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A horizontal mesh algorithm for posets with positive Tits form

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Abstract. Following our paper [Fund. Inform. 136 (2015), 345–379], we define a horizontal mesh algorithm that constructs a $\widehat{\Phi}_I$-mesh translation quiver $\Gamma(\widehat{\mathcal{R}}_I, \widehat{\Phi}_I)$ consisting of $\widehat{\Phi}_I$-orbits of the finite set $\widehat{\mathcal{R}}_I = \{v \in \mathbb{Z}^I\ ;\ \widehat{q}_I(v) = 1\}$ of Tits roots of a poset I with positive definite Tits quadratic form $\widehat{q}_I : \mathbb{Z}^I \to \mathbb{Z}$. Under the assumption that $\widehat{q}_I : \mathbb{Z}^I \to \mathbb{Z}$ is positive definite, the algorithm constructs $\Gamma(\widehat{\mathcal{R}}_I, \widehat{\Phi}_I)$ such that it is isomorphic with the $\widehat{\Phi}_D$-mesh translation quiver $\Gamma(\mathcal{R}_D, \Phi_D)$ of $\widehat{\Phi}_D$-orbits of the finite set $\mathcal{R}_D$ of roots of a simply laced Dynkin quiver D associated with I.

1. Introduction

The paper is mainly devoted to the existence of a $\widehat{\Phi}_I$-mesh root system $\Gamma(\widehat{\mathcal{R}}_I, \widehat{\Phi}_I)$ in the sense of [30], that is, a $\widehat{\Phi}_I$-mesh translation quiver $\Gamma(\widehat{\mathcal{R}}_I, \widehat{\Phi}_I)$ consisting of $\widehat{\Phi}_I$-orbits of the set $\widehat{\mathcal{R}}_I = \{v \in \mathbb{Z}^I\ ;\ \widehat{q}_I(v) = 1\}$ of Tits roots of a finite poset $I = (I, \preceq)$ with positive quadratic Tits form $\widehat{q}_I : \mathbb{Z}^I \to \mathbb{Z}$, where $\widehat{\Phi}_I : \mathbb{Z}^I \to \mathbb{Z}^I$ is the Coxeter-Tits transformation associated with I in [9, 28, 29, 34]. The reader is also referred to [14], [16], and [30]-[34] for analogous existence mesh root system theorems in the setting of positive edge-bipartite graphs and non-negative posets.

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Our interest in the $\widehat{\Phi}_I$-mesh analysis of $\widehat{\Phi}_I$-orbits of the set $\widehat{\mathcal{R}}_I$ of Tits roots is motivated by applications of matrix representations of posets in representation theory, where a matrix representation of a partially ordered set $T = \{p_1, \dots, p_n\}$, with a partial order $\preceq$, means a block matrix

$$M = [M_1|M_2|\dots|M_n]$$

(over a field K) of size $d_* \times (d_1, \dots, d_n)$ determined up to all elementary row transformations, elementary column transformations within each of the substrips $M_1, M_2, \dots, M_n$, and additions of linear combinations of columns of M_i to columns of M_j, if $p_i \prec p_j$, see Nazarova and Roiter [22]. In [9], Drozd proves that T has only a finite number of direct-sum-indecomposable representetions if and only if its quadratic Tits form

$$q(x_1, \dots, x_n, x_*) = x_1^2 + \dots + x_n^2 + x_*^2 + \sum_{p_i \prec p_j} x_i x_j - x_*(x_1 + \dots + x_n) \quad (1.1)$$

is weakly positive (i.e., $q(a_1, \dots, a_n, a_*) > 0$, for all non-zero vectors $(a_1, \dots, a_n, a_*)$ with integer non-negative coefficients). In this case, there exists an indecomposable representation M of size $d_* \times (d_1, \dots, d_n)$ if and only if $(d_1, \dots, d_n, d_*)$ is a root of q, i.e., $q(d_1, \dots, d_n, d_*) = 1$, see [10] and [26, Chapter 10] for more details.

In [5, 6], Bondarenko and Stepochkina give a complete list of posets T with positive Tits form $q(x_1, \dots, x_n, x_*)$; it consists of four infinite series and 108 exceptional posets, up to duality (see also [11, 12] for an alternative proof).

Throughout this paper, we assume that

$$I = (I, \preceq)$$

is a poset (i.e., a finite partially ordered set). We denote by $\max I$ the set of all maximal elements of I and let $I^- = I \setminus \max I$. For $i, j \in I$, we write $i \prec j$ if $i \preceq j$ and $i \neq j$. Moreover, for $i, j \in I$, we write $i \to j$, if $i \prec j$ and there is no s in I such that $i \prec s \prec j$. We denote by $\mathbb{Z}$ the ring of integers and by $\mathbb{M}_I(\mathbb{Z})$ the ring of I by I square matrices with integer coefficients.

Usually we define a poset I by presenting its Hasse quiver $\mathcal{H}(I) = (\mathcal{H}_0(I), \mathcal{H}_1(I))$, with the set of vertices $\mathcal{H}_0(I) = I$ and the set $\mathcal{H}_1(I)$ of arrows $i \to j$ defined earlier, for $i, j \in I$.

Following [26, 28, 29, 34], with any poset I, we associate the *incidence matrix* $C_I = [c_{ij}] \in \mathbb{M}_I(\mathbb{Z})$ and the *Tits matrix* $\widehat{C}_I \in \mathbb{M}_I(\mathbb{Z})$, where

$$c_{ij} = \begin{cases} 1 & \text{if } i \preceq j, \\ 0 & \text{otherwise,} \end{cases} \tag{1.2}$$

and

$$\widehat{C}_I = \begin{bmatrix} C_{I^-}^{tr} & U \\ 0 & E \end{bmatrix}, \tag{1.3}$$

where $U = [u_{iw}]_{i \in I^-; w \in \max I}$ and

$$u_{iw} = \begin{cases} -1 & \text{if } i \preceq w, \\ 0 & \text{otherwise,} \end{cases} \tag{1.4}$$

Following [11, 32, 34], we call a poset I *positive*, if the symmetric Gram matrix $G_I := \frac{1}{2}(\widehat{C}_I + \widehat{C}_I^{tr})$ is positive definite.

The following two sets of vectors associated with a poset I are playing an important role in the representation theory of algebras: the set of *Tits roots*

$$\widehat{\mathcal{R}}_I := \{v \in \mathbb{Z}^n;\ v \cdot \widehat{C}_I \cdot v^{tr} = 1\} \tag{1.5}$$

and the set of *Euler roots*

$$\overline{\mathcal{R}}_I := \{v \in \mathbb{Z}^n;\ v \cdot \overline{C}_I \cdot v^{tr} = 1\} \tag{1.6}$$

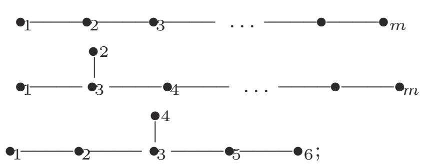
of a poset I, where

$$\overline{C}_I = C_I^{-1} \tag{1.7}$$

see [10, 21, 24, 26]. We recall from [30] that the sets of Tits roots $\widehat{\mathcal{R}}_I$ and Euler roots $\overline{\mathcal{R}}_I$ of I are finite, if I is positive. Moreover, if I is assumed to be connected then the sets $\widehat{\mathcal{R}}_I$ and $\overline{\mathcal{R}}_I$ are irreducible and reduced root systems in the sense of Bourbaki, see [24, p. 40] and [16], for more details.

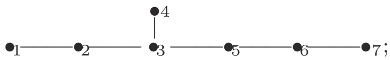
By [29, Corollary 1.8], given a positive poset I, the root systems $\widehat{\mathcal{R}}_I$ and $\overline{\mathcal{R}}_I$ are isomorphic, and we denote by DI the common Coxeter-Dynkin type of these root systems. One should note that DI is one of the simply laced Dynkin diagrams (see [24, p. 40] and [16])

$\mathbb{A}_m$:
$\mathbb{D}_m$:
$\mathbb{E}_6$:

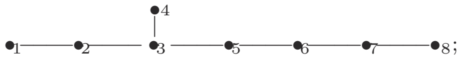


(m vertices, $m \geqslant 1$);
(m vertices, $m \geqslant 4$);

$\mathbb{E}_7$:



$\mathbb{E}_8$:



It follows from [16] that the Dynkin diagram DI can be determined by applying the inflation algorithm constructed in [20] and [32].

We recall from [29] that the square matrix

$$\widehat{\mathrm{Cox}}_I := -\widehat{C}_I \cdot \widehat{C}_I^{-tr} \in \mathbb{M}_n(\mathbb{Z}), \tag{1.8}$$

is called the *Coxeter-Tits matrix* of I. Here $\widehat{C}_I^{tr}$ is the transpose of $\widehat{C}_I$, and we set $\widehat{C}_I^{-tr} := (\widehat{C}_I^{tr})^{-1}$. The charactristic polynomial

$$\mathrm{cox}_I(t) := \det(t \cdot E - \widehat{\mathrm{Cox}}_I) \in \mathbb{Z}[t], \tag{1.9}$$

of $\widehat{\mathrm{Cox}}_I$ is called the *Coxeter polynomial* of I, the group isomorphism

$$\widehat{\Phi}_I : \mathbb{Z}^n \to \mathbb{Z}^n, \quad x \mapsto \widehat{\Phi}_I(x) := x \cdot \widehat{\mathrm{Cox}}_I, \tag{1.10}$$

is called the *Coxeter-Tits transformation* of I, and the *Coxeter number* $\mathbf{c}_I$ of I is the minimal integer $r \geqslant 1$ such that $\widehat{\Phi}_I^r$ is the identity map on $\mathbb{Z}^n$. If $\widehat{\Phi}_I^r \neq id$, for all $r \geqslant 1$, we set $\mathbf{c}_I = \infty$.

Recall also that the matrix

$$\overline{\mathrm{Cox}}_I := -\overline{C}_I \cdot \overline{C}_I^{-tr} \in \mathbb{M}_n(\mathbb{Z}), \tag{1.11}$$

is called the *Coxeter-Euler matrix* of I, and the group isomorphism

$$\overline{\Phi}_I : \mathbb{Z}^n \to \mathbb{Z}^n, \quad x \mapsto \overline{\Phi}_I(x) := x \cdot \overline{\mathrm{Cox}}_I, \tag{1.12}$$

is called the *Coxeter-Euler transformation* of I.

Following an idea introduced in [30, 31], we study in the paper a $\widehat{\Phi}_I$-mesh root system structure $\Gamma(\widehat{\mathcal{R}}_I, \widehat{\Phi}_I)$ on the set of roots $\widehat{\mathcal{R}}_I \subseteq \mathbb{Z}^n$ of any connected positive poset I, with $n \geqslant 2$ vertices, where $\widehat{\Phi}_I : \mathbb{Z}^n \to \mathbb{Z}^n$ is the Coxeter-Tits transformation defined by the Tits matrix $\widehat{C}_I \in \mathbb{M}_n(\mathbb{Z})$ of I.

One of the main aims of the paper is to present a combinatorial algorithm that constructs a $\widehat{\Phi}_I$-mesh root system structure $\Gamma(\widehat{\mathcal{R}}_I, \widehat{\Phi}_I)$ (see Definition 2.13) on the finite set of all $\widehat{\Phi}_I$-orbits of the irreducible root system $\widehat{\mathcal{R}}_I$. Moreover, in Corollary 4.6, we prove that the Coxeter polynomial $\mathrm{cox}_I(t)$ and the Coxeter number $\mathbf{c}_I$ of such poset I depend only on the simply laced Dynkin type DI of $\widehat{\mathcal{R}}_I$ and $\mathrm{cox}_I(t)$ coincides with the Coxeter polynomial $\mathrm{cox}_{DI}(t)$ of the Dynkin diagram DI, see [29, Example 3.12].

The idea of construction of our horizontal mesh algorithm is inspired by the method of a construction of postprojective component in some categories of modules (see [7, 8, 15, 26]). However, this well-known method computes only a mesh quiver consisting of the positive vectors. In the present paper we show that our modification of this algorithm computes a $\widehat{\Phi}_I$-mesh root system structure $\Gamma(\widehat{\mathcal{R}}_I, \widehat{\Phi}_I)$ for the set $\widehat{\mathcal{R}}_I$ of all roots (not only positive roots).

We recall that one of the motivations for the study of a $\widehat{\Phi}_I$-mesh root system structure $\Gamma(\widehat{\mathcal{R}}_I, \widehat{\Phi}_I)$ comes from the poset representation theory (see [9, 10, 21, 24, 26, 28, 29, 34]).

The sets of roots and Tits roots are playing an important role in many areas of mathematics. In the representation theory of finite dimensional algebras over a field the roots control categories of indecomposable modules for a large classes of algebras (see [1–3, 24, 25]), while in the theory of Lie groups and Lie algebras they are connected with root spaces (see [4, 13]). Moreover, they control linear bases, generators and relations of Ringel-Hall algebras (see [18, 19]).

Recall that in [17] the Tits roots were applied to get a classification of two-peak sincere posets of finite prinjective type. Therefore, it is of importance to have efficient combinatorial algorithms that compute roots, Tits roots and $\widehat{\Phi}_I$-mesh root system structures.

2. Preliminaries

Throughout this paper all posets are assumed to be connected.

2.1. Unit quadratic forms associated with a poset

Let I be a poset. By a *Tits quadratic form* and an *Euler quadratic form* of I we mean the unit quadratic forms

$$\widehat{q}_I, \overline{q}_I : \mathbb{Z}^I \to \mathbb{Z}$$

defined by the formulae

$$\widehat{q}_I(x) = x \cdot \widehat{C}_I \cdot x^{tr}, \quad \overline{q}_I(x) = x \cdot \overline{C}_I \cdot x^{tr}.$$

It is easy to see that

$$\widehat{q}_I(x) = \sum_{i \in I} x_i^2 + \sum_{i \prec j \in I^-} x_j x_i - \sum_{w \in \max I} \sum_{i \prec w} x_i x_w. \tag{2.1}$$

Note also that the Tits quadratic form $q(x_1,\dots,x_n,x_*)$ (1.1) of a partially ordered set $T=\{p_1,\dots,p_n\}$ (defined by Drozd [9]) coincides with the Tits form $\widehat{q}_I(x_1,\dots,x_n,x_*)$ (2.1) of the one-peak poset $I=T^*\cup\{*\}$ obtained from T by adding a unique maximal element $*$.

Recall from [29, Corollary 1.8] that one of the quadratic forms $\widehat{q}_I,\overline{q}_I$ is positive if and only if both of them are positive. Moreover, in this case we have

$$\overline{q}_I(x)=\sum_{i\in I}x_i^2-\sum_{i\to j}x_ix_j+\sum_{i\blacktriangleleft j}c_{ij}^{\bullet}x_ix_j, \tag{2.2}$$

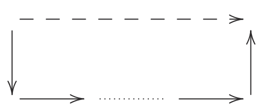
where the relation $i\blacktriangleleft j$ holds if there exists a minimal commutativity relation $w'-w''$ in I, where w',w'' are paths with the source i and the terminus j and $c_{ij}^{\bullet}$ is the maximal number of linearly independent minimal commutativity relations $w'-w''$ in I with the source i and the terminus j, see Corollary 1.8, Remark 3.5 and Proposition 4.2 in [29].

Remark 2.3. Let I be a positive poset. The formula (2.2) implies that the matrix $\overline{C}_I=(\overline{c}_{ij})$ satisfies the non-cycle condition defined in [14]. Let us recall this definition. With a poset I we associate the biquiver $\overline{Q}_I=(\overline{Q}_0,\overline{Q}_1)$ with the set of vertices $\overline{Q}_0=I$. Moreover, there are $-\overline{c}_{ij}$ solid arrows $i\longrightarrow j$, if $\overline{c}_{ij}<0$ and $\overline{c}_{ij}$ broken arrows $i\dashrightarrow j$, if $\overline{c}_{ij}>0$. Let $Q=(Q_0,Q_1)$ be a biquiver.

(a) We say that a (unoriented) cycle $(x_1,x_2,\dots,x_n,x_1)$ in Q is *simple* if for all $i,j\in\{1,\dots,n\}$, $i\neq j$ we have $x_i\neq x_j$.

(b) We say that a simple cycle $(x_1,x_2,\dots,x_n,x_1)$ is *chordless* if for any arrow (x_i,x_j) we have $i=j\pm 1$ (wherein $1\equiv n+1$).

(c) Further, consider a simple cycle in Q of the form



(2.4)

The biquiver Q satisfies the *non-cycle condition*, if every simple chordless cycle in Q containing a broken arrow has the form (2.4).

(d) Given a poset I the matrix $\overline{C}_I=(\overline{c}_{ij})$ satisfies the *non-cycle condition*, if the biquiver $\overline{Q}_I$ satisfies this condition.

For all $i\in I$, denote by $\widehat{p}_i$ the *Tits-projective* vector associated with i, i.e. $\widehat{p}_i$ is defined by the formula

$$\widehat{p}_i(j)=\begin{cases}1 & \text{for} \quad i=j;\\ 1 & \text{for} \quad i\preceq j\in\max I;\\ 0 & \text{otherwise.}\end{cases} \tag{2.5}$$

Let

$$\widehat{\mathcal{P}} = \widehat{\mathcal{P}}(I) = \{\widehat{p}_i \ ; \ i \in I\}$$

be the set of all Tits-projective vectors.

For all $i \in I$, denote by $\widehat{r}_i$ the *Tits-radical* vector associated with i, i.e. $\widehat{r}_i$ is defined by the formula

$$\widehat{r}_i(j) = \begin{cases} 1 & \text{for all} \quad i \to j; \\ 1 & \text{for} \quad i \prec j \in \max I; \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

Let

$$\widehat{\mathrm{Rad}} = \widehat{\mathrm{Rad}}(I) = \{\widehat{r}_i \ ; \ i \in I\}$$

be the set of all Tits-radical vectors.

Let $i \in I$ and let $\widehat{r}_i$ be the corresponding Tits-radical vector. Consider the convex subposet

$$I\text{-supp}(\widehat{r}_i) = conv.hull\{j \in I \ ; \ \widehat{r}_i(j) \neq 0\}$$

of I. Let $I_1, \dots, I_{k_i}$ be the set of all connected components of the Hasse quiver of I-supp(r_i). We define the vectors $\widehat{r}_i^1, \dots, \widehat{r}_i^{k_i}$ by the following formula:

$$\widehat{r}_i^t(j) = \begin{cases} \widehat{r}_i(j) & \text{if } i \in I_t; \\ 0 & \text{otherwise} \end{cases} \tag{2.7}$$

for all $t \in \{1, \dots, k_i\}$. We denote by $\widehat{\mathrm{Rad}}_{comp}$ the set of vectors $\widehat{r}_i^1, \dots, \widehat{r}_i^{k_i}$, where $i \in I$.

It is known that $\widehat{p}_i \in \widehat{\mathcal{R}}_I$ and $\widehat{r}_i^j \in \widehat{\mathcal{R}}_I$, for all i, j, see [23, 26, 27].

Denote by $\overline{p}_i$ the *Euler-projective* vector associated with i, i.e. $\overline{p}_i$ is defined by the formula

$$\overline{p}_i(j) = \begin{cases} 1 & \text{for all} \quad i \preceq j; \\ 0 & \text{otherwise.} \end{cases} \tag{2.8}$$

Let

$$\overline{\mathcal{P}} = \overline{\mathcal{P}}(I) = \{\overline{p}_i \ ; \ i \in I\}$$

be the set of all Euler-projective vectors.

For all $i \in I$, denote by $\overline{r}_i$ the *Euler-radical* vector associated with i, i.e. $\overline{r}_i$ is defined by the formula:

$$\overline{r}_i = \overline{p}_i - e_i. \tag{2.9}$$

Let

$$\overline{\mathrm{Rad}} = \overline{\mathrm{Rad}}(I) = \{\overline{r}_i \ ; \ i \in I\}$$

be the set of all Euler-radical vectors.

Let $i \in I$ and let $\overline{r}_i$ be the corresponding Euler-radical vector. Consider the convex subposet

$$I\text{-supp}(\overline{r}_i) = \{j \in I \ ; \ \overline{r}_i(j) \neq 0\}$$

of I. Let $I_1, \ldots, I_{k_i}$ be the set of all connected components of the Hasse quiver of I-supp$(\overline{r}_i)$. We define the vectors $\overline{r}_i^1, \ldots, \overline{r}_i^{k_i}$ by the following formula:

$$\overline{r}_i^t(j) = \begin{cases} \overline{r}_i(j) & \text{if } i \in I_t; \\ 0 & \text{otherwise} \end{cases} \tag{2.10}$$

for all $t \in \{1, \ldots, k_i\}$. We denote by $\overline{\mathrm{Rad}}_{comp}$ the set of vectors $\overline{r}_i^1, \ldots, \overline{r}_i^{k_i}$, where $i \in I$.

It is known that $\overline{p_i} \in \overline{\mathcal{R}}_I$ and $\overline{r}_i^j \in \overline{\mathcal{R}}_I$, for all i, j, see [14, 23, 26, 27].

2.2. Mesh translation quivers in $\mathbb{Z}^n$

We recall from [30, 31] the following definitions (see also [14]). They are inspired by the definition of the Auslander-Reiten quiver of an algebra (see [1, 2]).

Let $\Phi : \mathbb{Z}^n \to \mathbb{Z}^n$ be a group automorphism (e.g. the Coxeter-Tits transformation $\widehat{\Phi}_I$ or the Coxeter transformation $\overline{\Phi}_I$ of a poset I). A Φ-orbit $\Phi - \mathcal{O}rb(v) = \{\Phi^k(v)\}_{k\in\mathbb{Z}}$ of a vector $v \in \mathbb{Z}^n$ will be visualised as an infinite graph:

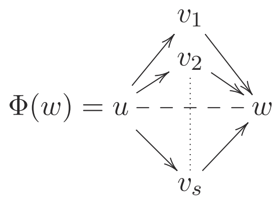
$$\ldots - - - \Phi(v) - - - v - - - \Phi^{-1}(v) - - - \Phi^{-2}(v) - - - \ldots$$

Definition 2.11. Let $\Phi : \mathbb{Z}^n \to \mathbb{Z}^n$ be a non-trivial group automorphism (e.g. the Coxeter-Tits transformation $\widehat{\Phi}_I$ or the Coxeter transformation $\overline{\Phi}_I$ of a poset I). We say that the vectors $u, v_1, \ldots, v_s, w \in \mathbb{Z}^n$ form a **Φ-mesh** starting from u and terminating at w, if the following two conditions are satisfied:

(i) $u = \Phi(w)$ and $u + w = \sum_{i=1}^{s} v_i$,

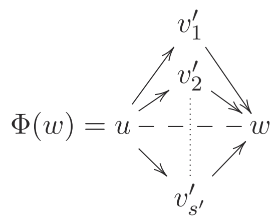
(ii) the vectors $v_1, \ldots, v_s$ are pairwise different, lie in pairwise different orbits of Φ and none of them lies in the Φ-orbit of u.

A Φ-mesh we visualise as the following triangular quiver:



(2.12)

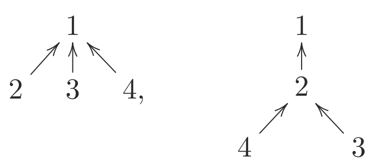
Definition 2.13. Let $n \geqslant 2$, let $\Phi : \mathbb{Z}^n \to \mathbb{Z}^n$ be a non-trivial group automorphism and let $\mathcal{R}$ be a Φ-invariant subset of $\mathbb{Z}^n$ (e.g. $\mathcal{R} = \widehat{\mathcal{R}}_I$ if $\Phi = \widehat{\Phi}_I$ or $\mathcal{R} = \overline{\mathcal{R}}_I$ if $\Phi = \overline{\Phi}_I$). We say that $\mathcal{R}$ admits a geometry of Φ-*mesh quiver* if there exists a quiver $\vec{\mathcal{R}} = (\vec{\mathcal{R}}_0, \vec{\mathcal{R}}_1)$ with $\vec{\mathcal{R}}_0 = \mathcal{R}$, such that $\vec{\mathcal{R}}$ together with the bijection $\Phi : \mathcal{R} \to \mathcal{R}$ induced by Φ is a triangular translation quiver $\Gamma(\mathcal{R}, \Phi)$ (see [1, IV.4.7]) with the following property: for every vector $w \in \mathcal{R}$, the full convex subquiver containing the vertices w and $\Phi(w)$ is a Φ-mesh of the form (2.12), and if



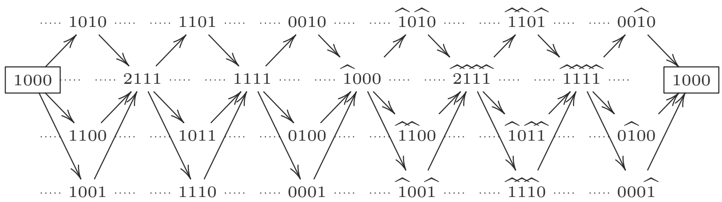
is a Φ-mesh, then $s' = s$ and $v_1 = v'_1, \ldots, v_s = v'_s$, up to permutation of the set $\{1, \ldots, s\}$.

Definition 2.14. Let $\Gamma(\mathcal{R}, \Phi)$ be a Φ-mesh quiver in $\mathbb{Z}^n$ as in Definition 2.13. A *slice* in $\Gamma(\mathcal{R}, \Phi)$ is a full convex connected subquiver $\Sigma = (\Sigma_0, \Sigma_1)$ of $\Gamma(\mathcal{R}, \Phi)$ such that for any $v \in \mathcal{R}$ the set $\Phi - \mathcal{O}rb(v) \cap \Sigma_0$ contains exactly one element.

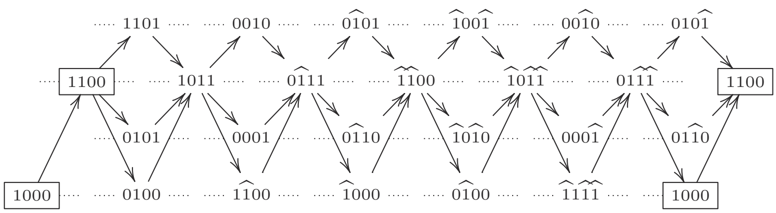
Example 2.15. Consider the posets I and I' defined by the following Hasse quivers:



respectively. Note that the set $\widehat{\mathcal{R}}_I \subseteq \mathbb{Z}^4$ of Tits roots of I consists of 24 vectors. One easily see that the set $\widehat{\mathcal{R}}_I$ admits the following geometry of $\widehat{\Phi}_I$-mesh quiver $\Gamma(\widehat{\mathcal{R}}_I, \widehat{\Phi}_I)$ (we identify the vectors in frames):

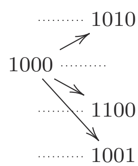


Moreover the set $\widehat{\mathcal{R}}_{I'}$ of Tits roots of I' consists of 24 vectors and admits the following geometry of $\widehat{\Phi}_{I'}$-mesh quiver $\Gamma(\widehat{\mathcal{R}}_{I'}, \widehat{\Phi}_{I'})$ (we identify the vectors in frames):

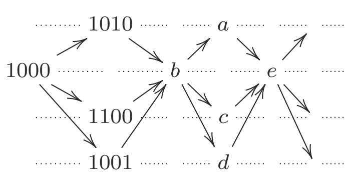


Here we set $\widehat{a} = -a$, for $a \in \mathbb{N}$.

In the algorithm presented in Section 3 first we look for a slice canditate Σ in $\Gamma(\widehat{\mathcal{R}}_I, \widehat{\Phi}_I)$. Then the remaining part of $\Gamma(\widehat{\mathcal{R}}_I, \widehat{\Phi}_I)$ is easy to compute. In $\Gamma(\widehat{\mathcal{R}}_I, \widehat{\Phi}_I)$ presented in Example 2.15 the quiver



is a slice. Applying definition of a $\widehat{\Phi}_I$-mesh we can construct now the $\widehat{\Phi}_I$-mesh translation quiver $\Gamma(\widehat{\mathcal{R}}_I, \widehat{\Phi}_I)$ by knitting $\widehat{\Phi}_I$-meshes as follows:



Indeed, we have

$b = (1\,0\,1\,0) + (1\,1\,0\,0) + (1\,0\,0\,1) - (1\,0\,0\,0)$,
$a = b - (1\,0\,1\,0)$,
$c = b - (1\,1\,0\,0)$,
$d = b - (1\,0\,0\,1)$,
$e = a + c + d - b$, and so on.

Note that $\widehat{\Phi}_I(a) = (1010)$, $\widehat{\Phi}_I(b) = (1000)$, $\widehat{\Phi}_I(c) = (1100)$, $\widehat{\Phi}_I(d) = (1001)$, and $\widehat{\Phi}_I(e) = b$.

3. A horizontal mesh algorithm

The idea of construction of a horizontal mesh algorithm that we present in this section is inspired by a construction of the postprojective component of the Auslander-Reiten quiver of an algebra or a poset (see [7, 8, 15]).

We would like to stress that the algorithm

$$(I, \widehat{\mathcal{P}}, \widehat{\mathrm{Rad}}, \widehat{\mathrm{Rad}}_{comp}, k) \mapsto \widehat{\Gamma} := \Gamma(\widehat{\mathcal{R}}_I, \widehat{\Phi}_I)$$

presented below, called *a horizontal mesh algorithm*, associates to an arbitrary poset I, with initial data $\widehat{\mathcal{P}}, \widehat{\mathrm{Rad}}, \widehat{\mathrm{Rad}}_{comp}, k$, a $\widehat{\Phi}_I$-mesh translation quiver $\Gamma(\widetilde{\mathcal{R}}_I, \widehat{\Phi}_I)$ such that $\widehat{\Gamma}$ defines a $\widehat{\Phi}_I$-mesh root system structure $\Gamma(\widehat{\mathcal{R}}_I, \widehat{\Phi}_I)$ on the set $\widehat{\mathcal{R}}_I$ of Tits roots of I, in case when I is positive (see Theorem 4.4 for a proof). The algorithm is a modification of a corresponding horizontal mesh algorithm presented in [14], for positive edge-bipartite graphs.

Algorithm 3.1. *Input:* A system $(I, \widehat{\mathcal{P}}, \widehat{\mathrm{Rad}}, \widehat{\mathrm{Rad}}_{comp}, k)$, where

- $I = (I, \preceq)$ is a poset such that $I = \{1, \ldots, n\}$,
- $\widehat{\mathcal{P}} = \{\widehat{p}_1, \ldots, \widehat{p}_n\}$ is the set of Tits-projective vectors,
- $\widehat{\mathrm{Rad}} = \{\widehat{r}_1, \ldots, \widehat{r}_n\}$ is the set of Tits-radical vectors,
- $\widehat{\mathrm{Rad}}_{comp} = \{\widehat{r}_1^1, \ldots, \widehat{r}_1^{k_1}, \ldots, \widehat{r}_n^1, \ldots, \widehat{r}_n^{k_n}\}$, where $\widehat{r}_i^j$ are defined by formula 2.7,
- $k \in \mathbb{N}$.

Output: The quiver $\widehat{\Gamma} = \Gamma(\widehat{\mathcal{R}}_I, \widehat{\Phi}_I)$.

Step 1. Inductively, we construct the following data:

- ordered lists $\widehat{L}[i]$, for any $i = 1, \ldots, n$;
- quivers $\widehat{G}^i = (\widehat{G}_0^i, \widehat{G}_1^i)$, for $i = 0, 1, 2, \ldots$;
- quivers $\widehat{\Gamma}^i = (\widehat{\Gamma}_0^i, \widehat{\Gamma}_1^i)$, for $i = 0, 1, 2, \ldots$;
- sets $\widehat{\mathcal{P}}_0 \subseteq \widehat{\mathcal{P}}_1 \subseteq \ldots \subseteq \widehat{\mathcal{P}}_k \subseteq \widehat{\mathcal{P}} = \{\widehat{p}_1, \ldots, \widehat{p}_n\}$;

in the following way.

Step 1.1. For any $i = 1, \ldots, n$, we put $\widehat{L}[i] := [\widehat{p}_i]$.

Step 1.2. Let

$$\widehat{\mathcal{P}}_0 = \widehat{G}_0^0 = \{\widehat{p}_i \in \widehat{\mathcal{P}} \ ; \ i \in \max I\} \quad \text{and} \quad \widehat{\Gamma}_0^0 = \widehat{\Gamma}_1^0 = \widehat{G}_1^0 = \varnothing.$$

Step 1.3. We put

$$\widehat{\mathcal{C}}_1 = \{\widehat{p}_i \ ; \ \widehat{r}_i \neq 0 \text{ and } \widehat{r}_i^j \in \widehat{G}_0^0 \text{ for all } j = 1, \ldots, k_i\},$$

$$\widehat{\mathcal{P}}_1 := \widehat{G}_0^1 := \widehat{G}_0^0 \cup \widehat{\mathcal{C}}_1 \text{ and } \quad \widehat{\Gamma}_0^1 = \widehat{\Gamma}_1^1 = \varnothing$$

$$\widehat{G}_1^1 = \{\widehat{r}_i^j \to \widehat{p}_i \ ; \ \text{ for all } \widehat{p}_i \in \widehat{\mathcal{C}}_1 \text{ and all } j = 1, \ldots, k_i\}.$$

Step 1.4. Assume that, for $i = 0, \ldots, m-1$, $m \geqslant 2$, data $\widehat{G}^i$, $\widehat{\Gamma}^i$, $\widehat{\mathcal{P}}_i$ are constructed. We set

$$\widehat{\mathcal{P}}'_m = \{\widehat{p}_i \in \widehat{\mathcal{P}} \setminus \widehat{\mathcal{P}}_{m-1} \ ; \ \widehat{r}_i \neq 0 \text{ and } \widehat{r}_i^j \in \widehat{G}_0^{m-1} \text{ for all } j = 1, \ldots, k_i\}$$

and

$$\widehat{\mathcal{P}}_m = \widehat{\mathcal{P}}'_m \cup \widehat{\mathcal{P}}_{m-1}.$$

We define

$$\widehat{\mathcal{C}}_m = \widehat{\mathcal{P}}'_m \cup \{z = -x + \sum_{x \to y} y \ ; \ y \in \widehat{\mathcal{C}}_{m-1}\},$$

$$\widehat{G}_0^m = \widehat{G}_0^{m-1} \cup \widehat{\mathcal{C}}_m$$

and

$$\widehat{G}_1^m = \{\widehat{r}_i^j \to \widehat{p}_i \ ; \ \text{ for all } \widehat{p}_i \in \widehat{\mathcal{C}}_m \text{ and all } j = 1, \ldots, k_i\} \cup$$

$$\cup \{y \to z \ ; \ \text{ for all } y \text{ such that } z = -x + \sum_{x \to y} y\}.$$

Moreover, if $\widehat{\mathcal{P}}_m \neq \widehat{\mathcal{P}}$, $z = -x + \sum_{x \to y} y$ and $x \in \widehat{L}[i]$, then we add z at the end of the list $\widehat{L}[i]$ and delete the first element of the list $\widehat{L}[i]$. If $\widehat{\mathcal{P}}_m \neq \widehat{\mathcal{P}}$, then we set $\widehat{\Gamma}_0^m = \widehat{\Gamma}_1^m = \varnothing$; otherwise we set

$$\widehat{\Gamma}_0^m = \widehat{\Gamma}_0^{m-1} \cup \widehat{\mathcal{C}}_m$$

and

$$\widehat{\Gamma}_1^m = \widehat{\Gamma}_1^{m-1} \cup \{y \to z \, ; \text{for all}\, y \to z \in \widehat{G}_1^m \, \text{such that } y, z \in \widehat{\Gamma}_0^{m-1} \cup \widehat{\Gamma}_0^m\}.$$

Moreover, if $\widehat{\mathcal{P}}_m = \widehat{\mathcal{P}}$, $z = -x + \sum_{x \to y} y$ and $x \in \widehat{L}[i]$, then we add z at the end of the list $\widehat{L}[i]$.

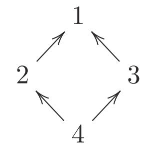
Step 2. If $m = k$, we finish and set $\widehat{\Gamma} = \widehat{\Gamma}^k$.

Remark 3.2. In this algorithm the set $\widehat{\mathcal{P}}$ of Tits-projective and the set $\widehat{\mathrm{Rad}}$ of Tits-radical can be replaced by the set $\overline{\mathcal{P}}$ of Euler-projective vectors and the set $\overline{\mathrm{Rad}}$ of Euler-radical vectors, respectively, i.e. as an input we put $(I, \overline{\mathcal{P}}, \overline{\mathrm{Rad}}, \overline{\mathrm{Rad}}_{comp}, k)$. In this way, we obtain an algorithm that for a positive poset I constructs a $\overline{\Phi}_I$-mesh root system structure $\Gamma(\overline{\mathcal{R}}_I, \overline{\Phi}_I)$, see Theorem 4.4.

In the description of Algorithm 3.1 with input $(I, \overline{\mathcal{P}}, \overline{\mathrm{Rad}}, \overline{\mathrm{Rad}}_{comp}, k)$ the data computed in Step 1 we denote by adding a dash over a corresponding symbol (e.g. we replace $\widehat{L}[i]$ by $\overline{L}[i]$, $\widehat{\Gamma}^k$ by $\overline{\Gamma}^k$ etc.).

We illustrate Algorithm 3.1 by the following example.

Example 3.3. Consider the following poset

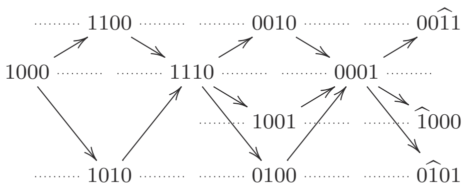


Note that

$$\widehat{\mathcal{P}} = \{\widehat{p}_1 = (1,0,0,0), \widehat{p}_2 = (1,1,0,0), \widehat{p}_3 = (1,0,1,0), \widehat{p}_4 = (1,0,0,1)\},$$

$$\widehat{\mathrm{Rad}} = \{\widehat{r}_2 = (1,0,0,0), \widehat{r}_3 = (1,0,0,0), \widehat{r}_4 = (1,1,1,0)\}$$

and $\widehat{\mathrm{Rad}}_{comp} = \{\widehat{r}_2^1 = \widehat{r}_2, \widehat{r}_3^1 = \widehat{r}_3, \widehat{r}_4^1 = \widehat{r}_4\}$. We set $k = 5$. Applying Algorithm 3.1 to $(I, \widehat{\mathcal{P}}, \widehat{\mathrm{Rad}}, \widehat{\mathrm{Rad}}_{comp}, k)$ we get



Indeed:

m=0: $\widehat{\mathcal{P}}_0 = \widehat{G}_0^0 = \{\widehat{p}_1 = (1,0,0,0)\}$; $\widehat{\Gamma}_0^0 = \widehat{\Gamma}_1^0 = \widehat{G}_1^0 = \varnothing$; $\widehat{L}[1] = [\widehat{p}_1]$, $\widehat{L}[2] = [\widehat{p}_2]$, $\widehat{L}[3] = [\widehat{p}_3]$, $\widehat{L}[4] = [\widehat{p}_4]$.

m=1: $\widehat{\mathcal{C}}_1 = \{\widehat{p}_2 = (1,1,0,0),\ \widehat{p}_3 = (1,0,1,0)\}$, $\widehat{\mathcal{P}}_1 = \widehat{G}_0^1 = \{\widehat{p}_1, \widehat{p}_2, \widehat{p}_3\}$, $\widehat{G}_1^1 = \{(\widehat{p}_1, \widehat{p}_2), (\widehat{p}_1, \widehat{p}_3)\}$, $\widehat{\Gamma}_0^1 = \widehat{\Gamma}_1^1 = \varnothing$. $\widehat{L}[1] = [\widehat{p}_1]$, $\widehat{L}[2] = [\widehat{p}_2]$, $\widehat{L}[3] = [\widehat{p}_3]$, $\widehat{L}[4] = [\widehat{p}_4]$.

m=2: $\widehat{\mathcal{P}}_2' = \varnothing$, $\widehat{\mathcal{P}}_2 = \widehat{\mathcal{P}}_1$, $\widehat{\mathcal{C}}_2 = \{\widehat{p}_2 + \widehat{p}_3 - \widehat{p}_1 = (1,1,1,0)\}$, $\widehat{G}_0^2 = \widehat{G}_0^1 \cup \widehat{\mathcal{C}}_2$, $\widehat{G}_1^2 = \{(\widehat{p}_2, (1,1,1,0)), (\widehat{p}_3, (1,1,1,0))\}$, $\widehat{\Gamma}_0^2 = \widehat{\Gamma}_1^2 = \varnothing$. $\widehat{L}[1] = [(1,1,1,0)]$, $\widehat{L}[2] = [\widehat{p}_2]$, $\widehat{L}[3] = [\widehat{p}_3]$, $\widehat{L}[4] = [\widehat{p}_4]$.

m=3: $\widehat{\mathcal{P}}'_3 = \{\widehat{p}_4 = (1,0,0,1)\}$, $\widehat{\mathcal{P}}_3 = \{\widehat{p}_1, \widehat{p}_2, \widehat{p}_3, \widehat{p}_4\}$, $\widehat{\mathcal{C}}_3 = \{\widehat{p}_4, (0,1,0,0), (0,0,1,0)\}$, $\widehat{G}^3_0 = \widehat{G}^2_0 \cup \widehat{\mathcal{C}}_3$, $\widehat{G}^3_1 = \{((1,1,1,0), \widehat{p}_4), ((1,1,1,0),(0,1,0,0)), ((1,1,1,0),(0,0,1,0))\}$, $\widehat{\Gamma}^3_0 = \widehat{\mathcal{C}}_3$, $\widehat{\Gamma}^3_1 = \varnothing$. $\widehat{L}[1] = [(1,1,1,0)]$, $\widehat{L}[2] = [(0,0,1,0)]$, $\widehat{L}[3] = [(0,1,0,0)]$, $\widehat{L}[4] = [\widehat{p}_4]$.

m=4: $\widehat{\mathcal{P}}'_4 = \varnothing$, $\widehat{\mathcal{P}}_4 = \widehat{\mathcal{P}}$, $\widehat{\mathcal{C}}_4 = \{(0,0,0,1)\}$, $\widehat{G}^4_0 = \widehat{G}^3_0 \cup \widehat{\mathcal{C}}_3$, $\widehat{G}^4_1 = \{(\widehat{p}_4, (0,0,0,1)), ((0,1,0,0),(0,0,0,1)), ((0,0,1,0),(0,0,0,1))\}$, $\widehat{\Gamma}^4_0 = \widehat{\Gamma}^3_0 \cup \widehat{\mathcal{C}}_3$, $\widehat{\Gamma}^4_1 = \widehat{G}^4_1$. $\widehat{L}[1] = [(1,1,1,0),(0,0,0,1)]$, $\widehat{L}[2] = [(0,0,1,0)]$, $\widehat{L}[3] = [(0,1,0,0)]$, $\widehat{L}[4] = [\widehat{p}_4]$.

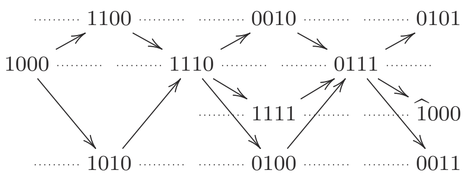
m=5: $\widehat{\mathcal{P}}'_5 = \varnothing$, $\widehat{\mathcal{P}}_5 = \widehat{\mathcal{P}}$, $\widehat{\mathcal{C}}_5 = \{(0,0,\widehat{1},1), (\widehat{1},0,0,0), (0,\widehat{1},0,1)\}$, $\widehat{G}^5_0 = \widehat{G}^4_0 \cup \widehat{\mathcal{C}}_4$, $\widehat{G}^5_1 = \{((0,0,0,1),(0,0,\widehat{1},1)), ((0,0,0,1),(\widehat{1},0,0,0)), ((0,0,0,1),(0,\widehat{1},0,1))\}$, $\widehat{\Gamma}^5_0 = \widehat{\Gamma}^4_0 \cup \widehat{\mathcal{C}}_4$, $\widehat{\Gamma}^5_1 = \widehat{\Gamma}^4_1 \cup \widehat{G}^5_1$. $\widehat{L}[1] = [(1,1,1,0),(0,0,0,1)]$, $\widehat{L}[2] = [(0,0,1,0),(0,0,\widehat{1},1)]$, $\widehat{L}[3] = [(0,1,0,0),(0,\widehat{1},0,1)]$, $\widehat{L}[4] = [\widehat{p}_4, (\widehat{1},0,0,0)]$.

Now we apply Algorithm 3.1 to $(I, \overline{\mathcal{P}}, \overline{\mathrm{Rad}}, \overline{\mathrm{Rad}}_{comp}, k)$. Note that

$$\overline{\mathcal{P}} = \{\overline{p}_1 = (1,0,0,0), \overline{p}_2 = (1,1,0,0), \overline{p}_3 = (1,0,1,0), \overline{p}_4 = (1,1,1,1)\},$$

$$\overline{\mathrm{Rad}} = \{\overline{r}_1 = (0,0,0,0), \overline{r}_2 = (1,0,0,0), \overline{r}_3 = (1,0,0,0), \overline{r}_4 = (1,1,1,0)\},$$

and $\overline{\mathrm{Rad}}_{comp} = \{\overline{r}^1_1 = \overline{r}_1, \overline{r}^1_2 = \overline{r}_2, \overline{r}^1_3 = \overline{r}_3, \overline{r}^1_4 = \overline{r}_4\}$. We set $k = 5$ and get



Indeed:

m=0: $\overline{\mathcal{P}}_0 = \overline{G}^0_0 = \{\overline{p}_1 = (1,0,0,0)\}$; $\overline{\Gamma}^0_0 = \overline{\Gamma}^0_1 = \overline{G}^0_1 = \varnothing$; $\overline{L}[1] = [\overline{p}_1]$, $\overline{L}[2] = [\overline{p}_2]$, $\overline{L}[3] = [\overline{p}_3]$, $\overline{L}[4] = [\overline{p}_4]$.

m=1: $\overline{\mathcal{C}}_1 = \{\overline{p}_2 = (1,1,0,0), \overline{p}_3 = (1,0,1,0)\}$, $\overline{\mathcal{P}}_1 = \overline{G}^1_0 = \{\overline{p}_1, \overline{p}_2, \overline{p}_3\}$, $\overline{G}^1_1 = \{(\overline{p}_1, \overline{p}_2), (\overline{p}_1, \overline{p}_3)\}$, $\overline{\Gamma}^1_0 = \overline{\Gamma}^1_1 = \varnothing$. $\overline{L}[1] = [\overline{p}_1]$, $\overline{L}[2] = [\overline{p}_2]$, $\overline{L}[3] = [\overline{p}_3]$, $\overline{L}[4] = [\overline{p}_4]$.

m=2: $\overline{\mathcal{P}}'_2 = \varnothing$, $\overline{\mathcal{P}}_2 = \overline{\mathcal{P}}_1$, $\overline{\mathcal{C}}_2 = \{\overline{p}_2 + \overline{p}_3 - \overline{p}_1 = (1,1,1,0)\}$, $\overline{G}^2_0 = \overline{G}^1_0 \cup \overline{\mathcal{C}}_2$, $\overline{G}^2_1 = \{(\overline{p}_2, (1,1,1,0)), (\overline{p}_3, (1,1,1,0))\}$, $\overline{\Gamma}^2_0 = \overline{\Gamma}^2_1 = \varnothing$. $\overline{L}[1] = [(1,1,1,0)]$, $\overline{L}[2] = [\overline{p}_2]$, $\overline{L}[3] = [\overline{p}_3]$, $\overline{L}[4] = [\overline{p}_4]$.

m=3: $\overline{\mathcal{P}}'_3 = \{\overline{p}_4 = (1,1,1,1)\}$, $\overline{\mathcal{P}}_3 = \{\overline{p}_1, \overline{p}_2, \overline{p}_3, \overline{p}_4\}$, $\overline{\mathcal{C}}_3 = \{\overline{p}_4, (0,1,0,0), (0,0,1,0)\}$, $\overline{G}^3_0 = \overline{G}^2_0 \cup \overline{\mathcal{C}}_3$, $\overline{G}^3_1 = \{((1,1,1,0), \overline{p}_4), ((1,1,1,0),(0,1,0,0)), ((1,1,1,0),(0,0,1,0))\}$, $\overline{\Gamma}^3_0 = \overline{\mathcal{C}}_3$, $\overline{\Gamma}^3_1 = \varnothing$. $\overline{L}[1] = [(1,1,1,0)]$, $\overline{L}[2] = [(0,0,1,0)]$, $\overline{L}[3] = [(0,1,0,0)]$, $\overline{L}[4] = [\overline{p}_4]$.

m=4: $\overline{\mathcal{P}}'_4 = \varnothing$, $\overline{\mathcal{P}}_4 = \overline{\mathcal{P}}$, $\overline{\mathcal{C}}_4 = \{(0,1,1,1)\}$, $\overline{G}_0^4 = \overline{G}_0^3 \cup \overline{\mathcal{C}}_3$, $\overline{G}_1^4 = \{(\overline{p}_4,(0,1,1,1)),((0,1,0,0),(0,1,1,1)),((0,0,1,0),(0,1,1,1))\}$, $\overline{\Gamma}_0^4 = \overline{\Gamma}_0^3 \cup \overline{\mathcal{C}}_3$, $\overline{\Gamma}_1^4 = \overline{G}_1^4$. $\overline{L}[1] = [(1,1,1,0),(0,1,1,1)]$, $\overline{L}[2] = [(0,0,1,0)]$, $\overline{L}[3] = [(0,1,0,0)]$, $\overline{L}[4] = [\overline{p}_4]$.

m=5: $\overline{\mathcal{P}}'_5 = \varnothing$, $\overline{\mathcal{P}}_5 = \overline{\mathcal{P}}$, $\overline{\mathcal{C}}_5 = \{(0,1,0,1),(\widehat{1},0,0,0),(0,0,1,1)\}$, $\overline{G}_0^5 = \overline{G}_0^4 \cup \overline{\mathcal{C}}_4$, $\overline{G}_1^5 = \{((0,1,1,1),(0,1,0,1)),((0,1,1,1),(\widehat{1},0,0,0)),((0,1,1,1),(0,0,1,1))\}$, $\overline{\Gamma}_0^5 = \overline{\Gamma}_0^4 \cup \overline{\mathcal{C}}_4$, $\overline{\Gamma}_1^5 = \overline{\Gamma}_1^4 \cup \overline{G}_1^5$.
$\overline{L}[1] = [(1,1,1,0),(0,1,1,1)]$, $\overline{L}[2] = [(0,0,1,0),(0,1,0,1)]$,
$\overline{L}[3] = [(0,1,0,0),(0,0,1,1)]$, $\overline{L}[4] = [\overline{p}_4,(\widehat{1},0,0,0)]$.

4. Correctness of Algorithm 3.1

Following [28, 29], we define a group isomorphism

$$\sigma_I^0 : \mathbb{Z}^I \to \mathbb{Z}^I \tag{4.1}$$

by the formula $\sigma_I^0(x) = x \cdot C_I^0$, where C_I^0 is the reduced incidence matrix

$$C_I^0 = \left[\begin{array}{c|c} C_{I^-} & 0 \\ \hline 0 & E \end{array}\right], \tag{4.2}$$

where E is the identity matrix. By [29, Proposition 3.13], σ_I^0 gives $\mathbb{Z}$-equivalence of $\widehat{q}_I$ and $\overline{q}_I$, i.e. $\overline{q}_I(\sigma_I^0(x)) = \widehat{q}(x)$.

Lemma 4.3. *For any poset I and for all $i \in I$, we have $(\sigma_I^0)^{-1}(\overline{p}_i) = \widehat{p}_i$ and $(\sigma_I^0)^{-1}(\overline{r}_i) = \widehat{r}_i$, where $\widehat{p}_i$, $\widehat{r}_i$, $\overline{p}_i$ and $\overline{r}_i$ are projective and radical vectors defined in Section 2.1, and $\sigma_I^0 : \mathbb{Z}^I \to \mathbb{Z}^I$ is the isomorphism* (4.1).

Proof. The proof is straightforward. □

Theorem 4.4. *Assume that I is a connected positive poset. Let $\widehat{\Gamma}$ be the quiver constructed by Algorithm 3.1 with input $(I, \widehat{\mathcal{P}}, \widehat{\mathrm{Rad}}, \widehat{\mathrm{Rad}}_{comp}, k)$ with k large enough (e.g. $k = |\widehat{\mathcal{R}}_I|$). The following conditions are satisfied.*

(a) $\widehat{\mathcal{P}} = \bigcup_k \widehat{\mathcal{P}}_k$, in particular there exists m such that $\widehat{\mathcal{P}}_m = \widehat{\mathcal{P}}$.

(b) The sequence $\widehat{\Gamma}^0 \subseteq \widehat{\Gamma}^1 \subseteq \ldots$ stabilizes.

(c) $\widehat{\mathcal{R}}_I = \bigcup_m \widehat{\Gamma}_0^m$.

(d) $\widehat{L}[1], \ldots, \widehat{L}[n]$ are the $\widehat{\Phi}_I$-orbits in $\widehat{\mathcal{R}}_I$ of the Coxeter-Tits transformation $\widehat{\Phi}_I$.

(e) The $\widehat{\Phi}_I$-mesh translation quiver $\widehat{\Gamma} = \Gamma(\widehat{\mathcal{R}}_I, \widehat{\Phi}_I)$ defines a $\widehat{\Phi}_I$-mesh root system structure $\Gamma(\widehat{\mathcal{R}}_I, \widehat{\Phi}_I)$ on the set $\widehat{\mathcal{R}}_I$ of Tits roots of I.

Proof. Assume that I is a connected positive poset. We apply Algorithm 3.1 to the system $(I, \overline{\mathcal{P}}, \overline{\text{Rad}}, \overline{\text{Rad}}_{comp}, k)$ defined in Remark 3.2. The formula (2.2) implies that the Euler matrix $\overline{C}_I = C_I^{-1}$ satisfies the non-cycle condition defined in [14], see Remark 2.3. Therefore, [14, Theorem 4.13] and [14, Section 5] yield:

($\bar{\text{a}}$) $\overline{\mathcal{P}} = \bigcup_k \overline{\mathcal{P}}_k$, in particular there exists m such that $\overline{\mathcal{P}}_m = \overline{\mathcal{P}}$.

($\bar{\text{b}}$) The sequence $\overline{\Gamma}^0 \subseteq \overline{\Gamma}^1 \subseteq \ldots$ stabilizes.

($\bar{\text{c}}$) $\overline{\mathcal{R}}_I = \bigcup_m \overline{\Gamma}_0^m$.

($\bar{\text{d}}$) $\overline{L}[1], \ldots, \overline{L}[n]$ are the $\overline{\Phi}_I$-orbits in $\overline{\mathcal{R}}_I$ of the Coxeter transformation $\overline{\Phi}_I$.

($\bar{\text{e}}$) The $\overline{\Phi}_I$-mesh translation quiver $\overline{\Gamma} = \Gamma(\overline{\mathcal{R}}_I, \overline{\Phi}_I)$ defines a $\overline{\Phi}_I$-mesh root system structure $\Gamma(\overline{\mathcal{R}}_I, \overline{\Phi}_I)$ on the set $\overline{\mathcal{R}}_I$ of Euler roots of I.

By Lemma 4.3, we have $(\sigma_I^0)^{-1}(\overline{p}_i) = \widehat{p}_i$ and $(\sigma_I^0)^{-1}(\overline{r}_i) = \widehat{r}_i$, see (4.1). It is easy to verify that the automorphism $(\sigma_i^0)^{-1}$ sends $\overline{\mathcal{P}}, \overline{\text{Rad}}, \overline{\text{Rad}}_{comp}$ to $\widehat{\mathcal{P}}, \widehat{\text{Rad}}, \widehat{\text{Rad}}_{comp}$, respectively. From [29, Proposition 3.13], it follows that $\widehat{\Phi}_I = (\sigma_I^0)^{-1} \circ \overline{\Phi}_I \circ \sigma_I^0$. Now, applying the linearity of σ_I^0, it is easy to deduce that the conditions ($\bar{\text{a}}$)-($\bar{\text{e}}$) imply the conditions (a)-(e), and the theorem follows. □

Remark 4.5. It follows from the proof of Theorem 4.4 that the $\widehat{\Phi}_I$-mesh quiver $\Gamma(\mathcal{R}_I, \widehat{\Phi}_I)$ is the image of $\Gamma(\overline{\mathcal{R}}_I, \overline{\Phi}_I)$ via the automorphism $(\sigma_I^0)^{-1} : \mathbb{Z}^I \to \mathbb{Z}^I$ (4.1).

We refer also to [11,12] for a discussion of Φ_I-mesh quivers of one-peak posets.

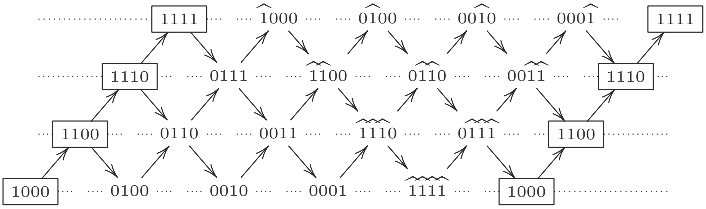
Corollary 4.6. *Let I be a positive connected poset and let DI be the Coxeter-Dynkin type of the root system $\widehat{\mathcal{R}}_I$. The Coxeter polynomial* $\text{cox}_I(t)$ *is equal to the Coxeter polynomial* $\text{cox}_{DI}(t)$ *of the Dynkin diagram DI and the Coxeter number* $\mathbf{c}_I$ *is equal to the Coxeter number* c_{DI} *of the Dynkin diagram DI; they are listed in* [29, Example 3.12].

Proof. By [14, Theorem 1.10] there exists a $\mathbb{Z}$-invertible matrix $B \in \mathbb{M}_I(\mathbb{Z})$ such that $\text{Cox}_{DI} = B \cdot \overline{\text{Cox}}_I \cdot B^{-1}$, where Cox_{DI} is the Coxeter matrix associated with the simply laced Dynkin diagram DI. Moreover by [29, Proposition 3.13], we have $\widehat{\text{Cox}}_I = C_I^0 \cdot \overline{\text{Cox}}_I \cdot (C_I^0)^{-1}$. Now it is easy to deduce that $\text{cox}_I(t) = \text{cox}_{DI}(t)$ and $\mathbf{c}_I = \mathbf{c}_{DI}$. □

Example 4.7. Consider the poset I given by the Hasse quiver

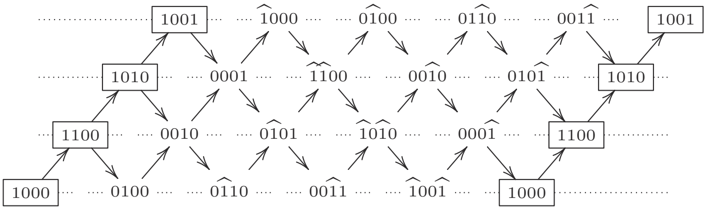
$$1 \longleftarrow 2 \longleftarrow 3 \longleftarrow 4 \tag{4.8}$$

By applying Algorithm 3.1 to $(I, \overline{\mathcal{P}}, \overline{\mathrm{Rad}}, \overline{\mathrm{Rad}}_{comp}, k = 6)$ we get the $\widehat{\Phi}_I$-mesh quiver $\Gamma(\overline{\mathcal{R}}_I, \overline{\Phi}_I)$:



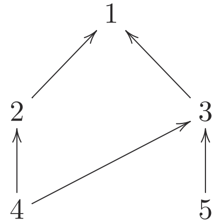
where vectors in frames lying in the same orbit are identified.

Moreover, by applying Algorithm 3.1 to $(I, \widehat{\mathcal{P}}, \widehat{\mathrm{Rad}}, \widehat{\mathrm{Rad}}_{comp}, k = 6)$ we get the $\widehat{\Phi}_I$-mesh quiver $\Gamma(\widehat{\mathcal{R}}_I, \widehat{\Phi}_I)$:



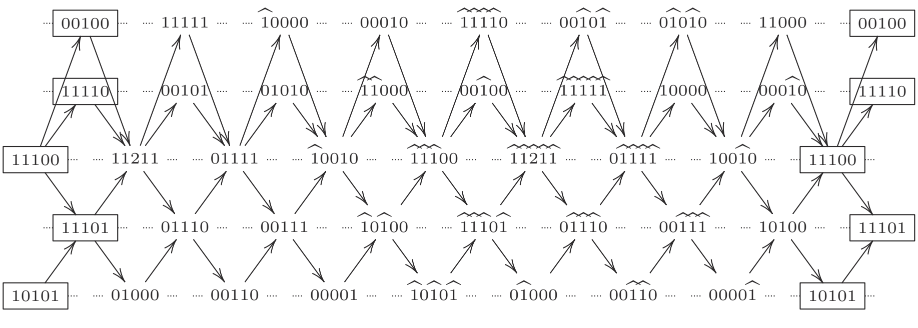
Note that the $\widehat{\Phi}_I$-mesh quiver $\Gamma(\overline{\mathcal{R}}_I, \overline{\Phi}_I)$ is isomorphic with the $\widehat{\Phi}_I$-mesh quiver $\Gamma(\widehat{\mathcal{R}}_I, \widehat{\Phi}_I)$ via the authomorphism $\sigma_I^0 : \mathbb{Z}^4 \to \mathbb{Z}^4$ (4.1).

Example 4.9. Consider the poset I given by the Hasse quiver

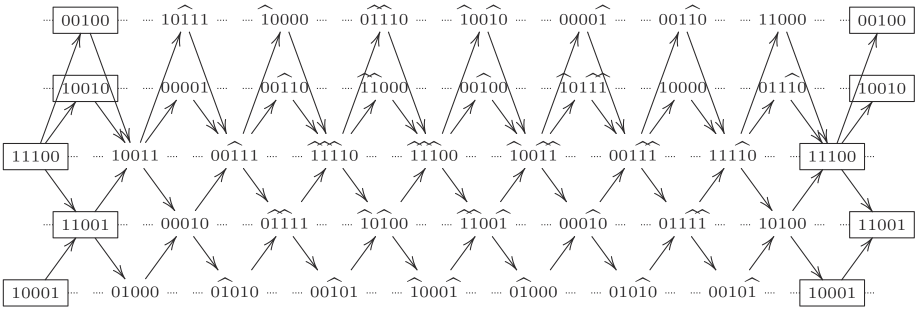


(4.10)

By applying Algorithm 3.1 we get the $\widehat{\Phi}_I$-mesh quiver $\Gamma(\overline{\mathcal{R}}_I, \overline{\Phi}_I)$:

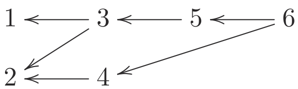


and the $\widehat{\Phi}_I$-mesh quiver $\Gamma(\widehat{\mathcal{R}}_I, \widehat{\Phi}_I)$:

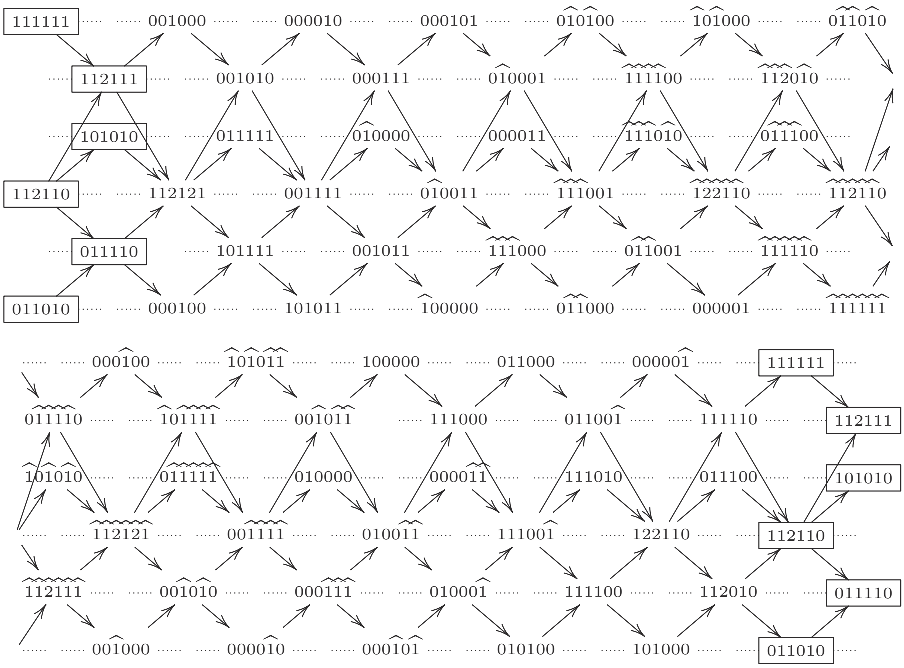


Note that the $\widehat{\Phi}_I$-mesh quiver $\Gamma(\overline{\mathcal{R}}_I, \overline{\Phi}_I)$ is isomorphic with the $\widehat{\Phi}_I$-mesh quiver $\Gamma(\widehat{\mathcal{R}}_I, \widehat{\Phi}_I)$ via the authomorphism $\sigma_I^0 : \mathbb{Z}^5 \to \mathbb{Z}^5$ (4.1).

Example 4.11. Consider the poset I given by the Hasse quiver

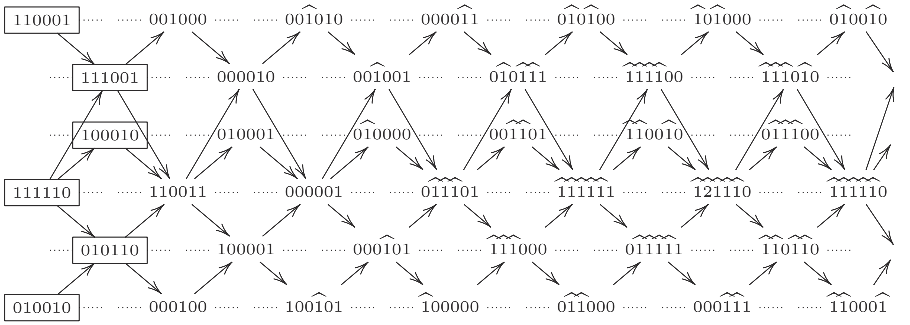


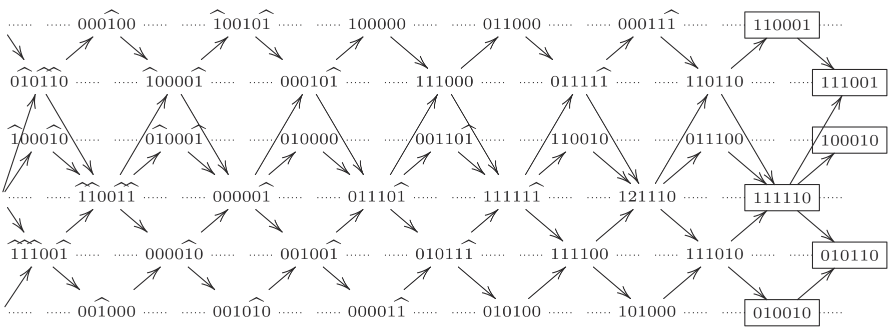
By applying Algorithm 3.1 to $(I, \overline{\mathcal{P}}, \overline{\mathrm{Rad}}, \overline{\mathrm{Rad}}_{comp}, k = 24)$ we get the $\widehat{\Phi}_I$-mesh quiver $\Gamma(\overline{\mathcal{R}}_I, \overline{\Phi}_I)$:



where vectors in frames lying in the same orbit are identified.

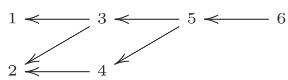
Moreover, by applying Algorithm 3.1 to $(I, \widehat{\mathcal{P}}, \widehat{\mathrm{Rad}}, \widehat{\mathrm{Rad}}_{comp}, k = 24)$ we get the $\widehat{\Phi}_I$-mesh quiver $\Gamma(\widehat{\mathcal{R}}_I, \widehat{\Phi}_I)$:



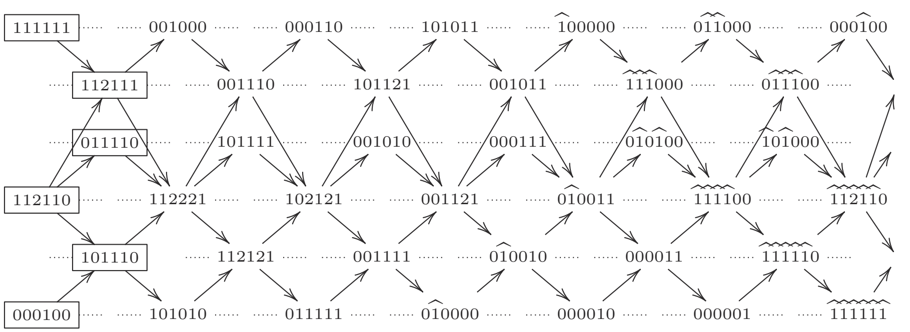


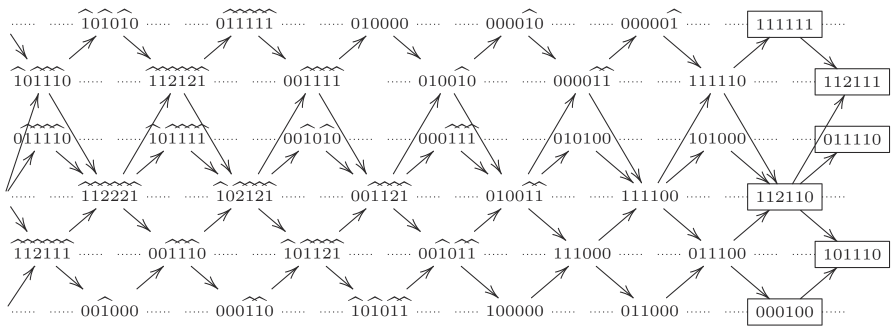
Note that the $\widehat{\Phi}_I$-mesh quiver $\Gamma(\overline{\mathcal{R}}_I, \overline{\Phi}_I)$ is isomorphic with the $\widehat{\Phi}_I$-mesh quiver $\Gamma(\widehat{\mathcal{R}}_I, \widehat{\Phi}_I)$ via the authomorphism $\sigma_I^0 : \mathbb{Z}^6 \to \mathbb{Z}^6$ (4.1).

Example 4.12. Consider the poset I given by the Hasse quiver



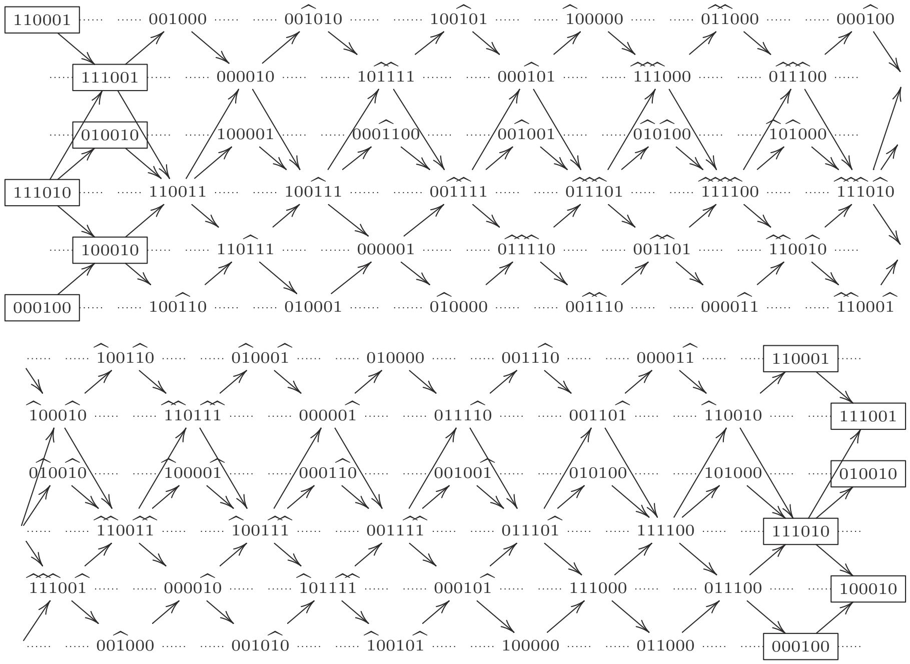
By applying Algorithm 3.1 to $(I, \overline{\mathcal{P}}, \overline{\mathrm{Rad}}, \overline{\mathrm{Rad}}_{comp}, k = 24)$ we get the$\widehat{\Phi}_I$-mesh quiver $\Gamma(\overline{\mathcal{R}}_I, \overline{\Phi}_I)$:





where vectors in frames lying in the same orbit are identified.

Moreover, by applying Algorithm 3.1 to $(I, \widehat{\mathcal{P}}, \widehat{\mathrm{Rad}}, \widehat{\mathrm{Rad}}_{comp}, k = 24)$ we get the $\widehat{\Phi}_I$-mesh quiver $\Gamma(\widehat{\mathcal{R}}_I, \widehat{\Phi}_I)$:



Note that the $\widehat{\Phi}_I$-mesh quiver $\Gamma(\overline{\mathcal{R}}_I, \overline{\Phi}_I)$ is isomorphic with the $\widehat{\Phi}_I$-mesh quiver $\Gamma(\widehat{\mathcal{R}}_I, \widehat{\Phi}_I)$ via the authomorphism $\sigma_I^0 : \mathbb{Z}^6 \to \mathbb{Z}^6$ (4.1).

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