

# Galois orders of symmetric differential operators

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*Dedicated to the memory of Sergey Ovsienko*

**ABSTRACT.** In this survey we discuss the theory of Galois rings and orders developed in ([20], [22]) by Sergey Ovsienko and the first author. This concept allows to unify the representation theories of Generalized Weyl Algebras ([4]) and of the universal enveloping algebras of Lie algebras. It also had an impact on the structure theory of algebras. In particular, this abstract framework has provided a new proof of the Gelfand-Kirillov Conjecture ([24]) in the classical and the quantum case for  $gl_n$  and  $sl_n$  in [18] and [21], respectively. We will give a detailed proof of the Gelfand-Kirillov Conjecture in the classical case and show that the algebra of symmetric differential operators has a structure of a Galois order.

## 1. Motivation

Throughout the paper  $\mathbf{k}$  will denote an algebraically closed field of zero characteristic. All considered rings are algebras over  $\mathbf{k}$ . In representation theory one often considers the following question: given an embedding of algebras  $\Gamma \subseteq U$ , relate representations of  $U$  and  $\Gamma$ . The functors of restriction and induction are very powerful tools in this study. In particular, in the representation theory of Lie algebras a concept of a Harish-Chandra module relates the universal enveloping algebra of

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\*Supported in part by CNPq grant (301320/2013-6) and by Fapesp grant (2014/09310-5)

\*\*Supported in part by Fapesp grant (2014/25612-1)

**2010 MSC:** 13N10, 16D30, 16S32, 16S85.

**Key words and phrases:** Weyl algebra, invariant differential operators, Galois order, field of fractions.

a reductive Lie algebra  $U$  and the universal enveloping algebra of its reductive subalgebra  $\Gamma$  [9]. On the other hand, when  $\Gamma$  is the universal enveloping algebra of a Cartan subalgebra, one obtains the so called generalized weight representations. The classification of irreducible weight modules whose weight spaces are finite dimensional was done in [13] and [31]. The problem remains open in general. To approach this problem, Drozd, Futorny and Ovsienko introduced the category of Gelfand-Tsetlin modules over  $U(\mathfrak{gl}_n)$  with respect to the Gelfand-Tsetlin subalgebra (a certain maximal commutative subalgebra) ([10]). This approach was inspired by a remarkable paper of Gelfand and Tsetlin ([24]) which gave a construction of irreducible representations of  $\mathfrak{gl}_n$  using as a basis a combinatorial object — Gelfand-Tsetlin tableaux, very much in the spirit of the representation theory of the symmetric groups [26]. A similar idea was used by Okunkov and Vershik in [37]. Using the natural embedding of  $S_{m-1}$  in  $S_m$  one introduces a subalgebra analogous to the Gelfand-Tsetlin subalgebra in the  $\mathfrak{gl}_n$  case. Namely, in this case  $U$  is  $\mathbf{k}S_n$  and  $\Gamma$  is the maximal commutative subalgebra generated by the Jucys-Murphy elements:

$$(1i) + \dots + (i-1i) i = 1, \dots, n.$$

Then  $\text{Specm } \Gamma$  parametrizes the irreducible representations of  $S_n$ , and the Young tableaux can be recovered.

For an account of the recent research in this area, including generalizations for Lie algebras of types B, C and D, see [34]. An excellent exposition of the classical material can be found in [41].

To understand better the phenomena of the Gelfand-Tsetlin formulas, the notion of an abstract Harish-Chandra subalgebra and Harish-Chandra module were introduced for an arbitrary associative algebras in [11]. In [20] it was noticed that using the Gelfand-Tsetlin formulas one can embed  $U(\mathfrak{gl}_n)$  into the skew group ring over a field  $L$  (a similar construction was also done by Khomenko [28]), where  $L$  is a finite Galois extension of the field of fractions of the Gelfand-Tsetlin subalgebra.

The appearance of skew group rings is also a phenomenon in the representation theory of another class of algebras - the Generalized Weyl Algebras [4]. In particular cases of the first Weyl Algebra and  $U(\mathfrak{sl}_2)$  (see [6]), and their quantum analogues, it is known that their irreducible modules are completely described modulo a classification of irreducible elements in certain skew polynomial rings in one variable over a skew field.

The main motivation of the development of this theory was an evolution of the ideas in [11] in the “semi-commutative” case for a pair of an associative algebra and its commutative subalgebra, and understanding of the role of skew group rings in the representation theory of infinite dimensional algebras. A key concept introduced in [20] is a notion of a noncommutative Galois order for skew monoid rings (cf. [33], Chapter 5). Known examples of Galois algebras include:

- Generalized Weyl algebras over integral domains with infinite order automorphisms, which include algebras, such as the  $n$ -th Weyl algebra  $A_n$ , the quantum plane, the  $q$ -deformed Heisenberg algebra, quantized Weyl algebras, the Witten—Woronowicz algebra among the others;
- The universal enveloping algebra  $U(gl_n)$  with respect to its Gelfand—Tsetlin subalgebra.
- It was shown in [16], [19] that shifted Yangians and finite  $W$ -algebras associated with  $gl_n$  are Galois orders with respect to the corresponding Gelfand—Tsetlin subalgebras;
- Certain invariant rings on the differential operators on the torus [20].

Representation theory of Galois orders was developed in [22]. In the case of  $gl_n$  the Galois order structure of the universal enveloping algebra led to a significant breakthrough in its representation theory in the remarkable paper [38].

## 2. Basic definitions

Let  $R$  be a ring,  $\mathfrak{M}$  a monoid acting on  $R$  by ring automorphisms. Consider the skew monoid ring  $R * \mathfrak{M}$ . Let  $G$  be a finite group acting on  $\mathfrak{M}$  by conjugation:  $g(m) = g(m)g^{-1}$ . We can define an action of  $G$  on  $R * \mathfrak{M}$  as  $g(rm) = g(r)g(m)$ ,  $g \in G, r \in R, m \in \mathfrak{M}$ . We denote the ring of invariants by the action of  $G$  by  $\mathfrak{K}$ .

Any element of  $R * \mathfrak{M}$  can be written in the form  $x = \sum_{m \in \mathfrak{M}} x_m m$ . Define  $\text{supp } x$  as the set of those  $m \in \mathfrak{M}$  for which  $x_m$  is not zero.

From now on we will restrict ourselves to the following case:  $R$  will be a field  $L$ , a finite Galois extension of a field  $K$  such that  $G = \text{Gal}(L, K)$ . The monoid  $\mathfrak{M}$  will be assumed to have the following property: if  $m, m' \in \mathfrak{M}$  and their restrictions to  $K$  coincide, then  $m = m'$ .

**Definition.** A finitely generated  $\Gamma$ -ring  $U$  embedded in  $\mathfrak{K}$  is called a Galois ring over  $\Gamma$  if  $KU = KU = \mathfrak{K}$ .

Note that  $\Gamma$  is not required to be central in  $U$ .

### 3. Structure of Galois rings

In this section we recall the structure theory of Galois rings following [20]. A very useful characterization of Galois rings in the following

**Proposition 1.** [20], Proposition 4.1] *Assume that a  $\Gamma$ -ring  $U \subset \mathfrak{K} = (L * \mathfrak{M})^G$  is generated by  $u_1, \dots, u_k$ . If  $\bigcup_{i=1}^k \text{supp} u_i$  generates  $\mathfrak{M}$  as a monoid then  $U$  is a Galois ring. In particular, if  $LU = L * \mathfrak{M}$  then  $U$  is a Galois ring.*

**Theorem 1** ([20], Theorem 4.1). *Let  $U$  be a Galois ring over  $\Gamma$  in  $\mathfrak{K}$ ,  $S = \Gamma \setminus \{0\}$ . Then*

- $U \cap K$  is a maximal commutative subalgebra in  $U$  and the center of  $U \cap K^{\mathfrak{K}}$ .
- $S$  is a left and right Ore denominator set, and the localization of  $U$  by  $S$  both at the left and the right are isomorphic to  $\mathfrak{K}$ .

**Definition.** A Galois ring is called a right (left) Galois order over  $\Gamma$  if for every right (left) finite dimensional  $K$  vector space  $W \subset \mathfrak{K}$ ,  $W \cap \Gamma$  is a finitely generated right (left)  $\Gamma$ -module. If it is both left and right, we will simply say Galois order.

We have the following characterization of Galois orders.

**Proposition 2.** *Let  $U$  is a Galois ring over  $\Gamma$ .*

- *If  $\Gamma$  is noetherian and  $U$  a left (right) projective  $\Gamma$ -module then  $U$  is a left (right) Galois order.*
- *If  $\Gamma$  is a finitely generated domain over  $\mathbf{k}$  and  $U$  a Galois order over  $\Gamma$  then  $\Gamma$  is a Harish-Chandra algebra in  $U$ .*

### 4. Noncommutative Noether's Problem and the Gelfand-Kirillov Conjecture

In this section we show how the theory of Galois rings can be used to prove the Gelfand-Kirillov Conjecture for  $gl_n$ . The noncommutative version of the the classical Noether's problem will also be required.

**Definition.** Let  $V$  be a finite dimensional vector space, of dimension  $n$  over  $\mathbf{k}$ ,  $G$  a finite subgroup of  $GL(V)$ . It acts on  $S(V^*)$  by  $\mathbf{k}$ -algebra automorphisms:  $g.f(v) = f(g^{-1}v)$ ,  $g \in G, f \in S(V^*), v \in V$ . After fixing a basis of  $V$ ,  $S(V^*)$  can be identified with  $\mathbf{k}[x_1, \dots, x_n]$ , where  $x_1, \dots, x_n$  are the duals of the basis elements in  $V^*$ . Automorphisms of the polynomial algebra arising this way will be called *linear*.

Hence the group  $G$  acts also on the field of rational functions  $K = \mathbf{k}(x_1, \dots, x_n)$  by extension. Then one can ask:

**Noether's Problem** ([36]). If  $G$  is a finite group of linear automorphisms, when  $K^G$  is a purely transcendental extension of  $\mathbf{k}$ ?

The following are some important cases when the Noether's Problem has a positive solution:

- $n = 1, n = 2$  or  $n = 3$  (these are classical results due to Luroth, Castelnuovo and Burnside).
- When  $V$  is a direct sum of one dimensional  $G$ -submodules. In particular, for abelian  $G$  (Theorem of Fischer).
- The action of  $G$  by pseudo-reflections (by the Chevalley-Shephard-Todd Theorem)
- for alternating groups  $A_3, A_4$  and  $A_5$  (by Maeda), permuting variables as usual. The question remains open for  $n > 5$ .

There are also counter-examples to the Noether's Problem, cf. [40], [15].

We will introduce now the Noncommutative Noether's Problem for the Weyl algebra,  $A_n(\mathbf{k})$  with generators  $x_i, \partial_i, i = 1, \dots, n$ , subject to the relations  $x_i x_j = x_j x_i, \partial_i \partial_j = \partial_j \partial_i$  and  $\partial_i x_j - x_j \partial_i = \delta_{ij}$  for all  $i, j$ . Recall that  $A_n(\mathbf{k})$  is a left and right noetherian simple domain which admits a total ring of fractions (skew field),  $F_n(\mathbf{k})$ , called the *Weyl field*. For our purposes it will be useful to identify the Weyl algebra with the ring of differential operators on the polynomial algebra in  $n$  variables.

Let  $A$  be a finitely generated commutative, regular  $\mathbf{k}$ -algebra. Then the ring of differential operators  $D(A)$  on  $A$  is the subalgebra of  $\text{End}_{\mathbf{k}}(A)$  generated by the  $\mathbf{k}$ -linear derivations of  $A$  and the scalar multiplications  $l_a$  that sends  $x \rightarrow ax, \forall a \in A$ . The set of multiplications gives an isomorphism of  $A$  with a subring of  $D(A)$ , allowing  $A$  to be viewed as a subring of it.

For our purposes it will be useful to identify the Weyl algebra with the ring of differential operators on the polynomial algebra in  $n$  variables. Let  $V$  a finite dimensional vector space of dimension  $n$ ,  $G$  be a finite subgroup of  $GL(V)$ . As previously, this induces an action on  $S(V^*) = \mathbf{k}[x_1, \dots, x_n]$ . This action can be extended to the ring of differential operators on  $S(V^*)$ : if  $d$  is such an operator,  $g.d(x) = g(d(g^{-1}x))$ , where  $x \in S(V^*)$ . This induces a  $\mathbf{k}$ -automorphism of the Weyl algebra  $A_n(\mathbf{k})$ . Such  $\mathbf{k}$ -automorphisms will be called linear.

The following Noncommutative Noether's Problem was formulated by Alev and Dumas, [3].

**Noncommutative Noether's Problem.** For a finite group of linear automorphisms  $G$ , when  $F_n(\mathbf{k})^G$  is isomorphic to  $F_n(\mathbf{k})$ ?

Some cases with known positive solution for the Noncommutative Noether's Problem are:

- For  $n = 1$  or  $n = 2$  and arbitrary  $G$  (Alev, Dumas, [3]).
- When  $V$  is a direct sum of one dimensional  $G$ -submodules (Alev, Dumas, [3]).
- When  $G$  acts by pseudo-reflections (Eshmatov, Futorny, Ovsienko, Schwarz, [12]).

Positive solution of the Noncommutative Noether's Problem in the context of the structure theory of Galois rings provides a new proof of the celebrated Gelfand-Kirillov conjecture for the  $gl_n$  and  $sl_n$  cases.

The Gelfand-Kirillov conjecture [24] states that if  $g$  be a finite dimensional algebraic Lie algebra then the skew field of fractions of the universal enveloping algebra  $U(g)$  is isomorphic to a Weyl field over a purely transcendental extension (of finite transcendence degree) of  $\mathbf{k}$ . The important cases with a positive solution are:

- $g = gl_n, sl_n$  or nilpotent [24];
- $g$  is solvable [5], [27], [32];
- $g$  has dimension at most 8 [2].

The first counter-example to this conjecture was found by Alev, Ooms, Van den Bergh in [1]. For simple finite dimensional Lie algebras the question was almost solved by Premet [39]: the conjecture is true for algebras of type  $A$  and  $G_2$ , unknown for type  $C$  and false for all other types.

We are going to present two proofs of the Noncommutative Noether's Problem in the case of the symmetric group. One of them is a simplified version of the proof found in [16] and [12], while the other is elementary — it involves only the Cramer's rule.

Let  $\Delta = (\prod_{i < j} (x_i - x_j))^2$ . It is clearly an  $S_n$ -invariant element and  $F_n(\mathbf{k}) = \text{Frac } A_n(\mathbf{k})_\Delta$ , the skew field of fractions of the localized algebra by  $\Delta$ . In the following we denote the polynomial algebra in  $n$  variables just by  $\Lambda$  for the sake of simplicity. The following holds:

**Proposition 3.**

- Let  $S$  be any multiplicatively closed set in  $\Lambda$ . Then  $D(\Lambda_S) = (D(\Lambda))_S$ .
- $(D(\Lambda)_\Delta)^{S_n} \cong ((D(\Lambda))^{S_n})_\Delta$ .
- $(\Lambda_\Delta)^{S_n} = (\Lambda^{S_n})_\Delta$ .
- $\text{Frac } A_n(\mathbf{k})^{S_n} \cong (\text{Frac } A_n(\mathbf{k}))^{S_n}$ .

*Proof.* The first item follows from Theorem 15.1.25 of [33]. For the second statement note that if  $d \in (D(\Lambda)_\Delta)^{S_n}$ , then  $d_1 = \Delta^k d \in D(\Lambda)^{S_n}$  for some  $k \geq 0$ . The third item is proved similarly. The fourth statement follows from [35], Theorem 5.3(4).  $\square$

Now we need the following crucial lemma:

**Lemma 1.**  $(D(\Lambda_\Delta))^{S_n} = D(\Lambda_\Delta^{S_n})$ .

*Proof.* First we follow [12]. Recall that if  $X$  is a normal irreducible affine variety and  $G$  a finite group of automorphisms that acts freely on  $X$  then  $D(X)^G \cong D(O(X/G))$  (cf. [8]). This applies to  $S_n$  acting on  $\Lambda_\Delta$ , and hence the lemma follows.

Now we show how to obtain this result algebraically. Note that  $S_n$  has no non-trivial inner automorphisms. Therefore,  $A_n(\mathbf{k})_\Delta^{S_n}$  is simple by [35], Corollary 2.6.

Let  $\sigma_i$  be the  $i$ -th symmetrical polynomial in  $x_1, \dots, x_n$ ,  $i = 1, \dots, n$ ,  $\Lambda^{S_n} = \mathbf{k}[\sigma_1, \dots, \sigma_n] \subset \Lambda$ . Let  $M$  be the  $n \times n$  matrix whose  $ij$ 's entry is  $\partial_j(\sigma_i)$ , and let  $J$  be it's determinant.

**Claim.**  $J = \prod_{i < j} (x_i - x_j)$ .

Indeed,  $J$  has degree  $n(n-1)/2$ . Clearly,  $\prod_{i < j} (x_i - x_j)$  divides  $J$ . Since both have the same degree we have  $J = a \prod_{i < j} (x_i - x_j)$  for some scalar  $a$ . Note that in both polynomials the monomial  $x_1^n x_2^{n-1} \dots x_n$  appears with coefficient 1. So  $a = 1$ .

Let  $d \in D(\Lambda_\Delta)^{S_n}$ , and  $f \in \Lambda_\Delta^{S_n}$ . For all  $\pi \in S_n$ ,  $\pi(d(f)) = (\pi.d)(\pi f) = d(f)$ , that is,  $d(f)$  also belongs to  $\Lambda_\Delta^{S_n}$ . In this way, by restricting the domain, we have a ring homomorphism  $\phi : D(\Lambda_\Delta)^{S_n} \rightarrow D(\Lambda_\Delta^{S_n})$ . We need is to show it is an isomorphism. The injectivity follows from the simplicity of  $D(\Lambda_\Delta)^{S_n}$ , as shown above. We discuss the surjectivity of  $\phi$ . The ring  $D(\Lambda_\Delta^{S_n})$  is generated over  $\Lambda_\Delta^{S_n}$  by  $\partial'_1, \dots, \partial'_n$  such that  $\partial'_i(\sigma_j) = \delta_{ij}$ ,  $i, j = 1, \dots, n$ . Hence, it is enough to construct  $S_n$ -invariant differential operators  $d_1, \dots, d_n : \Lambda_\Delta \rightarrow \Lambda_\Delta$ , whose restriction onto  $\Lambda_\Delta^{S_n}$  coincide with  $\partial'_1, \dots, \partial'_n$  above.

Let

$$E_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix},$$

be a vector of size  $n$ , with 1 in the position  $i$  and 0 elsewhere, and let

$$F_i = \begin{pmatrix} f_{i1} \\ \vdots \\ f_{in} \end{pmatrix}$$

be a solution of the system  $MF_i = E_i$ . By the Kramer rule,  $f_{ij} \in \Lambda_\Delta$ ,  $1 \leq i, j \leq n$ .

Let  $d_i = \sum_{k=1}^n f_{ik} \partial_k$ . We have  $d_i(\sigma_j) = \delta_{ij}$ , and that  $d_i \in D(\Lambda_\Delta) = D(\Lambda)_\Delta$ . What is left is to show that  $d_i$  is  $S_n$ -invariant.

It is sufficient to show that for any  $\pi \in S_n$  we have  $\pi f_{ij} = f_{i\pi(j)}$  for  $1 \leq i, j \leq n$ , since  $\pi(\partial_i) = \partial_{\pi(i)}$ . We shall use the Kramer's rule. Let  $v_i$  be the vector

$$\begin{pmatrix} \partial_i(\sigma_1) \\ \vdots \\ \partial_i(\sigma_n) \end{pmatrix}.$$

It is clear that  $\pi(v_i) = v_{\pi(i)}$  and

$$f_{ij} = \frac{\det(v_1, \dots, E_i, \dots, v_n)}{\det(v_1, \dots, v_n)},$$

with  $E_i$  in the  $j$ 's position.

Then

$$\pi f_{ij} = \frac{\det(v_{\pi(1)}, \dots, E_i, \dots, v_{\pi(n)})}{\det(v_{\pi(1)}, \dots, v_{\pi(n)})}$$

$$= \text{sign}(\pi) \det(v_1, \dots, E_i, \dots, v_n) / \text{sign}(\pi) \det(v_1, \dots, v_n),$$

now with  $E_i$  in the position  $\pi(j)$ . This clearly equals  $f_{i\pi(j)}$ .  $\square$

Now we are in the position to prove the Gelfand-Kirillov conjecture.

*Proof of the Gelfand-Kirillov conjecture.* The Galois ring structure of the universal enveloping algebra  $U(\mathfrak{gl}_n)$  over the Gelfand-Tsetlin subalgebra  $\Gamma$  implies the embedding of  $U(\mathfrak{gl}_n)$  into the tensor product

$$\mathcal{A}_1^{S_1} \otimes \mathcal{A}_2^{S_2} \otimes \dots \otimes \mathcal{A}_{n-1}^{S_{n-1}} \otimes \mathbb{k}[t_1, \dots, t_n]^{S_n},$$

where  $\mathcal{A}_k$  is a certain localization of the  $k$ -th Weyl algebra  $A_k$ . Since  $(F_k)^{S_k} \simeq \text{Frac}(A_k^{S_k}) \simeq F_k$  by the Noether's Problem, we have that

$$\text{Frac}(U(\mathfrak{gl}_n)) \simeq F_1 \otimes \dots \otimes F_{n-1} \otimes \mathbf{k}(y_1, \dots, y_n) \simeq F_{\frac{n(n-1)}{2}} \otimes \mathbf{k}(y_1, \dots, y_n),$$

that  $U(\mathfrak{gl}_n)$  is birationally equivalent to  $A_m$  over  $\mathbb{k}(y_1, \dots, y_n)$ ,  $m = n(n-1)/2$  (see [18], Proposition 5.2 for details).  $\square$



## 5. Symmetric differential operators

Let  $t_i = \partial_i x_i \in A_n(\mathbf{k})$ ,  $i = 1, \dots, n$ . It is well known that  $\mathbf{k}[t_1, \dots, t_n]$  is a maximal commutative algebra of  $A_n(\mathbf{k})$ , and  $A_n(\mathbf{k})$  is a free left and right module over  $\mathbf{k}[t_1, \dots, t_n]$  (which can be seen, for example, using the theorem from [17]). In this section we construct a new example of a Galois order given by the algebra of symmetric differential operators. Set  $\Gamma = \mathbf{k}[x_1, \dots, x_n]^{S_n}$ .

**Theorem 2.**  $\Gamma$  is a Harish-Chandra subalgebra of  $A_n(\mathbf{k})^{S_n}$  and  $A_n(\mathbf{k})^{S_n}$  is a Galois order over  $\Gamma$ .

*Proof.* By the result of Levasseur and Stafford ([29], Theorem 5) we have that  $A_n(\mathbf{k})^{S_n}$  is generated as an algebra by  $\mathbf{k}[x_1, \dots, x_n]^{S_n}$  and  $\mathbf{k}[\partial_1, \dots, \partial_n]^{S_n}$ . Denote  $K = \text{Frac} \Gamma$  and  $L = \text{Frac} \mathbf{k}[x_1, \dots, x_n]$ . Let  $\mathbb{Z}^n$  be generated by  $\delta_1, \dots, \delta_n$ , acting on  $L$  in the following way:  $\delta_i(t_j) = t_j - \delta_{ij}$ . Consider an action of  $S_n$  on  $\mathbb{Z}^n$  by conjugation, and set  $\mathfrak{K} = (L * \mathbb{Z}^n)^{S_n}$ . Recall that  $A_n(\mathbf{k})^{S_n}$  is simple. Hence we have an embedding

$$A_n(\mathbf{k})^{S_n} \rightarrow \mathfrak{K}$$

induced by the homomorphism  $A_n(\mathbf{k}) \rightarrow L * \mathbb{Z}^n$ , which sends  $x_i$  to  $\delta_i$  and  $\partial_i$  to  $t_i \delta_i^{-1}$ .

Consider the elements  $x_1 + \dots + x_n$  and  $\partial_1 + \dots + \partial_n$ . Their images in  $\mathfrak{K}$  have supports that generate  $\mathbb{Z}^n$  as a monoid. So, by Proposition 1,  $A_n(\mathbf{k})^{S_n}$  is a Galois ring over  $\Gamma$ . Moreover, the canonical embedding of  $\Gamma$  modules

$$A_n(\mathbf{k})^{S_n} \rightarrow A_n(\mathbf{k})$$

splits, with inverse being the symmetrizer map  $\frac{1}{n!} \sum_{\pi \in S_n} \pi$ . Since  $A_n(\mathbf{k})$  is free over  $\mathbf{k}[t_1, \dots, t_n]$ , and the latter algebra is free over  $\Gamma$  we have that  $A_n(\mathbf{k})^{S_n}$  is a left and right projective  $\Gamma$  module. Applying Proposition 2 we conclude that  $A_n(\mathbf{k})^{S_n}$  is a Galois order over  $\Gamma$  and  $\Gamma$  is a Harish-Chandra subalgebra.  $\square$

We finish with the following conjecture.

**Conjecture.**  $A_n(\mathbf{k})^{S_n}$  is a free left (right)- module over  $\Gamma$ .

**Remark.** One way to prove the conjecture above would be to use the analog of the Kostant theorem from [17]. For that one would need to show in particular that the associated graded algebra of  $A_n(\mathbf{k})^{S_n}$  is a complete intersection ring. However, we were communicated by Gregor Kemper, that this fails already for  $n = 3$ .

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Received by the editors: 21.03.2017.