

# Quadratic residues of the norm group in sectorial domains

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**ABSTRACT.** In the article the distribution of quadratic residues in the ring  $G_{p^n}$ , in the norm subgroup  $E_n$  of multiplicative group  $G_{p^n}^*$ , is investigated. The asymptotic formula for the number  $R(x, \phi)$  of quadratic residues in the sectorial domain of a special form has been constructed.

## 1. Introduction

In 1918 I.M. Vinogradov and G. Polya built the asymptotic formula for the number of quadratic residues modulo prime number on the segment  $1 \leq n \leq x < p$ , which was nontrivial for every  $x > \sqrt{p} \log p$ . It was the first result about incomplete residue system in analytic number theory. Henceforth Vinogradov-Polya theorem was firstly sharpened by D. Burgess [1]. After this on the assumption under extended Weil hypothesis H. Montgomery and R. Vaughan [3] got the unimprovable result for the theorem.

The research of analogous issue over the ring of Gaussian integers is, evidently, a difficult problem by the virtue of the fact, that geometry of points of a plane is richer than geometry of points on a line. In this article the distribution of quadratic residues in the norm subgroup  $E_n$

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of multiplicative group  $G_{p^n}^*$ , is investigated. Here  $p$  is a prime rational number of the type  $p = 3 + 4k$  and  $E_n$  can be written in the form:

$$E_n := \{\alpha \in G_{p^n} \mid N(\alpha) \equiv \pm 1 \pmod{p^n}\}.$$

This subgroup is cyclic, its order is equal to  $2(p+1)p^{n-1}$ . The numbers  $p = 2$  and  $p \equiv 1 \pmod{4}$  are not prime in  $G$ . Thus, for  $p \equiv 1 \pmod{4}$  we have  $p = \pi \cdot \bar{\pi}$ ;  $\pi, \bar{\pi} \in \mathbb{Z}[i]$ , and the residue class rings in  $\mathbb{Z}[i]$  modulo  $p^n$  (respectively,  $\pi^n$ ) are isomorphic. So, this case was investigated in the works mentioned above. Similarly we have for  $p = 2$ . That is why we don't consider these  $p$ .

If  $(u_0 + iv_0)$  is a generating element of the group  $E_n$ , then  $N(u_0 + iv_0) \equiv -1 \pmod{p^n}$ . It follows that only the elements of the type  $(u_0 + iv_0)^{2a}$ , where  $a = 0, 1, \dots, (p+1)p^{n-1}$ , are quadratic residues modulo  $p^n$  in  $E_n$ .

Our aim is to prove Theorems 1 and 2 stated in Section 3, and to obtain an asymptotic formula for the number  $R(x, \phi)$  of quadratic residues in the sectorial domain

$$S(x, \phi) = \left\{ \phi_1 \leq \arg w < \phi_2, 0 < N(w) \leq x, \phi_2 - \phi_1 = \phi < \frac{\pi}{2} \right\}. \quad (1)$$

The formula for  $R(x, \varphi)$  is contained in Theorem 2 and has the following form

$$R(x; \phi) = \frac{\phi_2 - \phi_1}{2} \cdot \frac{p+1}{p} \cdot \frac{x}{p^n} + O\left(3^n \frac{x^{1-s}}{p^n} \log x\right).$$

The most interesting case is the case, when  $\phi_2 - \phi_1 \rightarrow 0$  with  $x \rightarrow \infty$ , because the case  $\phi_2 - \phi_1 \geq C$ ,  $C > 0$  is a fixed constant, follows from the work [5] about the distribution of values of the function  $r(n)$  (the number of representations of  $n$  by the sum of two squares) in the arithmetic progression.

**Notations.** We will use the following notations:

- $G := \{a + bi \mid a, b \in \mathbb{Z}, i^2 = -1\}$  is the ring of Gaussian integers;
- $G_\gamma$  is the ring of residues of Gaussian integers modulo  $\gamma$ ;
- $G_\gamma^* = \{\alpha \in G_\gamma, (\alpha, \gamma) = 1\}$ ;
- for  $\alpha \in G$  we denote  $N(\alpha) = |\alpha|^2$ ,  $\text{Sp}(\alpha) = 2\Re(\alpha)$ ;
- $E_n \subset G_{p^n}$  is the norm group;
- $\chi$  stands for a character of the group  $E_n$ ;
- for  $a \in \mathbb{Z}$  (or  $\alpha \in G$ )  $\nu_p(a)$  (or  $\nu_p(\alpha)$ ) stands that  $p^{\nu_p(a)} \mid a, p^{\nu_p(a)+1}$  does not divide  $a$ ;

- $s \in \mathbb{C}, s = \sigma + it, \sigma = \Re s, t = \Im s;$
- $\Gamma(z)$  is the Euler gamma-function;
- by  $f \ll g (f = O(g))$  for  $x \in X$ , where  $X$  is an arbitrary set, on which  $f$  and  $g$  are defined, we mean that there exists a constant  $C > 0$  such that  $|f(x)| \leq C \cdot g(x)$  for all  $x \in X;$
- $\exp(x) = e^x$  for  $x \in \mathbb{C}$  (sometimes, instead of  $e^x$  we will use  $\exp(x)$ ).

Let us denote

$$E_n^+ := \left\{ \alpha \in G_{p^n}^* \mid N(\alpha) \equiv 1 \pmod{p^n} \right\} \\ = \left\{ \alpha \in E_n \mid \alpha = (u_0 + iv_0)^{2a}, a = 0, 1, \dots, (p+1)p^{n-1} \right\}.$$

Then

$$R(x, \phi) = \sum_{\alpha \in E_n^+} \sum_{\substack{w \in G \\ w \equiv \alpha \pmod{p^n} \\ w \in S(x, \phi)}} 1. \tag{2}$$

We consider Dirichlet series

$$F_m(s) = \sum_{\alpha \in E_n^+} \sum_{w \equiv \alpha \pmod{p^n}} \frac{e^{4mi \arg w}}{N(w)^s}, \quad \Re s > 1.$$

We have

$$F_m(s) = \sum_{\alpha \in E_n^+} \frac{1}{N(p^n)^s} \zeta_m \left( s; \frac{\alpha}{p^n}, 0 \right), \quad \Re s > 1, \tag{3}$$

where  $\zeta_m \left( s; \frac{\alpha}{p^n}, 0 \right)$  is a special case of Hecke zeta-function  $\zeta_m(s; \delta_0, \delta)$  with a shift. In the domain  $\Re s > 1$  the last is defined by absolutely convergent Dirichlet series

$$\zeta_m(s; \delta_0, \delta) = \sum_{w \in G} \frac{e^{4mi \arg(w + \delta_0)}}{N(w + \delta_0)^s} e^{\pi i Sp(\delta w)},$$

where  $\delta_0, \delta$  are Gaussian numbers from the field  $\mathbb{Q}(i); Sp(\beta)$  is a trace of an element  $\beta$  from  $\mathbb{Q}(i)$  to  $\mathbb{Q}$ .

## 2. Auxiliary results

In the following lemmas we bring necessary information about Hecke zeta-function for the next steps.

**Lemma 1.** *The Hecke zeta-function  $\zeta_m(s; \delta_0, \delta)$  satisfies the functional equation*

$$\begin{aligned} &\pi^{-s}\Gamma(2|m| + s)\zeta_m(s; \delta_0, \delta) \\ &= \pi^{-(1-s)}\Gamma(2|m| + 1 - s) \cdot \zeta_{-m}(1 - s; \delta_0, -\delta) e^{-\pi i \operatorname{Sp}(\delta\bar{\delta}_0)}. \end{aligned}$$

Moreover,  $\zeta_m(s; \delta_0, \delta)$  is an entire function if  $m \neq 0$  or  $m = 0$  and  $\delta$  is not a Gaussian integer. For  $m = 0$  and  $\delta \in G$  it is holomorphic except for the point  $s = 1$ , where it has a simple pole with the residue  $\pi$ .

*Proof.* For  $\delta_0 = \delta = 0$  and  $m = 4m_1$ , we get the well-known Hecke zeta-function  $Z_m(s)$  with the Hecke character of the first kind with the exponent  $m$  (see, [2]). In [8] this lemma has been stated without a proof. But for the completeness of treatment we restore a proof of this statement.

In the general case, for the proof of statement of the lemma we start from the relation

$$\Gamma(s) |w\delta_0|^{-2s} = \int_0^\infty \exp(-x|w + \delta_0|^2) x^{s-1} dx.$$

It is evident that for  $\Re s > 1$  and  $m \in \mathbb{Z}$  we can write

$$\Gamma(2|m| + s)Z_m(s; \delta_0; \delta) = \int_0^\delta \sum_{\substack{w \in G \\ w \neq -\delta_0}} e^{-x|w+\delta_0|^2} x^{s-1} dx.$$

Let us denote  $\delta_0 = \delta_{01} + \delta_{02}$ . Then a groundtruthing shows that the functions

$$\begin{aligned} f(u_1, u_2) &= \exp(-x(u_1^2 + u_2^2) + 2\pi i(\delta_{01}u_1 + \delta_{02}u_2)), \\ \hat{f}(v_1, v_2) &= \frac{\pi}{x} \exp\left(-\frac{\pi^2}{x} [(\delta_{01} + v_1)^2 + (\delta_{02} + v_2)^2]\right) \end{aligned}$$

satisfy the conditions of Poisson summation formula (see, e.g. [6], Ch. VII, Corollary 2.6).

Hence, denoting

$$\Theta_m(x, \delta_0, \delta) = \sum_{w \in G} \exp(-x|w + \delta_0|^2) (w + \delta_0)^{4m} \exp(\pi i \operatorname{Sp}(\delta\bar{\delta}w))$$

and using Poisson summation formula, we find

$$\Theta_0(x, \delta_0, \delta) = \frac{\pi}{x} \Theta_0\left(\frac{\pi^2}{x}, \delta, -\delta_0\right) \exp(-\pi i \operatorname{Sp}(\delta_0\bar{\delta})).$$

Consider the operator

$$\frac{d}{d\delta_0} := \frac{\partial}{\partial\delta_{01}} + i\frac{\partial}{\partial\delta_{02}}, \quad \delta_0 = \delta_{01} + \delta_{02}.$$

Then the following equalities for  $m \geq 0$

$$(-2x)^{4m} \Theta_m(x, \delta_0, \delta) = \frac{d^m}{d\delta^m} \Theta_0(x, \delta_0, \delta)$$

and

$$\begin{aligned} & \frac{\pi}{x} (-2\pi i)^{4m} \Theta_m\left(\frac{\pi^2}{x}, \delta_0, -\delta\right) \exp\left(-\pi i \operatorname{Sp}(\delta_0 \bar{\delta})\right) \\ &= \frac{d^m}{d\delta_0^m} \left( \frac{\pi}{x} \Theta_0\left(\frac{\pi^2}{x}, \delta, -\delta_0\right) \exp\left(-\pi i \operatorname{Sp}(\delta_0 \bar{\delta})\right) \right) \end{aligned}$$

hold.

So, for any  $m \in \mathbb{Z}$  the following functional equation

$$\Theta_m(x, \delta_0, \delta) = \left(\frac{\pi}{x}\right)^{4m+1} \Theta_m\left(\frac{\pi^2}{x}, \delta, \delta_0\right) \exp\left(-\pi i \operatorname{Sp}(\delta_0 \bar{\delta})\right) \quad (4)$$

is true.

Now, applying reasoning used for the proof of the functional equation for Riemann zeta-function by the functional equation for a theta-function  $\Theta_m$  we easily infer

$$\Gamma(2|m| + s) \zeta_m(s, \delta_0, \delta) = \pi^{-(1-2s)} \exp\left(-\pi i \operatorname{Sp}(\bar{\delta}_0 \delta)\right) \mathfrak{I}_m(\delta_0, \delta),$$

where

$$\begin{aligned} & \mathfrak{I}_m(\delta_0, \delta) \\ &= \int_0^\infty \sum_{\substack{w \\ w \neq -\delta_0}} \exp(-x|w + \delta_0|^2) (w + \delta_0)^{4m} \exp(\pi i \operatorname{Sp}(\bar{\delta} w)) x^{s+2m-1} dx \\ &= \int_0^\pi + \int_\pi^\infty := \mathfrak{I}_{m,1} + \mathfrak{I}_{m,2}. \end{aligned}$$

In the integral  $\mathfrak{J}_{m,1}$  we apply the functional equation (4) for  $\Theta_m(x, \delta_0, \delta)$  and make the substitution  $x = \pi^2 y^{-1}$ . This gives the equality

$$\begin{aligned} \Gamma(2|m| + s)\zeta_m(s, \delta_0, \delta) &= \pi^{2s-1} \exp(-\pi i \operatorname{Sp}(\bar{\delta}_0 \delta)) \times \\ &\times \int_{\pi}^{\infty} \sum_{\substack{w \in G \\ w \neq -\delta}} \exp(-x|w + \delta|^2)(w + \delta)^{4m} \exp(-\pi i \operatorname{Sp}(\bar{\delta}_0 w)) x^{-s+2m} dx \\ &+ \int_{\pi}^{\infty} \sum_{\substack{w \in G \\ w \neq -\delta_0}} \exp(-x|w + \delta_0|^2)(w + \delta_0)^{4m} \exp(-\pi i \operatorname{Sp}(\bar{\delta} w)) x^{s+2m-1} dx \\ &+ \varepsilon(m, \delta) \frac{\pi^s}{s-1} - \varepsilon(m, \delta_0) \exp(-\pi i \operatorname{Sp}(\delta_0, \bar{\delta})) \frac{\pi^s}{s}, \end{aligned} \tag{5}$$

where

$$\varepsilon(m, a) = \begin{cases} 1 & \text{if } m = 0 \text{ and } a \in G \\ 0 & \text{otherwise.} \end{cases}$$

The equality (5) was obtained for  $\Re s > 1$ . However, the right part of this equality is an analytic function in all complex  $s$ -planes except maybe the points  $s = 0$  and  $s = 1$ , which can be the poles.

Now, multiplying the equality (5) by  $\exp(\pi i \operatorname{Sp}(\bar{\delta}_0 \delta))\pi^{-2s+1}$  and making the substitution  $s \rightarrow 1 - s$ ,  $\delta_0 \rightarrow \delta$ ,  $\delta \rightarrow \delta_0$ , we obtain that the right part doesn't vary, and hence, we have proved the following functional equation

$$\begin{aligned} \pi^{-s}\Gamma(2|m| + s)\zeta_m(s; \delta_0, \delta) \\ = \pi^{-(1-s)}\Gamma(2|m| + 1 - s)\zeta_m(1 - s; -\delta, \delta_0) \exp(-\pi i \operatorname{Sp}(\delta_0 \bar{\delta})). \end{aligned}$$

If  $m = -m'$ ,  $m' > 0$ , we put  $\delta_0 = -\delta'_0$ ,  $\delta = -\delta'$ , and then we have

$$\zeta_m(s, \delta, \delta_0) = \zeta_{m'}(s, -\delta, -\delta_0) \Rightarrow \zeta_{m'}(1 - s, \delta_0, -\delta) = \zeta_m(1 - s, -\delta_0, \delta).$$

So, for any  $m \in \mathbb{Z}$ ,

$$\begin{aligned} \pi^{-s}\Gamma(2|m| + s)\zeta_m(s; \delta, \delta_0) &= \pi^{-(1-s)}\Gamma(2|m| + 1 - s)\zeta_{-m}(1 - s, -\delta_0, \delta) \\ &= \pi^{-(1-s)}\Gamma(2|m| + 1 - s)\zeta_{-m}(1 - s; \delta_0, -\delta). \end{aligned}$$

This completes the proof of Lemma 1. □

**Corollary 1.** *If  $\delta$  is not a Gaussian integer, then  $\zeta_0(0; \delta_0, \delta) = 0$ .*

**Lemma 2.** *In the strip  $\varepsilon \leq \Re s \leq 1 + \varepsilon$ ,  $\varepsilon > 0$ , the following estimate*

$$(s - 1) \cdot \zeta_m(s; \delta_0, \delta) \ll (|t| + 1)(t^2 + m^2)^{\frac{(1-2\sigma)(1+\varepsilon-\sigma)}{1+2\varepsilon}} |N(\delta)|^{-\frac{\sigma+\varepsilon}{1+2\varepsilon}}$$

holds.

This lemma follows from Phragmen-Lindelof principle and the estimates for  $\zeta_m(s; \delta_0, \delta)$  on the boundaries of the strip  $\varepsilon \leq \Re s \leq 1 + \varepsilon$ , which can be received with the usage of the functional equation for  $\zeta_m(s; \delta_0, \delta)$  and Stirling formula for  $\Gamma(z)$ .

**Lemma 3.** *Let  $y \geq k \in \{0, 1, 2\}$ . Let  $a$  be a real number,  $-1 < a \leq \frac{5}{4}$ ,  $\eta(a) = \min_{j=0,1,\dots,k} |a - j| \neq 0$ . Then for any real numbers  $u, v$  the following estimate*

$$\int_{a+iu}^{a+iv} \frac{y^s \psi(s, m)}{s(s+1)\dots(s+k)} ds \ll N(\gamma)^{\frac{1}{2}} M \left( \left( \frac{y}{N(\gamma)} \cdot \frac{1}{M} \right)^a (\eta^{-1}(a) + \log M) + \left( \frac{y}{N(\gamma)M} \right)^{\frac{1}{2} - \frac{2k+1}{4}} \right),$$

holds, where  $\psi(s, m) = \left( \frac{1}{\pi} N(\gamma)^{\frac{1}{2}} \right)^{1-2s} \frac{\Gamma(2|m|+1-s)}{\Gamma(2|m|+s)}$ ,  $M = |m| + 10$ .

*Proof.* Apply [3, Lemma 8]. □

**Lemma 4** ([7], Theorem 1). *Upon the condition  $D^{\frac{1}{2}} \leq x < D^2$  the asymptotic formula*

$$\begin{aligned} \sum_{\substack{n \equiv 1 \pmod{D} \\ n \leq x}} r(n) &= \frac{\pi x}{D} \gamma_0 \prod_{p|D} \left( 1 - \frac{\chi_4(p)}{p} \right) \\ &+ O \left( D^{\frac{1}{2}} \exp \left( c \frac{(\log D)^{\frac{1}{2}}}{\log \log D} \right) \right) + O \left( \frac{x^{\frac{1}{2}}}{D^{\frac{1}{2}}} \tau(D) \right), \\ \gamma_0 &= \begin{cases} 1 & \text{if } D \not\equiv 0 \pmod{4}, \\ 2 & \text{if } D \equiv 0 \pmod{4} \end{cases} \end{aligned}$$

is true.

**Lemma 5.** *Let  $p \equiv 3 \pmod{4}$ . Then for  $n = 1, 2, 3, \dots$  the estimate*

$$\sum_{\alpha \in E_n^+} e^{\pi i \operatorname{Sp} \frac{\alpha^2}{p^n}} \ll p^{\frac{n}{2}}$$

holds.

*Proof.* In the articles [5] and [9] the following description of elements  $\alpha \in E_n^+, n \geq 2$  was given:

$$\alpha = (u_0 + iv_0)^{2(p+1)t+k} \equiv \sum_{j=0}^{n-1} (A_j(k) + iB_j(k)) t^j \pmod{p^n}.$$

Here  $(u_0 + iv_0)$  is a generator of the group  $E_n^+$ ,  $t = 0, 1, \dots, p^{n-1}$ ,  $k = 0, 1, \dots, 2p + 1$ . Moreover,

$$\begin{aligned} A_0(k) &= u(k), & B_0(k) &= v(k); \\ A_1(k) &\equiv -py_0v(k) \pmod{p^3}, & B_1(k) &\equiv py_0u(k) \pmod{p^3}, \\ A_2(k) &\equiv -\frac{1}{2}p^2y_0^2u(k) \pmod{p^3}, & B_2(k) &\equiv -\frac{1}{2}p^2y_0^2v(k) \pmod{p^3}, \\ & & (u_0 + iv_0)^k &\equiv u(k) + iv(k) \pmod{p^n}, (y_0, p) = 1. \end{aligned}$$

Furthermore,

$$\begin{aligned} u(k) &\equiv 0 \pmod{p}, & \text{when } k &= \frac{p+1}{2}, k = \frac{3(p+1)}{2}; \\ v(k) &\equiv 0 \pmod{p}, & \text{when } k &= 0, k = p+1; \\ A_j(k) &\equiv B_j(k) \equiv 0 \pmod{p^3}, & j &= 3, 4, \dots, m-1, k = 0, 1, \dots, 2p+1. \end{aligned}$$

Hence we easily conclude

$$\begin{aligned} \Re(\alpha^2) &\equiv (A_0^2(k) - B_0^2(k)) + 2(A_0(k)A_1(k) - B_0(k)B_1(k))t \\ &\quad + (A_1^2(k) - B_1^2(k)) + A_0(k)A_2(k) - B_0(k)B_2(k)t^2 \pmod{p^3}. \end{aligned}$$

Then  $\Re(\alpha^2) \equiv C_0 + C_1t + C_2t^2 \pmod{p^3}$  with the coefficients

$$\begin{aligned} C_1 &\equiv -2py_0u(k)v(k) \pmod{p^3}, & C_2 &\equiv \frac{1}{2}p^2y_0^2(u^2(k) - v^2(k)) \pmod{p^3} \\ & & \text{or } C_2 &\equiv \frac{1}{2}p^2y_0^2(1 - 2v^2(k)) \pmod{p^3}. \end{aligned}$$

Let us note that  $u(k)^2 + v(k)^2 \equiv (-1)^k \pmod{p}$ . Therefore, it follows that  $u(k)$  and  $v(k)$  can not divide  $p$  simultaneously. It is obvious that  $\nu_p(C_2) \geq 2$  (the strict inequality holds for the cases  $k=0, \frac{p+1}{2}, \frac{3(p+1)}{2}, p+1$ ). That is why, when  $\nu_p(C_1) < \nu_p(C_2)$ ,  $S = 0$ . So, from the well-known relation, for  $(b, p) = 1, f(x) \in \mathbb{Z}[x]$ ,

$$\left| \sum_{x \in \mathbb{Z}_{p^n}} e^{2\pi i \frac{ax + pbx^2 + p^2f(x)}{p^n}} \right| = \begin{cases} 0 & \text{if } (a, p) = 1, \\ 2p^{\frac{n-1}{2}} & \text{if } a \equiv 0 \pmod{p} \end{cases}$$



we get

$$\left| \sum_{\alpha \in E_n^+} e^{\pi i \operatorname{Sp} \frac{\alpha^2}{p^n}} \right| \leq 4p^{\frac{n}{2}}.$$

In case  $n = 1$  we take into account that

$$E_1 = \left\{ \pm 1, \pm i, \frac{a-i}{a+i}, i \frac{a-i}{a+i} \mid a = 1, 2, \dots, p-1 \right\}.$$

Thus, we conclude that  $\operatorname{Sp}(\alpha^2)$  can be represented as the ratio of the polynomials of degree 2. Then, following Weil [11], we have

$$\left| \sum_{\alpha \in E_1} e^{\pi i \operatorname{Sp} \frac{\alpha^2}{p}} \right| \leq 2\sqrt{p}.$$

Hence, the assertion of lemma follows. □

**Lemma 6** ([7], Lemma 5). *Let  $p$  be a prime number, let  $u_1, u_2$  be integers and  $(u_1, u_2, p^n) = p^m$ . Then*

$$\left| \sum_{l_1^2 + l_2^2 \equiv 1 \pmod{p^n}} e^{2\pi i \frac{u_1 l_1 + u_2 l_2}{p^n}} \right| \leq 2p^{\frac{n+m}{2}}.$$

**Corollary 2.** *For  $m \neq 0$  the following estimate*

$$\sum_{\alpha \in E_n^+} \zeta_m \left( 0; \frac{\alpha}{p^n}, 0 \right) \ll p^{\frac{3}{2}n} M \log M, \quad M = |m| + 10,$$

*holds.*

This statement follows immediately from the functional equation for  $\zeta_m(s; \delta_0, \delta)$  for  $m \neq 0$  and Lemma 6.

The following Lemma was proved in [10] (see Lemma 11, pp. 259–260).

**Lemma 7** (Vinogradov’s ‘glasses’). *Let  $r \in \mathbb{N}$ ,  $\Omega > 0$ ,  $0 < \Delta < \frac{1}{2}\Omega$  and let  $\phi_1, \phi_2$  be real numbers,  $\Delta \leq \phi_2 - \phi_1 \leq \Omega - 2\Delta$ . Then there exists a periodic function  $f(\phi)$  with the period  $\Omega$  such that:*

- (i)  $f(\phi) = 1$ , in the segment  $\phi \in [\phi_1, \phi_2]$ ;
- $0 \leq f(\phi) \leq 1$  in the segments  $[\phi_1 - \Delta, \phi_1]$  and  $[\phi_2, \phi_2 + \Delta]$ ;
- $f(\phi) = 0$ , in the segment  $[\phi_2 + \Delta, \phi_1 + \Omega - \Delta]$ ;

(ii)  $f(\phi)$  has the expansion in a Fourier series

$$f(\phi) = \sum_{m=-\infty}^{+\infty} a_m e^{2\pi i \frac{m\phi}{\Omega}},$$

where  $a_0 = \frac{1}{\Omega}(\phi_2 - \phi_1 + \Delta)$ , and for  $m \neq 0$  and  $r \in \mathbb{N}$  each of the following inequalities holds

$$|a_m| \leq \begin{cases} \frac{1}{\Omega}(\phi_2 - \phi_1 + \Delta), \\ \frac{2}{\pi|m|}, \\ \frac{2}{\pi|m|} \left( \frac{r\Omega}{\pi|m|\Delta} \right)^r. \end{cases}$$

### 3. Main results

Let us consider the function of a natural argument

$$r_m(k) = \sum_{\substack{u, v \in \mathbb{Z} \\ u^2 + v^2 = k}} e^{4mi \arg(u+iv)}.$$

In view of (3) we can write

$$F_m(s) = \sum_{\substack{k \leq x \\ k \equiv 1 \pmod{p^n}}}^{\infty} \frac{r_m(k)}{k^s}.$$

**Theorem 1.** *Let  $m \neq 0$ ,  $p^n \leq x \leq p^{2n}$ . Then*

$$\sum_{\substack{k \leq x \\ k \equiv 1 \pmod{p^n}}} r_m(k) \ll \frac{\sqrt{x}}{p^{\frac{n}{2}}} + p^{\frac{n}{2}} \log x + p^{\frac{n}{2}} M \log M.$$

*Proof.* Our assertion is trivial for  $x \ll p^n M$ . That is why we will assume that  $x \geq C \cdot Mp^n$ ,  $C > 0$ . It follows from Lemma 1 that  $\zeta_m(s; \delta_0, \delta)$  is an entire function. In view of the fact

$$\frac{1}{p^{2ns}} \zeta_m \left( s; \frac{\alpha}{p^n}, 0 \right) = \sum_{\substack{w \in G \\ w \equiv \alpha \pmod{p^n}}} e^{\frac{4mi \arg w}{N(w)^s}}$$

for  $\Re s > 1$  and every  $\alpha \in G$  the usage of the theorem of the residues gives

$$\begin{aligned} & \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{x^{s+2} \zeta_m \left( s; \frac{\alpha}{p^n}, 0 \right)}{p^{2ns} s(s+1)(s+2)} ds \\ &= \frac{x^2}{2} \delta_m \left( s; \frac{\alpha}{p^n}, 0 \right) + \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{x^{s+2} \zeta_m \left( s; \frac{\alpha}{p^n}, 0 \right)}{p^{2ns} s(s+1)(s+2)} ds \end{aligned} \quad (6)$$

for every  $-1 < a < 0$ .

Let us denote

$$S_2(x, \alpha) = \frac{1}{2} \sum_{\substack{0 < N(w) \leq x \\ w \equiv \alpha \pmod{p^n}}} e^{4mi \arg w} (x - N(w))^2. \tag{7}$$

Using the relation

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{y^{s+l}}{s(s+1)\dots(s+k)} = \begin{cases} \frac{1}{l!} (y-1)^l & \text{if } y > 1 \\ 0 & \text{if } 0 < y < 1 \end{cases}$$

and taking into account the uniform convergence of the series for zeta-function  $\zeta_m\left(s; \frac{\alpha}{p^n}, 0\right)$  in the semiplane  $\Re s \geq 1 + \varepsilon$ ,  $\varepsilon > 0$ , we get

$$\begin{aligned} & \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{x^{s+2} \zeta_m\left(s; \frac{\alpha}{p^n}, 0\right)}{p^{2ns} s(s+1)(s+2)} ds \\ &= \sum_{w \equiv \alpha \pmod{p^n}} \frac{e^{4mi \arg w}}{N(w)^{-2}} \cdot \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{\left(\frac{x}{N(w)}\right)^{s+2}}{s(s+1)(s+2)} ds \\ &= \frac{1}{2} \sum_{\substack{w \equiv \alpha \pmod{p^n} \\ N(w) \leq x}} e^{4mi \arg w} (x - N(w))^2 = S_2(x, \alpha). \end{aligned} \tag{8}$$

The application of the functional equation for  $\zeta_m(s; \delta_0, \delta)$  (see Lemma 1) and the estimate  $\zeta_m(s; \delta_0, \delta)$  in critical strip (see Lemma 2) give

$$\begin{aligned} & \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{x^{s+2} \zeta_m\left(s; \frac{\alpha}{p^n}, 0\right)}{p^{2ns} s(s+1)(s+2)} \\ &= \sum_{w \in G \setminus \{0\}} e^{-4mi \arg w} e^{\pi i \operatorname{Sp}\left(\frac{\alpha w}{p^n}\right)} N(w)^{-s} \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{(xN(w))^{s+2} \frac{\Gamma(2|m|+1-s)}{\Gamma(2|m|+s)}}{\pi^{1-2s} s(s+1)(s+2) p^{2ns}} ds. \end{aligned} \tag{9}$$

From (6)–(9) we deduce the formula:

$$\begin{aligned} S_2(x, \alpha) &= \frac{x^2}{2} \zeta_m\left(0; \frac{\alpha}{p^n}, 0\right) \\ &+ \sum_{w \in G \setminus \{0\}} e^{-4mi \arg w} e^{\pi i \operatorname{Sp}\left(\frac{\alpha w}{p^n}\right)} N(w)^{-s} W\left(\frac{xN(w)}{p^{2n}}\right) p^{2(n+1)}, \end{aligned}$$

where

$$W(y) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{y^{s+2} \Gamma(2|m|+1-s)}{s(s+1)(s+2)\Gamma(2|m|+s)} ds.$$

We consider the following operator

$$\Delta_z F(x) = \sum_{j=0}^2 (-1)^j F(x+jz) = \int_x^{x+z} dy_1 \int_{y_1}^{y_1+z} F''(y_2) dy_2.$$

Then

$$\Delta_z \left( \frac{x^2}{2} \zeta_m \left( 0; \frac{\alpha}{p^n}, 0 \right) \right) = z^2 \zeta_m \left( 0; \frac{\alpha}{p^n}, 0 \right).$$

It is obvious that for every  $b$ ,  $-1 < b < 0$ , we have

$$W(y) = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{y^{s+2} \Gamma(2|m|+1-s)}{s(s+1)(s+2)\Gamma(2|m|+s)} ds.$$

We put  $b = -1 + \frac{1}{\log y}$ , if  $y > 1$ . Using Lemma 3, we conclude that

$$W(y) \ll K(y, m), \tag{10}$$

where

$$K(y, m) = p^{3n} M^3 y (\log y + \log M).$$

It means that

$$\Delta_z W \left( \frac{xN(w)}{p^{2n}} \right) \ll K \left( \frac{xN(w)}{p^{2n}}, m \right), \tag{11}$$

if only  $z \ll \frac{xN(w)}{p^{2n}}$ .

The value  $\Delta_z W \left( \frac{xN(w)}{p^{2n}} \right)$  may be defined in a different way. We put

$$\Phi(y) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{y^{s+2} \Gamma(2|m|+1-s)}{s(s+1)(s+2)\Gamma(2|m|+s)} ds, \quad c > 1.$$

Then

$$\Phi(y) = \frac{y^2 \Gamma(2|m|+1)}{2 \Gamma(2|m|)} + W(y).$$

For all  $y > 0$  the integrals

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{y^{s+2}\Gamma(2|m|+1-s)}{s \cdot \dots \cdot (s+2-j)\Gamma(2|m|+s)} ds, \quad j = 0, 1, 2,$$

converge absolutely and uniformly. Hence, for the derivatives of  $\Phi(y)$  we have

$$\Phi^{(j)}(y) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{y^{s+2-j}\Gamma(2|m|+1-s)}{s \cdot \dots \cdot (s+2-j)\Gamma(2|m|+s)} ds, \quad j = 0, 1, 2.$$

Thus,

$$W''(y) = -\frac{\Gamma(2|m|+1)}{\Gamma(2|m|)} + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{y^s \Gamma(2|m|+1-s)}{s \Gamma(2|m|+s)} ds. \quad (12)$$

Now we will take into account that the subintegral function doesn't have singularities in the semiplane  $\Re s > 0$ . Then, transferring the contour of the integration in (12) to the line  $\Re s = \frac{1}{\log y}$  and using Lemma 3, Stirling formula for the gamma-function, we get

$$W''(y) \ll L(y, m),$$

where  $L(y, m) = p^n(M \log M + y^{\frac{1}{4}})$ . But then

$$\begin{aligned} \Delta_z \left( W \left( \frac{N(w)x}{p^{2n}} \right) \right) &= \int_{\frac{N(w)}{p^{2n}}x}^{\frac{N(w)}{p^{2n}}(x+z)} dy_1 \int_{y_1}^{y_1 + \frac{N(w)}{p^{2n}}z} W''(y_2) dy_2 \\ &\ll L \left( \frac{xN(\alpha)}{p^{2n}}, m \right) \frac{z^2 N(w)^2}{p^{4n}}. \end{aligned} \quad (13)$$

Let us denote as  $S_2(x)$  the following sum

$$S_2(x) = \sum_{\alpha \in E_n^+} S_2(x, \alpha).$$

We have

$$\begin{aligned}
 S_2(x) &= \frac{x^2}{2} \sum_{\alpha \in E_n^+} \zeta_m \left( 0; \frac{\alpha}{p^n}, 0 \right) \\
 &\quad + \sum_{\alpha \in E_n^+} \sum_{w \equiv \alpha \pmod{p^n}} e^{4mi \arg w} e^{\pi i \operatorname{Sp} \left( \frac{\alpha w}{p^n} \right)} \frac{W \left( \frac{xN(w)}{p^{2n}} \right)}{N(w)^3} \\
 &= \frac{x^2}{2} \sum_{\alpha \in E_n^+} \zeta_m \left( 0; \frac{\alpha}{p^n}, 0 \right) + \sum_{\chi \in E_n} \frac{1}{|E_n|} \sum_{\alpha \in E_n} \bar{\chi}(\alpha) \\
 &\quad \times \sum_{N(w) \equiv 1 \pmod{p^n}} \frac{\chi(w) e^{4mi \arg w}}{N(w)^3} \cdot e^{\pi i \operatorname{Sp} \left( \frac{\alpha w}{p^n} \right)} W \left( \frac{xN(w)}{p^n} \right).
 \end{aligned} \tag{14}$$

Applying the operator  $\Delta_z$  to both parts of (14), we obtain

$$\begin{aligned}
 \Delta_z(S_2(x)) &= z^2 \sum_{\alpha \in E_n^+} \zeta_m \left( 0; \frac{\alpha}{p^n}, 0 \right) \\
 &\quad + \sum_{\substack{w \in G \\ N(w) \equiv 1 \pmod{p^n}}} e^{-4mi \arg w} N(w)^{-3} W \left( \frac{xN(w)}{p^{2n}} \right) \sum_{\alpha \in E_n^+} e^{\pi i \operatorname{Sp} \frac{\alpha^2}{p^n}}.
 \end{aligned}$$

In virtue of (10), (11) and (13), Lemma 5 and Corollary 2 we infer

$$\begin{aligned}
 \Delta_z(S_2(x)) &\ll z^2 p^{\frac{3}{2}n} M \log M + p^{\frac{n}{2}} z^2 \sum_{N(w) \leq x} N(w)^{-1} L \left( \frac{xN(w)}{p^{2n}}, m \right) \\
 &\quad + p^{\frac{n}{2}} \sum_{N(w) > x} N(w)^{-3} K \left( \frac{xN(w)}{p^{2n}}, m \right) \\
 &\ll z^2 p^{\frac{n}{2}} p^n M \log M + z^2 p^{\frac{n}{2}} \sum_{N(w) \leq x} p^n \left( M \log M + \frac{N(w)^{\frac{1}{4}} x^{\frac{1}{4}}}{p^{\frac{n}{2}}} \right) N(w)^{-1} \\
 &\quad + z^2 p^{\frac{n}{2}} \sum_{N(w) > x} p^{3n} M^3 N(w)^{-2} p^{-2n} \log N(w).
 \end{aligned} \tag{15}$$

From this we get

$$\begin{aligned}
 \Delta_z(S_2(x)) &\ll \\
 &\ll p^{\frac{3}{2}} \left\{ z^2 M \log M + z^2 M \log M \log x + z^2 p^{-\frac{n}{2}} \sqrt{x} + M^3 \log x \right\}. \tag{16}
 \end{aligned}$$

The application of the estimates (10), (14) requires that  $z \ll \frac{xN(w)}{p^{2n}}$ . Thus the condition  $N(w) > x$  in the second sum of (15) allows to assume  $z = p^n M \leq \frac{x^2}{p^{2n}}$  for  $M \ll \frac{x^2}{p^{2n}}$ . Then the following inequality

$$\Delta_z(S_2(x)) \ll z^2 p^{\frac{3}{2}n} M \log M$$

holds.

Let  $H_2(x)$  stands for the sum

$$H_2(x) = \sum_{\alpha \in E_n^+} \sum_{\substack{w \in G \\ w \equiv \alpha \pmod{p^n} \\ N(w) \leq x}} e^{4mi \arg w}. \tag{17}$$

Then from the definition of  $S_2(x)$  we easily find

$$H_2(x) = \frac{d^2}{dx^2}(S_2(x)).$$

It is clear that

$$\int_x^{x+z} dy_1 \int_{y_1}^{y_1+z} H_2(y_2) dy_2 = \Delta_z(S_2(x)).$$

By  $x \leq y_1 \leq x + 2z$  and Lemma 4 we have

$$\begin{aligned} |H_2(y_2) - H_2(x)| &= |E_n^+| \cdot \left| \sum_{\substack{x < N(w) \leq y_2 \\ N(w) \equiv 1 \pmod{p^n}}} e^{4mi \arg w} \right| \\ &\leq (p+1)p^{n-1} \sum_{\substack{x < n \leq x+2z \\ n \equiv 1 \pmod{p^n}}} r(n) \\ &\leq \frac{\pi z}{p^n} \cdot \frac{p+1}{p} + O\left(\frac{\sqrt{x}}{p^{\frac{n}{2}}}\right) + O\left(p^{\frac{n}{2}} \exp\left(c \frac{(\log p^n)^{\frac{1}{2}}}{\log \log p^n}\right)\right). \end{aligned}$$

Consequently,

$$|H_2(y_2) - H_2(x)| = O\left(\frac{z}{p^n}\right) + O\left(\sqrt{x}p^{-\frac{n}{2}}\right) + O\left(p^{\frac{n}{2}} \exp\left(c \frac{(\log p^n)^{\frac{1}{2}}}{\log \log p^n}\right)\right).$$

It follows that

$$H_2(y_2) = H_2(x) + O\left(\frac{z}{p^n}\right) + O\left(\sqrt{x}p^{-\frac{n}{2}}\right) + O\left(p^{\frac{n}{2}} \exp\left(c \frac{(\log p^n)^{\frac{1}{2}}}{\log \log p^n}\right)\right). \tag{18}$$

Now from (17) and (18) we get

$$z^2 \left( H_2(x) + O\left(\frac{z}{p^n}\right) + O\left(\sqrt{x}p^{-\frac{n}{2}}\right) + O\left(p^{\frac{n}{2}} \exp\left(c\frac{(\log p^n)^{\frac{1}{2}}}{\log \log p^n}\right)\right) \right) = O(z^2 p^{\frac{n}{2}} M \log M).$$

Thus,

$$H_2(x) = x^{\frac{1}{2}} p^{-\frac{n}{2}} + p^{\frac{n}{2}} \exp\left(c\frac{(\log p^n)^{\frac{1}{2}}}{\log \log p^n}\right) + p^{\frac{n}{2}} M \log M.$$

So, the proof of Theorem 1 is completed. □

Now we can investigate the distribution of quadratic residues modulo  $p^n$  in narrow sectors.

**Theorem 2.** *Let  $p^{\frac{3}{2}n} \leq x \leq p^{2n}$ ,  $0 \leq \phi_1 < \phi_2 \leq \frac{\pi}{2}$  and let  $0 < s \leq \frac{1}{8}$ . Then for  $\phi_2 - \phi_1 \geq x^{-s}$  the asymptotic formula*

$$R(x; \phi) = \frac{\phi_2 - \phi_1}{2} \cdot \frac{p+1}{p} \cdot \frac{x}{p^n} + O\left(3^n \frac{x^{1-s}}{p^n} \log x\right)$$

holds.

*Proof.* It is known that the distribution of the arguments of Gaussian integers (being considered up to the association) has the period  $\frac{\pi}{2}$ . In view of this fact the application of Lemma 7 with  $\Omega = \frac{\pi}{2}$  gives for every  $T \geq 1$

$$\sum_{\substack{\alpha \in E_n^+ \\ \phi_1 \leq \alpha < \phi_2 \\ N(\alpha) \leq x}} 1 = \Phi(\phi_1, \phi_2) + \theta_1 \Phi(\phi_1 - \Delta, \phi_1) + \theta_2 \Phi(\phi_2, \phi_2 + \Delta),$$

$$|\theta_i| \leq 1, \quad i = 1, 2, \quad \Phi(\phi_1, \phi_2) = \frac{1}{4} \sum_{\substack{w \in E_n^+ \\ N(w) \leq x}} f(\arg w)$$

and  $f(\phi)$  is the function from Lemma 7,  $0 < \Delta = \frac{1}{2}\Omega$ .

Furthermore

$$\Phi(\phi_1, \phi_2) = \sum_{\substack{w \in E_n^+ \\ N(w) \leq x}} \sum_{m=-\infty}^{+\infty} a_m e^{4mi \arg \alpha} = \sum_{m=-\infty}^{+\infty} a_m \sum_{\substack{k \equiv 1 \pmod{p^n} \\ k \leq x}} r_m(x),$$

where  $a_m$  is the Fourier coefficient from Lemma 7.



We put  $\delta = x^s$ ,  $0 < s < 1$  (we will find the more precise estimate for  $s$  later). Let us use the estimates for the coefficients  $a_m$  (see Lemma 7 with  $r = 2$ ):

$$|a_m| \ll \begin{cases} \frac{1}{|m|}, & |m| \leq \delta = \Delta^{-1}; \\ \frac{1}{|m|^3 \Delta^2}, & |m| > \delta. \end{cases}$$

After simple calculations we get

$$\begin{aligned} \Phi(\phi_1, \phi_2) &= \frac{\phi_2 - \phi_1}{2} \cdot \frac{p+1}{p} \cdot \frac{x}{p^n} + O\left(\frac{x^{1-s}}{p^n}\right) + O\left(s \frac{\sqrt{x}}{p^{\frac{n}{2}}} \log^2 x\right) \\ &+ O\left(sp^{\frac{n}{2}} \log^2 x\right) + O\left(\frac{x^{\frac{1}{2}+s}}{p^{\frac{n}{2}}}\right) + O\left(3^n p^{\frac{n}{2}} x^s \log x\right). \end{aligned}$$

In view of the assumption of the theorem the following inequalities

$$\frac{x^{1-s}}{p^n} \gg p^{\frac{n}{2}} x^s, \quad \frac{x^{1-s}}{p^n} \gg \frac{x^{\frac{1}{2}+s}}{p^{\frac{n}{2}}}$$

hold. Therefore,

$$\Phi(\phi_1, \phi_2) = \frac{\phi_2 - \phi_1}{2} \cdot \frac{p+1}{p} \cdot \frac{x}{p^n} + O\left(3^n \frac{x^{1-s}}{p^n} \log x\right). \quad (19)$$

It follows from (19) that

$$\Phi(\phi_1 - \Delta, \phi_1), \Phi(\phi_2, \phi_2 + \Delta) \ll 3^n \frac{x^{1-s}}{p^n} \log x. \quad (20)$$

The relations (19) and (20) show that Theorem 2 is proved for every  $s$ ,  $0 < s \leq \frac{1}{8}$ .  $\square$

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