Algebra and Discrete Mathematics
Volume **22** (2016). Number 2, pp. 301–303 RESEARCH ARTICLE

On n-stars in colorings and orientations of graphs

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Abstract. An n-star S in a graph G is the union of geodesic intervals $I_1, \dots, I_k$ with common end O such that the subgraphs $I_1 \setminus \{O\}, \dots, I_k \setminus \{O\}$ are pairwise disjoint and $l(I_1)+\ldots+l(I_k) = n$. If the edges of G are oriented, S is directed if each ray I_i is directed. For natural number n, r, we construct a graph G of $\operatorname{diam}(G) = n$ such that, for any r-coloring and orientation of $E(G)$, there exists a directed n-star with monochrome rays of pairwise distinct colors.

Let G be a finite connected graph (with the set of vertices $V(G)$ and the set of edges $E(G)$) endowed with the path metric d ($d(u,v)$ is the length of a shortest path between u and v). A path $v_0v_1\dots v_n$ is called a *geodesic interval* if $d(v_0, v_n) = n$.

For a natural number n, we say that a subgraph S of G is an *n-star centered at the vertex O* if S is the union of geodesic intervals $I_1, \dots, I_k$ with common end O such that the subgraphs $I_1 \setminus \{O\}, \dots, I_k \setminus \{O\}$ are pairwise disjoint and $l(I_1) + \ldots + l(I_k) = n$, $l(I_i) > 0, i \in \{1, \dots, k\}$, where $l(I)$ is the length of I. Each I_i is called a *ray* of S. We say that S is *isometrically embedded* (in G) if for any two vertices u, v of S, $d(u,v)$ is equal to the distance between u and v inside S.

If each edge from $E(G)$ is oriented, we say that S is *directed* if each edge v_iv_{i+1} in its ray $v_0\dots v_t$ is oriented as $\overrightarrow{v_iv_{i+1}}$.

We recall that $\operatorname{diam}(G)$ is the length of a longest geodesic interval in G, and introduce *directed* and *chromatic diameters* by

$\operatorname{diam}(G) :=$ maximal p such that, in each orientation of $E(G)$, there is a directed geodesic interval of length p;

2010 MSC: 05C55.
Key words and phrases: n-star, coloring, orientation.

r-diam(G) := maximal p such that, in each r-coloring of $E(G)$, there is a monochrome interval of length p.

Theorem 1. For any natural numbers n, r, there exists a graph G of $\mathrm{diam}(G) = n$ such that, for any r-coloring and orientation of $E(G)$, there is a directed isometrically embedded n-star S with monochrome rays of pairwise distinct colors. In particular, $\overrightarrow{\mathrm{diam}}(G) = n$ and r-$\mathrm{diam}(G) \geqslant n/r$.

Proof. We fix r and proceed by induction on n. For $n = 1$, we take $G = K_2$, the complete graph with two vertices.

We assume that a graph G satisfies the conclusion for some n. We put $m = r^{|E(G)|}\ 2^{|E(G)|} + 1$ and consider the Cartesian product $H = G \times K_m$, $V(H) = V(G) \times V(K_m)$ and $(a, c)(b, d) \in E(H)$ if and only if either $a = b$ or $c = d$ and $ab \in E(G)$.

Now we take arbitrary orientation $\mathcal{O}$ of $E(H)$ and coloring $\chi : E(H) \to \{1, \dots, r\}$. By the choice of m there are $c, d \in V(K_m)$ such that the restrictions of χ and $\mathcal{O}$ onto $G \times \{c\}$, $G \times \{d\}$ coincide.

By the inductive assumption, there is an n-star S in G centered at O such that the n-star $S \times \{c\}$ is directed and has monochrome rays of pairwise distinct colors. We suppose that the edge $(O, c)(O, d)$ is directed from (O, c) to (O, d) (the opposite case is analogous), look at the color $i = \chi((O, c)(O, d))$ and replace the ray of color i in $S \times \{c\}$ with the ray $(O, c)(O, d)I$, where I is the ray of color i in $S \times \{d\}$. After that, we get the desired $(n + 1)$-star in H. $\square$

By the construction, G is the Cartesian product of n complete graphs. Analyzing the proof with vertex-colorings in place of edge-colorings, we get

Theorem 2. For any natural numbers n, r, there exists a graph G of $\mathrm{diam}(G) = n$ such that, for any r-coloring of $V(G)$ and orientation of $E(G)$, there is a directed monochrome geodesic interval of length $n - 1$.

Comments. The story began when Taras Banakh transferred me the following question of Krzysztof Pszczoła: can every graph be oriented so that each directed path $v_0v_1v_2v_3$ has the shortcut $\overrightarrow{v_0v_2}$ or $\overrightarrow{v_1v_3}$. In the case $r = 1$, Theorem 1 gives maximally negative answer to this question (perhaps, motivated by comparability graphs). We mention only three somehow related results from *Ramsey Graph Theory*. For every acyclic directed graph G, there exists [1] a graph H such that, for every orientation of $E(H)$, there is an induced copy of G. In the case G is a tree, see [3] for bounds on $|V(H)|$ and $|E(H)|$. For every graph G and a natural number r, there exists a graph H such that, for every r-coloring of $E(H)$, there is a monochrome isometric copy of G. This statement can be extracted from Theorem 3.1 in [2].

Applying above Ramsey-isometric fact and Theorem 2, we conclude: *For any natural numbers m, r, there exists a graph H such that, for every r-coloring of $E(H)$, every r-coloring of $V(H)$ and any orientation of $E(H)$, there is a directed, edge-monochrome, vertex-monochrome geodesic interval of length m.*

Acknowledgement

I thank Taras Banakh and Oleg Pikhurko for infosupport and discussions roundabout this note.

References

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Received by the editors: 30.09.2016
and in final form 03.10.2016.