest elements, that accordingly are such supernatural numbers*

$$\mathbb{I} = \prod_{p\in\mathbb{P}} p^{\infty} \qquad 1 = \prod_{p\in\mathbb{P}} p^{0}. \tag{3}$$

The proof of this statement is not difficult.

The exact lower and the exact upper bounds of supernatural numbers u, v, that are given by their decompositions (2), are defined by the equalities

$$u \vee v = \prod_{p\in\mathbb{P}} p^{\max(k_p, l_p)} \tag{4}$$

$$u \wedge v = \prod_{p\in\mathbb{P}} p^{\min(k_p, l_p)} \tag{5}$$

where $\max(k, \infty) = \infty$, $\min(k, \infty) = k$ for $k \in \mathbb{N} \cup \{0\}$.

Definition 2. Lattice $(\mathbb{SN}, \wedge, \vee)$ will be called Steinitz's lattice.

The following lemma follows from equations (4), (5).

Lemma 2. *Steinitz's lattice is a complete distributive lattice.*

In the set of supernatural numbers we select two subsets.

Definition 3. Supernatural number $u = \prod_{p \in \mathbb{P}} p^{k_p}$ is called complete, if for each $p \in \mathbb{P}$ there is inclusion $k_p \in \{0, \infty\}$.

According to the definition a complete supernatural number u is uniquely determined by the subset $\mathcal{O}(u)$ of that primes p from $\mathbb{P}$, for that $k_p = \infty$. The set $\mathcal{C}$ of complete supernatural numbers is closed on the operations $\vee$, $\wedge$ and contains the numbers $\mathbb{I}$ and 1 that is defined by (3). Moreover, on the set $\mathcal{C}$ one can define the operation of addition, namely the addition $\bar{u}$ of the number $u \in \mathcal{C}$ is called complete supernatural number that is determined by subset $\mathcal{O}(\bar{u}) = \mathbb{P} \setminus \mathcal{O}(u)$. It is obvious, that $u \vee \bar{u} = \mathbb{I}$, $u \wedge \bar{u} = 1$.

Lemma 3. *The set* $\mathcal{C}$ *with the operations* $\vee, \wedge, ^{-}$ *forms a Boolean algebra with* 1 *as a zero element and* $\mathbb{I}$ *as an unit element. The Boolean algebra* $(\mathcal{C}, \vee, \wedge, ^{-})$ *is isomorphic to the algebra of subsets of a countable set.*

Proof. Define the mapping φ from the algebra of subsets of the set $\mathbb{P}$ to the algebra $\mathcal{C}$ in such way. For any subset $\mathcal{X} \subset \mathbb{P}$ put $\varphi(\mathcal{X}) = u$, were $\mathcal{O}(u) = \mathcal{X}$. From mentioned above it follows that φ is a bijection. In addition, for any $\mathcal{X}_1, \mathcal{X}_2 \subset \mathcal{P}$ for which $\varphi(\mathcal{X}_1) = u_1$, $\varphi(\mathcal{X}_2) = u_2$ we have

$$\varphi(\mathcal{X}_1 \cup \mathcal{X}_2) = u_1 \vee u_2, \quad \varphi(\mathcal{X}_1 \cap \mathcal{X}_2) = u_1 \wedge u_2, \quad \varphi(\overline{\mathcal{X}}_1) = \overline{u}_1.$$

So, the mapping φ is an isomorphism, and this, in particular, means that $(\mathcal{C}, \vee, \wedge, ^{-})$ is a Boolean algebra. □

Definition 4. A supernatural number is called notsquare one, if indicators of powers in its canonical decomposition (1) takes on only two values $0, 1$.

Let $\mathcal{B}$ be a set of all notsquare supernatural numbers. It is clear, that $\mathcal{B}$ is closed regarding to the operations $\vee$ and $\wedge$. Moreover, on $\mathcal{B}$ can also be defined the complement operation by the rule: for the number $u = \prod_{p \in \mathcal{X}} p$, we define

$$\overline{u} = \prod_{p \in \mathbb{P} \setminus \mathcal{X}} p.$$

The number $\mathrm{J} = \prod_{p \in \mathbb{P}} p$ is the largest element in the set $\mathcal{B}$.

Lemma 4. *Sublattice* $\mathcal{B}$ *of the lattice* $(\mathbb{SN}, \wedge, \vee)$ *with the additional operation, namely complement one, forms a Boolean algebra, the largest element of which is number* J*, and the least element is* 1*. The algebra* $(\mathcal{B}, \wedge, \vee, ^{-})$ *is isomorphic to the subalgebra of subsets of the set* $\mathbb{P}$*.*

Proof. Define the mapping, which puts the subset $\mathcal{X}$ in correspondence to the number $\prod_{p\in\mathcal{X}} p$. This mapping is an isomorphism. $\square$

2.2. The sequence of positive integers $\chi = \langle k_1, k_2, \ldots\rangle$ will be called divisible if $k_i \mid k_{i+1}$ for $i = 1, 2, \ldots$. Let DS be a set of the most possible of divisible sequences over $\mathbb{N}$.

Definition 5. The sequence $\chi = \langle k_i\rangle_{i\in\mathbb{N}}$ divides the sequence $\chi' = \langle k'_i\rangle_{i\in\mathbb{N}}$, if for any $i \in \mathbb{N}$ there are $j \in \mathbb{N}$ for which $k_i \mid k'_j$.

Let $\mid$ denote the divisibility of sequences. The relation $\mid$ on DS is:
(i) reflexive, i.e. $\chi|\chi$ for arbitrary sequence $\chi \in DS$;
(ii) transitive, i.e. from $\chi_1|\chi_2$ and $\chi_2|\chi_3$ follows $\chi_1|\chi_3$ for any $\chi_1, \chi_2, \chi_3 \in DS$.

But the relation of divisibility is not symmetric or antisymmetric relation.

Definition 6. Sequences χ and χ' are called exactly divisible if at the same time $\chi|\chi'$ and $\chi'|\chi$.

The exactly divisible relation is equivalence on DS, which we denote by the symbol $\sim$. An arbitrary sequence $\chi \in DS$ determine a supernatural number $\operatorname{char}\chi$ (characteristic χ), which is defined thus
(i) each member of the sequence χ be a divisor of $\operatorname{char}\chi$;
(ii) every natural divisor of $\operatorname{char}\chi$ be a divisor of some member of the sequence χ.

For example, if $\chi = \langle 1, p, p^2, \ldots\rangle$, $p \in \mathbb{P}$, then $\operatorname{char}\chi = p^\infty$, and when $\chi = \langle 1, 2!, 3!, 4!, \ldots\rangle$, then $\operatorname{char}\chi = \mathbb{I}$. From the definition of characteristic we get easy

Lemma 5. 1) *For arbitrary $\chi_1, \chi_2 \in DS$ the divisibility $\chi_1|\chi_2$ holds if and only if* $\operatorname{char}\chi_1|\operatorname{char}\chi_2$.
2) *The sequences $\chi_1, \chi_2 \in DS$ are exactly divisible if and only if when* $\operatorname{char}\chi_1 = \operatorname{char}\chi_2$.

So, sets of exactly divisible sequences are characterized by supernatural numbers, moreover the correspondence between these classes of objects is a bijective.

3. Relational structures

3.1. Recall that n-arity relation over the set A is called an arbitrary subset of Cartesian degree A^n. A relational structure over the set A is called an a pair of the type

$$\Re = \langle A, \{\Phi_\alpha^{k_\alpha}\}_{\alpha \in I}\rangle, \tag{6}$$

where I is some set of indices, and for every $\alpha \in I$ $\Phi_\alpha^{k_\alpha}$ is some relation of the arity k_α over A ($k_\alpha \in \mathbb{N} \cup \{0\}$). The set A is called a support of the relational structure $\Re$, the set $\{\Phi_\alpha^{k_\alpha}\}_{\alpha \in I}$ is called its signature, and the set $(k_\alpha)_{\alpha \in I}$ is called its type.

Examples

1) Every directed graph without multiple edges with the set of vertices V and the set of edges $E \subset V \times V$ is a relational structure (V, E) with the support V and the signature E of the type (2).

2) Every colored graph with the set of vertices V, whose edges are colored in k colors, is a relational structure with the support V and the signature E_1, …, E_k of the type $(\underbrace{2, \ldots, 2}_{k})$.

3) Every metric space (X, d) with the set I of values of the metric is a relational structure $\langle X, \{D_\alpha\}_{\alpha \in I}\rangle$, where

$$D_\alpha = \{(x, y) \mid x, y \in X, d(x, y) = d(y, x) = \alpha\}.$$

This relational structure has the type $(2_\alpha)_{\alpha \in I}$.

4) Every universal algebra

$$\pounds = \langle A, \{\varphi_\alpha^{k_\alpha}\}_{\alpha \in \mathrm{I}}\rangle, \quad \text{where} \quad \varphi_\alpha^{k_\alpha} : A^{k_\alpha} \to A$$

is an operation of the arity k_α on A, can seen as a relational structure of the form (6) of the type $(k_\alpha + 1)_{\alpha \in I}$, where

$$\Phi_\alpha^{k_\alpha + 1}(x_1, \ldots, x_{k_\alpha}, y)$$

occurs if and only if, then $\varphi_\alpha^{k_\alpha}(x_1, \ldots, x_{k_\alpha}) = y$.

The relational structures

$$\Re = \langle A, \{\Phi_\alpha^{k_\alpha}\}_{\alpha \in I}\rangle \quad \text{and} \quad \Re' = \langle B, \{\Psi_\beta^{l_\beta}\}_{\beta \in J}\rangle$$

have the same type, if the sets I and J have the same cardinality and there is a bijection $f : I \leftrightarrow J$ so that for every $\alpha \in I$ the equality $k_\alpha = l_{f(\alpha)}$ holds.

Let the relational structures $\Re$ and $\Re'$ have the same type. The bijection $F : A \to B$ is called an isomorphism of these structures, if for any $\alpha \in I$ the relations

$$\Phi_{\alpha}^{k_\alpha}(x_1, x_2, \ldots, x_{k_\alpha}), \quad x_1, x_2, \ \ldots, x_{k_\alpha} \in A,$$
$$\text{and} \quad \Psi_{f(\alpha)}^{l_{f(\alpha)}}(F(x_1), F(x_2), \ldots, F(x_{k_\alpha}))$$

are isomorphic.

In particular, an isomorphism of universal algebras or graphs as relational structures is their isomorphism in the usual sense, but an isomorphism of relational structures related to metric spaces means that these spaces are isometric.

An isomorphism of relational structure itself is called an automorphism. All automorphisms of relational structure $\Re$ form a group with the operation of superposition of automorphisms, which denoted by the symbol $\mathrm{Aut}\Re$ and named the group of automorphisms of the structure $\Re$.

Let A' be an arbitrary nonempty subset of the set A. For the subset A' we can consider restriction $\Phi_{\alpha}^{k_\alpha}|_{A'}$ of the relation $\Phi_{\alpha}^{k_\alpha}$ $(\alpha \in I)$ on the set A':

$$\Phi_{\alpha}^{k_\alpha}|_{A'}(x_1, x_2, \ldots, x_{k_\alpha}) = \Phi_{\alpha}^{k_\alpha}(x_1, x_2, \ldots, x_{k_\alpha}) \cap (A')^{k_\alpha}.$$

Note, that in this case some restrictions $\Phi_{\alpha}^{k_\alpha}|_{A'}$ may be equal to empty relations.

The relational structure $\langle A', \{\Phi_{\alpha}^{k_\alpha}|_{A'}\}_{\alpha \in I}\rangle$ is called *a substructure* of the relational structure $\Re = \langle A, \{\Phi_{\alpha}^{k_\alpha}\}_{\alpha \in I}\rangle$.

An isomorphism of $\Re$ onto some substructure of the structure $\Re'$ is called isomorphic emmbeding of the structure $\Re$ in the structure $\Re'$.

3.2. The partially ordered set $(I, \geqslant)$ is called a directed to the right, if for any $a, b \in I$ there exist such element $c \in I$, that $a \leqslant c$, $b \leqslant c$.

Definition 7. The family of structures $\{\Re_i\}_{i \in I}$ and embedding $f_{i,k} : \Re_i \to \Re_k$, $i, k \in I$, $i \leqslant k$, satisfying the following requirements:

1) I be a directed to the right partially ordered set;
2) for arbitrary indices $i, k \in I$, $i \leqslant k$, there exist emmbedings $f_{i,k} : \Re_i \to \Re_k$, where $f_{i,i}$ be identity isomorphism;
3) if $i, j, k \in I$ and $i \leqslant j \leqslant k$, then $f_{i,j} \cdot f_{j,k} = f_{i,k}$,

is called an inductive family Σ of relational structures over a set of indices I.

Every inductive family of relational structures

$$\Sigma = \langle \Re_i, f_{i,k} \rangle_{i,k \in I}$$

defines the limit structure which is called inductive limit of family Σ and is denoted by the symbol

$$\Re(\Sigma) = \varinjlim_{i} (\Re_i, f_{i,k}), \quad i, k \in I. \tag{7}$$

Elements of the structure (7) are the so-called strings, the relation $\Phi_\alpha^{k_\alpha}$ extends to their by the standard way ([13], pp. 151–156).

We will apply the construction of an inductive limit in a special case when the index set I be the set of positive integers with the natural order. In this case, the inductive family is the sequence $\Re_1$, $\Re_2$, …, and morphisms $f_{i,k}$ will be defined as compositions of morphisms $f_i = f_{i,i+1}$ $(i, k \in \mathbb{N})$. Moreover, sequences of the type $u = a_k a_{k+1} \ldots$ $(k \in \mathbb{N})$ are strings, if the following conditions hold:

(i) $a_i \in A_i$ $(i \geqslant k)$;

(ii) $f_i(a_i) = a_{i+1}$ $(i \geqslant k)$;

(iii) there is no an element $a_{k-1} \in A_{k-1}$, for which $f_{k-1}(a_{k-1}) = a_k$.

Let $(u_i = a_{l_i}^{(i)} a_{l_i+1}^{(i)} \ldots,\ 1 \leqslant i \leqslant k_\alpha)$, and let $\Phi_\alpha^{k_\alpha}$ be a relation from the signature of relational structures $\Re_i$, $i \in \mathbb{N}$. From the definition follows that the tuple of strings $(u_1, u_2, \ldots, u_{k_\alpha})$ is in the relation $\Phi_\alpha^{k_\alpha}$ if and only if for $l \geqslant \max\{l_1, l_2, \ldots, l_{k_\alpha}\}$ the tuples $(a_l^{(1)}, a_l^{(2)}, \ldots, a_l^{(k_\alpha)})$ is in this relation. Let the subset $\Re^{(k)}$ of the support of the structure $\Re(\Sigma)$, be the set of all strings that was began with the elements from A_l, $l \leqslant k$. Then subset $\Re^{(k)}$ determines a substructure of the structure $\Re(\Sigma)$, and the equality

$$\Re(\Sigma) = \bigcup_{k=1}^{\infty} \Re^{(k)}$$

takes place.

4. Construction of "lattice completion"

Let $\Re = \langle A, \{\Phi_\alpha^{k_\alpha}\}_{\alpha \in I} \rangle$ be relational structures with the signature $(k_\alpha)_{\alpha \in I}$. Suppose that for every natural number n the structure $\Re$ has the only one its own substructure $\Re(n)$, moreover for the family of substructures $\langle \Re(n), n \in \mathbb{N} \rangle$ following conditions hold:

(A) if $n_1 \neq n_2$, then $\Re(n_1) \neq \Re(n_2)$;

(B) the inclusion $\Re(n_1) \subseteq \Re(n_2)$ is satisfied if and only if, when $n_1|n_2$;

It follows from properties (A), (B) that family of substructures $\Re(n)$, $n \in \mathbb{N}$, forms a lattice under the inclusion. A correspondence $n \leftrightarrow \Re(n)$, $n \in \mathbb{N}$, is an isomorphism between this lattice and lattice $(\mathbb{N}, |)$ of natural numbers.

Using the family $\Re(n)$, $n \in \mathbb{N}$ of substructures we define a new family from $\Re$ as follows. Let $\chi = \langle n_1, n_2, \ldots \rangle \in DS$ be an arbitrary divisible sequence of natural number. Construct now by sequence χ a growing chain of substructures of structure $\Re$ of the form

$$\Re(n_1) \subseteq \Re(n_2) \subseteq \cdots .$$

If sequence χ is bounded, then

$$\Re(\chi) = \bigcup_{i=1}^{\infty} \Re(n_i)$$

coincides with one of substructures $\Re(n_k)$, $k \in \mathbb{N}$. Therefore it suffices to consider only divisible unbounded sequences. Suppose that for any χ as unbounded divisible sequences the family of substructures $\Re(\chi)$ satisfies the following conditions:

(C) union $\Re(\chi)$ is its own substructure of structure $\Re$;

(D) $\Re(\chi)$ does not coincide with any substructures $\Re(n)$, $n \in \mathbb{N}$.

Definition 8. The family of substructures $\Re(\chi)$, $\chi \in DS$, of structure $\Re$ will be called the lattice completion of lattice $\Re(n)$, $n \in \mathbb{N}$.

Since DS contains arbitrary bounded divisible sequences, the family $\Re(\chi)$, $\chi \in DS$, contains sublattice $\Re(n)$, $n \in \mathbb{N}$.

Theorem 1. *Suppose, that the family of substructures $\Re(n)$, $n \in \mathbb{N}$, of relational structure $\Re$ satisfies properties (A)–(D). Then*

(i) *substructures $\Re(\chi_1)$ and $\Re(\chi_2)$, χ_1, $\chi_2 \in DS$, coincide if and only if* $\operatorname{char}\chi_1 = \operatorname{char}\chi_2$;

(ii) *the family of substructures $\Re(\chi)$, $\chi \in DS$, forms a lattice under the inclusion, which is isomorphic to Steinitz's lattice.*

Proof. (i) Suppose $\Re(\chi_1)$ and $\Re(\chi_2)$ are determined by divisible sequences of natural numbers $\chi_1 = \langle n_i^{(1)} \rangle_{i \in \mathbb{N}}$, $\chi_2 = \langle n_i^{(2)} \rangle_{i \in \mathbb{N}}$. If $\Re(\chi_1) = \Re(\chi_2)$, then $\Re(\chi_1) \subseteq \Re(\chi_2)$ and $\Re(\chi_2) \subseteq \Re(\chi_1)$. The inclusion $\Re(\chi_1) \subseteq \Re(\chi_2)$ holds if and only if for any natural i there exists a number j, such that $\Re(n_i^{(1)}) \subseteq \Re(n_j^{(2)})$. According to the property (B) it means that

$n_i^{(1)}|n_j^{(2)}$, that is the sequence χ_1 is a divisor of the sequence χ_2. On the other hand, the inclusion $\Re(\chi_2) \subseteq \Re(\chi_1)$ means that for any $j \in \mathbb{N}$ substructures $\Re(n_j^{(2)})$ is contained into some substructures $\Re(n_i^{(1)})$, namely for an arbitrary $j \in \mathbb{N}$ there exists such $i \in \mathbb{N}$, that $n_j^{(2)}|n_i^{(1)}$. This means that the relation $\chi_2|\chi_1$ holds. Thus, sequences χ_2 and χ_1 are exactly divisible. So, $\operatorname{char}\chi_1 = \operatorname{char}\chi_2$ by lemma 5.

Now suppose $\operatorname{char}\chi_1 = \operatorname{char}\chi_2$. Then sequences χ_1 and χ_2 are exactly divisible, that is $\chi_1|\chi_2$ and $\chi_2|\chi_1$. As properties (A) and (B) hold for the family $\Re(n)$, $n \in \mathbb{N}$, using the considerations similar to above we obtain that $\Re(\chi_1) \subseteq \Re(\chi_2)$ and $\Re(\chi_2) \subseteq \Re(\chi_1)$. In other words, these substructures coincide.

(ii) Let $DS^{(0)}$ be a set of fixed representatives of each class of exactly divisible sequences from DS. Then

$$\{\Re(\chi) \mid \chi \in DS\} = \{\Re(\chi) \mid \chi \in DS^{(0)}\}.$$

We shall show, that the family of substructures on the right side of this equality satisfies the condition (ii) of this theorem. Then the subset of $DS^{(0)}$ is determined by classes of exactly divisible sequences on DS. Hence, the mapping $\lambda : \mathbb{SN} \to DS^{(0)}$, such that $\lambda(u) = \chi \quad \text{iff} \quad \operatorname{char}\chi = u$, is a bijective by using lemma 5. So, properties (C), (D) of the family $\Re(\chi)$, $\chi \in DS$, imply that a mapping $\bar{\lambda} : DS^{(0)} \to \{\Re(\chi) \mid \chi \in DS\}$ defined by

$$\bar{\lambda}(\chi) = \bigcup_{n\in\chi} \Re(n) = \Re(\chi), \quad \chi \in DS^{(0)},$$

is a bijective too. Thus the mapping $\mu = \lambda \cdot \bar{\lambda} : \mathbb{SN} \to \{\Re(\chi) \mid \chi \in DS^{(0)}\}$ will also be a bijective. So that the image of arbitrary supernatural number will be own substructure of the structure $\Re$. It is also clear that $\mu(1) = \Re(1)$, $\mu(\mathbb{I}) = \cup_{n\in\chi}\Re(n)$, where χ be such divisible sequence, that $\operatorname{char}\chi = \mathbb{I}$. It remains to show that the mapping μ is consistent with the lattice operations $\vee$ and $\wedge$. To this end, we are going to check that μ is consistent with the divisibility $|$ on the set $\mathbb{SN}$ and the inclusion of $\subseteq$ on substructures $\{\Re(\chi) \mid \chi \in DS^{(0)}\}$. Indeed, let the condition $u_1|u_2$ hold for supernatural numbers u_1, u_2. Then $\lambda(u_1)|\lambda(u_2)$. It follows from (i) that $\bar{\lambda}(\lambda(u_1)) \subseteq \bar{\lambda}(\lambda(u_2))$. So $(\lambda \cdot \bar{\lambda})(u_1) \subseteq (\lambda \cdot \bar{\lambda})(u_2)$. The theorem is proved. $\square$

Theorem 1 can be reformulated in terms of inductive limits as follows. Let $\Re(n)$, $n \in \mathbb{N}$ be a family of structures. This family satisfies the condition (A). There is an embedding φ_{n_1,n_2} of the structure $\Re(n_1)$ into the structure $\Re(n_2)$ for arbitrary $n_1, n_2 \in \mathbb{N}$, such that $n_1|n_2$. Assume that for monomorphisms φ_{n_1,n_2} following standard requirements:

(a) $\varphi_{n,n} = Id$ for any $n \in \mathbb{N}$;

(b) $\varphi_{n_1,n_3} = \varphi_{n_1,n_2} \cdot \varphi_{n_2,n_3}$ for arbitrary n_1, n_2, $n_3 \in \mathbb{N}$ such, that $n_1|n_2$ and $n_2|n_3$;

hold.

Then we have the direct spectrum $\langle \Re(n),\ \varphi_{n,m} \rangle_{n,m \in \mathbb{N}}$ of relational structures, and let $\Re$ be a limit of this spectrum. For arbitrary sequence $\chi = \langle n_i \rangle_{i \in \mathbb{N}}$ it can be considered a direct spectrum $\langle \Re(n_i),\ \varphi_{n_i,n_{i+1}} \rangle_{i \in \mathbb{N}}$ and its direct limit $\Re(\chi)$. This structure can considered as a substructure of structure $\Re$ in an obvious way.

Theorem 2. 1) *Direct limits, defined by divisible sequences χ_1 and χ_2, coincide as substructures of $\Re$ if and only if* $\mathrm{char}\, \chi_1 = \mathrm{char}\, \chi_2$.

2) *All boundary structures considered as substructures of $\Re$ and defined by divisible sequences form a lattice with respect to inclusion. This lattice is isomorphic to the Steinitz's lattice.*

We shall show examples, described how such construction works in two different situations: for an universal algebras and metric spaces.

5. Groups and semigroups of periodically defined transformations of natural numbers

Let PT_n be a semigroup of all partial defined transformations of set the $\{1, 2, \ldots, n\}$, let $PT(N)$ be a semigroup of all partial defined transformations of the set of natural numbers $\mathbb{N}$.

Definition 9. A transformation $\widehat{\pi} \in PT(N)$ is called periodically defined expansion of the transformation $\pi \in PT_n$ on the set $\mathbb{N}$, if its act on natural numbers is defined by the following table

$$\left(\begin{array}{cccc|cccc|c} 1 & 2 & \ldots & n & n+1 & n+2 & \ldots & 2n & \ldots\ldots \\ 1^\pi & 2^\pi & \ldots & n^\pi & n+1^\pi & n+2^\pi & \ldots & n+n^\pi & \ldots\ldots \end{array} \right).$$

A transformation $\alpha \in PT(N)$ is called periodically defined with the period of definition n, if there is a transformation $\pi \in PT_n$ such, that $\alpha = \widehat{\pi}$.

Every periodically defined transformation has infinitely many of different periods that are multiples of the minimal period. Let T_n, IS_n, S_n be, respectively, the complete semigroup of everywhere defined transformations of the set $\{1, 2, \ldots, n\}$, the complete inverse semigroup of partial permutations and the complete symmetric group over this set. Thus let $T(N)$, $IS(N)$, $S(N)$ denote these semigroups over the set of natural numbers.

Lemma 6. *For an arbitrary $n \in \mathbb{N}$ the mapping $\varphi_n : \pi \to \widehat{\pi}$, $\pi \in PT_n$, is a homomorphic emmbeding of the semigroup PT_n into $PT(N)$ and its restrictions on T_n, IS_n, S_n, respectively, is an emmbeding into $T(N)$, $IS(N)$ and $S(N)$.*

Proof. The injection of the mapping φ_n follows directly from its definition and its consistency with the operation of the multiplication of permutations is obvious. Moreover, for any partial permutation $\pi \in IS_n$ and its inverse permutation π^* we have $\varphi_n(\pi^*) = \widehat{\pi^*} = \widehat{\pi}^*$. In particular, for any everywhere defined permutation $\pi \in S_n$ we shall get $\varphi_n(\pi^{-1}) = \widehat{\pi}^{-1}$. So, φ_n will be a monomorphism IS_n into $IS(N)$ and S_n into $S(N)$. □

We will denote by the symbol G_n one of semigroups PT_n, T_n, IS_n, S_n, $n \in \mathbb{N}$, and will denote by the symbol G one of the corresponding semigroups $PT(N)$, $T(N)$, $IS(N)$ or $S(N)$. Thus, all formulated statements will take place simultaneously for all four series of semigroups.

Let $\widehat{G}_n$ be the image $\varphi_n(G_n)$ of the semigroup G_n. Then $\widehat{G}_n$ be a subsemigroup in G, which is isomorphic to G_n. A family of semigroups $\widehat{G}_n$, $n \in \mathbb{N}$, in G is partially ordered by the inclusion. The main property of this partially ordered set will be given in following lemma.

Lemma 7. *A partially ordered set $(\{\widehat{G}_n, n \in \mathbb{N}\}, \subseteq)$ is a lattice, which is isomorphic to the lattice of positive integers with a relation of the divisibility.*

Proof. We shall verify that the correspondence $n \leftrightarrow \widehat{G}_n$, $n \in \mathbb{N}$, will be an isomorphism of partially ordered sets $(\mathbb{N}, |)$ and $(\{\widehat{G}_n, n \in \mathbb{N}\}, \subseteq)$. This correspondence is bijective, hence it is enough to show that for any $m, n \in \mathbb{N}$ the condition $m|n$ holds if and only if $\widehat{G}_m \subseteq \widehat{G}_n$. Let $m|n$ and $n = km$. So that the permutation $\widehat{\pi}^{(k)}$ given by the equality

$$\widehat{\pi}^{(k)} = \left(\begin{array}{cccc|c|ccc} 1 & 2 & \dots & m & \dots & (k-1)m+1 & \dots & km \\ 1^\pi & 2^\pi & \dots & m^\pi & \dots & (k-1)m+1^\pi & \dots & (k-1)m+m^\pi \end{array} \right)$$

contains into G_n, and the equality holds:

$$\varphi_m(\pi) = \varphi(\widehat{\pi}^{(k)}) = \widehat{\pi}.$$

So, for any transformation $\alpha \in \widehat{G}$ such, that $\alpha \in \widehat{G}_m$, we obtain $\alpha \in \widehat{G}_n$. Hence, $\widehat{G}_m \subseteq \widehat{G}_n$.

On the other hand, let $\widehat{G}_m \subseteq \widehat{G}_n$. In semigroup $\widehat{G}_m$ there are permutations with the minimal period of a definition m. Any such permutation has the period of a definition n because it belongs to $\widehat{G}_n$. It follows that $m|n$. □

Since the conditions (A)–(B) from section 4 for the family $\{G_n, n \in \mathbb{N}\}$ hold, then it is possible to apply the construction of lattice completion. Namely, we introduce $G(\chi)$ by setting

$$G(\chi) = \bigcup_{i=1}^{\infty} \widehat{G}_{n_i}$$

for an arbitrary sequence $\chi = \langle n_1, n_2, \ldots \rangle \in DS$. Hence, for so determined subsemigroups of the semigroup G the conditions (C) and (D) from section 4 hold. This allows us to get the following result.

Theorem 3. *The family of subsemigroups $G(\chi)$, $\chi \in DS$, forms a lattice regarding to an inclusion in the semigroup G, which is a Steinitz's lattice. Different elements of this lattice are pairwise non-isomorphic semigroups.*

Proof. The first part of the statement is a direct consequence of Theorem 1. So, we have to show its second part. Let first $G = S(N)$ be the group of permutations on the set $\mathbb{N}$, and let $G(u) = S(u)$ be the corresponding group of periodically defined permutations ($u \in \mathbb{SN}$). Due to [6] groups $G(u_1)$ and $G(u_2)$ for $u_1, u_2 \in \mathbb{SN}$, $u_1 \neq u_2$, are non-isomorphic, because one of them contains a permutation a centralizator that can be non isomorphic to a centralizator of any permutation from another group. Suppose now that $G(u)$ is a subsemigroup of such periodically defined permutations from G, that minimum periods are divisors of a supernatural number u. Then the group $G(u)$ of inverse elements coincides with $S(u)$. Hence, for $u_1 \neq u_2$ semigroups $G(u_1)$ and $G(u_2)$ are non-isomorphic because of groups of their inverse elements are non-isomorphic too. □

The semigroup $G(u)$, $u \in \mathbb{SN}$, can be defined as a limit of the direct spectrum of semigroups with so-called diagonal emmbeding. According to [14] the emmbeding of the transitive transformations group (G, X) into the transformations group (H, Y) is called a diagonal emmbeding if orbits of G onto the set Y either are trivial (consist of one point), or have the same orbit cardinality $|X|$. Note, that the act G on such orbit is isomorphic to the permutation group (G, X). A diagonal emmbeding of a group (G, X) into a permutation group (H, Y) is called a strictly diagonal emmbeding, if there are no trivial orbits of group G onto Y. In this case, with some $k \in \mathbb{N}$ we have $|Y| = k|X|$. Note that strictly diagonal emmbeding can be defined for arbitrary, not necessarily transitive permutations groups. So, its can be defined for transformations semigroups too.

Definition 10. An emmbeding of a transformation semigroup (V, X) into a transformation semigroup (W, Y) will be called (strictly) diagonal emmbeding, if there is a partition of Y onto subsets of the capacity $|X|$, which are invariant under the action of the image of V. Moreover, the action of the image of V onto each of these subsets is isomorphic as a semigroup of transformations to semigroup (V, X).

As before, let $G_n \in \{PT_n, T_n, IS_n, S_n\}$.

Lemma 8. *Let the mapping $\delta_k : G_n \to G_{nk}$ is defined by equality*

$$\delta_k(\pi) = \widehat{\pi}^{(k)},$$

for any permutation $\pi \in G_n$. Then δ_k is an isomorphic emmbeding of the semigroup G_n into the semigroup G_{nk}.

A proof is done by a direct check as at lemma 6.

For an arbitrary divisible sequence $\chi = \langle n_1, n_2, \ldots \rangle \in \mathbb{SN}, n_{i+1}/n_i = k_i$ $(i = 1, 2, \ldots)$ we define a direct spectrum of semigroups

$$G_{n_i} \longrightarrow^{\delta_{k_1}} G_{n_2} \longrightarrow^{\delta_{k_2}} G_{n_3} \longrightarrow \cdots .$$

So, we also have to consider the limit semigroup of the spectrum

$$G[\chi] = \lim_{\to_i} (G_{n_i}, \delta_{k_i}).$$

Theorem 4. *For any supernatural number $\chi \in \mathbb{SN}$ semigroups $G[\chi]$ and $G(\chi)$ are isomorphic as semigroups of transformations.*

Proof. An isomorphism of semigroups is constructed in the standard way, and the set of an action of $G[\chi]$ is naturally identified with $\mathbb{N}$. Finally, we note that our semigroups act equally on the set $\mathbb{N}$. $\square$

6. Steinitz's lattice of subspaces in Besicovitch's space

A normalized Hamming metric on the set H_n of $(0, 1)$-sequences of a length n is called a metric d_H, defined by the following equality

$$d_H(x, y) = \frac{1}{n} \sum_{i=1}^{n} |x_i - y_i|, \quad x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in H_n. \quad (8)$$

So, let now $\{0,1\}^{\mathbb{N}}$ be a set of infinite $(0,1)$-sequences. We will represent the distance function $\widehat{d}_B$ by the rule

$$\widehat{d}_B(x,y) = \lim_{n\to\infty} \sup d_H((x_1,\ldots,x_n),(y_1,\ldots,y_n)), \tag{9}$$

$$x = (x_1,\ldots,x_n), y = (y_1,\ldots,y_n) \in \{0,1\}^n.$$

Since (8) is a metric, it is easy to verify that the function $\widehat{d}_B$, defined by (9), is a pseudometric, i.e., it differs from a metric so that there are infinite sequences with the distance between them equals to 0. The binary relation

$$x \sim_B y \Leftrightarrow \widehat{d}_B(x,y) = 0$$

is an equivalence on $\{0,1\}^{\mathbb{N}}$. Define

$$\mathcal{X}_B = \{0,1\}^{\mathbb{N}}/_{\sim_B}.$$

The function $\widehat{d}_B$ is consistent with the equivalence $\sim_B$. Hence, it determines a function d_B on $\mathcal{X}_B$ as follows:

$$d_B([x],[y]) = \widehat{d}_B(x,y), \tag{10}$$

where $[x]$, $[y]$ are equivalence classes of $\sim_B$, and $x \in [x]$, $y \in [y]$ are arbitrary representatives of these classes. Defined by the equality (10) the function d_B is a metric. So, a metric space $(\mathcal{X}_B, d_B)$ is called Besicovitch's space (see [15]).

For any natural number n normalized Hamming space H_n is isometric embedded into Besicovitch's space $\mathcal{X}_B$.

Lemma 9. *The mapping $h_n : H_n \to \mathcal{X}_B$, defined by setting*

$$h_n((x_1,\ldots,x_n)) = [(x_1,\ldots,x_n,x_1,\ldots,x_n,\ldots)],\ (x_1,\ldots,x_n) \in H_n,$$

is an isometric emmbeding.

Proof. By the definition the distance (10) between classes of the equivalence $\sim_B$, which defined by periodic sequences

$$\overline{x} = x_1,\ldots,x_n,x_1,\ldots,x_n,\ldots \quad \text{and} \quad \overline{y} = y_1,\ldots,y_n,y_1,\ldots,y_n,\ldots,$$

is equal to

$$\frac{1}{n} d_H((x_1,\ldots,x_n),(y_1,\ldots,y_n)).$$

$\square$

Denote by $\widetilde{H}_n$ the image of Hamming space H_n for the mapping h_n.

Lemma 10. *The inclusion $\widetilde{H}_n \subseteq \widetilde{H}_m$ $(n, m \in \mathbb{N})$ holds if and only if then $n|m$.*

Proof. It is obvious. $\square$

So that, Besicovitch's space $\mathcal{X}_B$ contains the family of subspaces $\widetilde{H}_n$, $n \in \mathbb{N}$. It is easy to see, that for this family conditions $(A) - (D)$ of the lattice completion hold. It follows from Theorem 1, that the space $\mathcal{X}_B$ contains a family of subspaces $\widetilde{H}_u$, $u \in \mathbb{SN}$, indexing by supernatural numbers. Note, that the subspace $\widetilde{H}_u$ consists of all possible periodic $(0,1)$-sequences, such that lengthes of their minimum periods are divisors of a supernatural number u. By properties of the general construction we obtain the following

Theorem 5. *A family of subspaces $\widetilde{H}_u$, $u \in \mathbb{SN}$, of the space $\mathcal{X}_B$ forms a lattice over the inclusion which is isomorphic to the lattice of supernatural numbers. If $u_1 \neq u_2$, then subspaces $\widetilde{H}_{u_1}$ and $\widetilde{H}_{u_2}$ are not isometric.*

Proof. The first part of the assertion follows from Theorem 1. The proof of the third part given in the article [16]. $\square$

By the Theorem 1 each of spaces $\widetilde{H}_u$, $u \in \mathbb{SN} \setminus \mathbb{N}$, is isometric to inductive limit of the sequence of finite Hamming spaces H_{m_1}, H_{m_2}, $\ldots$, where $\chi = \langle m_1, m_2, \ldots \rangle$ is such divisible sequence that $\operatorname{char} \chi = u$. Monomorphisms are the diagonal emmbedings $f_i : H_{m_i} \to H_{m_{i+1}}$, defined by following equalities

$$f_i((x_1, \ldots, x_{m_i})) = (\underbrace{x_1, \ldots, x_{m_i}, \ldots, x_1, \ldots, x_{m_i}}_{k_i m_i}), \ (x_1, \ldots, x_{m_i}) \in H_{m_i}, \tag{11}$$

where $k_i = m_{i+1}/m_i$, $i = 1, 2, \ldots$. Note, that a construction of limit cube, that is built as inductive limit of the sequence of Hamming spaces H_{2^i} of the dimension 2^i with emmbedings of following doubling coordinates:

$$\delta_i((x_1, \ldots, x_{2^i})) = (x_1, x_1, x_2, x_2, \ldots, x_{2^i}, x_{2^i}), \quad i = 1, 2, \ldots,$$

is considered in [17]. So that, it is easy to understand that the limit space of such direct spectrum is isometric to the space $\widetilde{H}_{2^\infty}$ with emmbedings of the type (11). Note, that in [18] was considered continuum family of subspaces of the Besicovitch space on some alphabet B, naturally parametrized by supernatural numbers. Every subspace is defined as a diagonal limit of finite Hamming spaces on the alphabet B. So, our construction is more general.

Other generalizations of this construction are proposed in [19], and a generalization of construction of lattice completion for linear groups is considered in the article [14].

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