Algebra and Discrete Mathematics
Volume **23** (2017). Number 1, pp. 47–61
© Journal "Algebra and Discrete Mathematics"
RESEARCH ARTICLE

On the representation type of Jordan basic algebras

Iryna Kashuba*, **Serge Ovsienko and Ivan Shestakov****

Abstract. A finite dimensional Jordan algebra J over a field $\mathbf{k}$ is called *basic* if the quotient algebra $J/\operatorname{Rad} J$ is isomorphic to a direct sum of copies of $\mathbf{k}$. We describe all basic Jordan algebras J with $(\operatorname{Rad} J)^2 = 0$ of finite and tame representation type over an algebraically closed field of characteristic 0.

1. Introduction

Jordan algebras were first introduced by P. Jordan, J. von Neumann and E. Wigner in the early 1930's in the search of a new algebraic setting for quantum mechanics [10]. A *Jordan algebra* J is a commutative algebra such that for any $a, b \in J$

$$(a^2 \cdot b) \cdot a = a^2 \cdot (b \cdot a).$$

In their fundamental paper the authors classified all finite-dimensional formally real algebras. In particular, they showed that any simple formally real finite-dimensional Jordan algebra is either an algebra of Hermitian matrices $H(A)$ over a composition algebra A, or a so-called Jordan algebra of non-degenerated bilinear form $J(V, f)$; for more details see [9, Corollary

*Supported by the Brazilian FAPESP Proc. 2016/08740-1 and CNPq Proc. 307998/2016-9.
**Supported by the Brazilian FAPESP Proc. 2014/09310-5 and CNPq Proc. 303916/2014-1.
2010 MSC: 16G60, 17C55, 17C99.
Key words and phrases: Jordan algebra, Jordan bimodule, Representation type, Quiver of an algebra.

V.6.2]. Later on, A.Albert developed a structure theory of finite dimensional Jordan algebras over arbitrary field of characteristic $\neq 2$ [1] (see also the book [9]).

The fundamentals of representation theory of Jordan algebras were developed by N.Jacobson [8, 9]. He introduced Jordan bimodules, defined their enveloping algebras and described representations of simple Jordan algebras. Jacobson used Eilenberg's definition for bimodules in the variety of algebras, as in [2]. In this approach, the notion of the *universal multiplicative enveloping algebra* proved to be very useful. Jacobson introduced the universal multiplicative enveloping algebra $U(J)$ of a Jordan algebra J as a quotient of the tensor algebra $T(J)$ by the certain ideal defined by Jordan representations of J. The fundamental property of the universal algebra $U(J)$ is that the category of Jordan bimodules J-bimod over J is isomorphic to the category $U(J)$-mod of left modules over the (associative) algebra $U(J)$. Moreover, Jacobson showed that for a unital Jordan algebra J we have the following decomposition of $U(J)$

$$U(J) = U_0(J) \oplus U_{\frac{1}{2}}(J) \oplus U_1(J),$$

where each direct summand is an ideal of $U(J)$, with $U_0(J)$ being a one-dimensional algebra which corresponds to the trivial J-module, $U_{\frac{1}{2}}(J)$ is the *universal associative enveloping algebra*, corresponding to so-called *special* or *one-sided* bimodules, and finally $U_1(J)$ is the *unital universal multiplicative enveloping algebra*, corresponding to unital bimodules. (For more details look Proposition 3.3 and Proposition 5.1, [12]). As a consequence, we have a decomposition

$$J\text{-bimod} = J\text{-bimod}_0 \oplus J\text{-bimod}_{\frac{1}{2}} \oplus J\text{-bimod}_1$$

of the category J-bimod into a direct sum of the three corresponding subcategories.

Furthermore, Jacobson showed that for any finite-dimensional Jordan algebra J the algebra $U(J)$ is of finite dimension as well. Finally, he described the enveloping algebras $U_{\frac{1}{2}}(J)$ and $U_1(J)$ and the irreducible J-bimodules for all finite dimensional simple Jordan algebra J. It turned out that for any simple algebra J both $U_{\frac{1}{2}}(J)$ and $U_1(J)$ are semi-simple algebras, thus both have only finite number of irreducible non-isomorphic Jordan bimodules and are completely reducible. Since the categories $U(J)$-mod and J-bimod are isomorphic, the same is valid for J-bimod. To describe the universal enveloping algebras for algebras of Hermitian matrices, Jacobson used the Coordinatization Theorem and classification of bimodules over composition algebras, [9, VII, 1-2], while for a Jordan algebra of bilinear form $J = J(V, f)$ the enveloping algebras $U_{\frac{1}{2}}(J)$ and $U_1(J)$ turned out to be the Clifford algebra and the meson algebra defined by a bilinear form f, respectively, see[9, VII,4-5].

Until recent time, there were no results which would describe bimodules for any class of Jordan algebras other then semi-simple algebras. In 2002 Serge Ovsienko was visiting São Paulo University, and the authors decided to look on representations of non-semisimple finite-dimensional Jordan algebras. One of the most known tools in the representation theory of finite-dimensional associative algebra A is to construct a quiver Q with relations R such that A-mod is Morita equivalent to the path algebra $\mathbf{k}[Q]$ modulo the ideal generated by relations R. Moreover, all finite-dimensional associative algebras can be divided into three types: finite, tame and wild. For the first two classes one can provide a complete description of indecomposable finite-dimensional left modules. We define a representation type of Jordan algebra as being a representation type of the corresponding associative universal enveloping algebra. Our objective was to describe all finite and tame Jordan algebras.

We started with considering basic Jordan algebra J of small dimension, calculating $U(J)$ as a quotient tensor algebra modulo relations (2)–(3). Our first examples and basic relations between Jordan algebra and its quiver were result of intensive repeated calculations. Ovsienko suggested to consider a class of algebras which proved to be very useful in the case of associative algebras, namely algebras J with $(\operatorname{Rad} J)^2 = 0$. Although long but straightforward method allowed us to describe finite and tame basic Jordan algebras in this class.

Unfortunately, for arbitrary non-basic Jordan algebra J to calculate $U(J)$ and then find basic algebra Morita equivalent to $U(J)$ was much harder task, and we were forced to find (and fortunately found) new methods, relaying on Jordan theory rather than associative one. Generalizing Jacobson's Coordinatization Theorem, we described Jordan algebras J such that $(\operatorname{Rad} J)^2 = 0$, semi-simple part of J is a direct sum of algebras of Hermitian matrices of order > 1, for which $U_{\frac{1}{2}}(J)$ is of finite or tame representation type, see [11]. In [9], Kashuba and Serganova, using a connection between Jordan and Lie algebras (the famous Tits-Kantor-Koecher construction), classified Jordan algebras J, such that semi-simple part of J is a direct sum of algebras of bilinear form and $U_1(J)$ is of finite or tame representation type.

In this paper we describe all basic Jordan algebras J with $(\mathrm{Rad}\, J)^2 = 0$ of finite and tame representation type by calculating the basic algebra $U(J)$ and then constructing its quiver. It was a starting point of our project in 2002–2003, which eventually was not included in [11]. By writing it we want to recall all joy and pleasure we had while working with Sergey Adamovich Ovsienko.

In Section 2 we present both definition and basic properties of Jordan bimodules over J and define the universal enveloping algebra $U(J)$. We also define the representation type of Jordan algebra and recall how to construct the quiver with relations corresponding to a basic associative algebra. In Section 3, we start with the technical lemma which describes relations between the elements of J in $U(J)$ based on the Peirce decomposition of J. Then we describe quivers for two basic Jordan algebras J^{ii} and J^{ij} such that both algebras have the dimension of the radical equals to one (one could think of these examples as "cells"). We finish Section 3 with the theorem classifying finite and tame basic Jordan algebras with $(\mathrm{Rad}\, J)^2 = 0$. Finally, Section 4 is the Appendix where we collect the results from the representation theory which we use to determine representation type of quivers.

2. Jordan bimodules and the universal enveloping algebras

We work over an algebraically closed field $\mathbf{k}$ of characteristic 0. Let J be a Jordan algebra over $\mathbf{k}$, M be a $\mathbf{k}$-vector space endowed with a pair of linear mapping $r : M \otimes_{\mathbf{k}} J \to M$, $m \otimes a \mapsto m \cdot a$ and $l : J \otimes_{\mathbf{k}} M \to M$, $a \otimes m \mapsto a \cdot m$. Then we consider $\mathbf{k}$-algebra $\Omega = J \oplus M$ with the following multiplication $* : \Omega \times \Omega \to \Omega$

$$(a_1 + m_1) * (a_2 + m_2) = a_1 \cdot a_2 + a_1 \cdot m_2 + m_1 \cdot a_2, \tag{1}$$

for $a_1, a_2 \in J$, $m_1, m_2 \in M$. We will say that M is a *Jordan bimodule* over J if Ω is a Jordan algebra with respect to $*$. Following [8], the (two-sided) action of the Jordan algebra J on the bimodule M can be rewritten as a (one-sided) action of the associative algebra $U(J)$, the *universal multiplicative envelope* of J. The algebra $U(J)$ can be constructed in the following way. Consider the free associative $\mathbf{k}$-algebra $F\langle J\rangle$ generated by the vector space J, and let I be the ideal of $F\langle J\rangle$ generated by the elements

$$abc + cba + (a \cdot c) \cdot b - a(b \cdot c) - b(a \cdot c) - c(a \cdot b), \tag{2}$$

$$a(b\cdot c)-(b\cdot c)a+b(a\cdot c)-(a\cdot c)b+c(a\cdot b)-(a\cdot b)c, \qquad (3)$$

where $a,b,c\in J$ and $a\cdot b$, ab denote the products of a,b in J and in $F\langle J\rangle$, respectively. Put $U(J):=F\langle J\rangle/I$. Then one may endow every left $U(J)$-module M with the canonical structure of J-bimodule via $a\cdot m=m\cdot a:=(a+I)m$, $a\in J, m\in M$. This defines an isomorphism of the category of Jordan bimodules over J and the category of associative left modules over $U(J)$. Moreover, by [8] if $\dim_{\mathbf{k}}J<\infty$ then $\dim_{\mathbf{k}}U(J)<\infty$, i.e. $U(J)$ is a finite-dimensional associative algebra (with 1).

Let A be a finite dimensional associative $\mathbf{k}$-algebra with 1, and denote by $\mathbf{k}\langle x,y\rangle$ the free associative algebra in x,y. Then we say that A has

- a *finite type* if there are finitely many isomorphism classes of indecomposable A-modules;
- a *tame type* if it is not of finite type and for each dimension d there exists finitely many one-parameter families $F_1,\dots,F_N$, such that every indecomposable module of dimension d is isomorphic to a module from some F_i;
- a *wild type* if there exists an A–$\mathbf{k}\langle x,y\rangle$-bimodule M, finitely generated and free as a $\mathbf{k}\langle x,y\rangle$-module, such that the functor $N\mapsto M\otimes_{\mathbf{k}\langle x,y\rangle}N$ from $\mathbf{k}\langle x,y\rangle$-mod to A-mod keeps indecomposability and isomorphism classes.

The following theorem allows us to classify algebras with respect to their representation type.

Theorem 2.1. [4] *A finite dimensional algebra A has either finite or tame or wild type.*

Having in mind the above isomorphism J-bimod $\simeq U(J)$-mod, we will say that a finite dimensional Jordan algebra J is of *finite, tame or wild* type if it is true for the finite dimensional associative algebra $U(J)$. By the Theorem 2.1, every finite dimensional Jordan algebra has either finite or time or wild type.

Now we recall briefly the notion of a quiver for associative algebra. For further details and examples we refer to [11].

An associative algebra A is called *basic* or *Morita reduced* if $A/\mathrm{Rad}A\simeq\mathbf{k}^n$ for some positive integer n. Two algebras A and B are called *Morita equivalent* if the categories A-mod and B-mod are equivalent. In any class of Morita equivalent algebras there exists a basic algebra and it is unique up to isomorphism.

Recall that a *quiver* Q is defined as a set of *vertices* Q_0 and a set of *arrows* Q_1 together with two mappings $s,e:Q_1\to Q_0$ which send an arrow to its start and end vertex correspondingly. For any basic algebra Λ we define its quiver $Q = Q(\Lambda)$ in the following way. Let $\Lambda = S \oplus \mathrm{Rad}\Lambda$, where $S \simeq \mathbf{k}^n$ is the semi-simple part of Λ and $\mathrm{Rad}\Lambda$ is the Jacobson radical. Choose a set of orthogonal primitive idempotents $e_1 + \dots + e_n = 1$ and denote by $k_{ij} = \dim_{\mathbf{k}} e_j(\mathrm{Rad}\Lambda/\mathrm{Rad}^2\Lambda)e_i$. Then the vertices of Q are labelled by the set of idempotents $Q_0 = \{1, \dots, n\}$ and from vertex i to vertex j lead k_{ij} arrows.

Reciprocally, one can construct the *path algebra* $\mathbf{k}[Q]$ starting with a quiver Q. This algebra has a basis formed by the set of orthogonal idempotents Q_0 and all oriented paths in Q, that is, the sequences $x_1 \dots x_k$, $x_i \in Q_1$, $k > 0$ such that $s(x_i) = e(x_{i+1})$, $i = 1, \dots, k-1$. The product of two paths $x_1 \dots x_k$ and $y_1 \dots y_s$ is the path $x_1 \dots x_k y_1 \dots y_s$ if $s(x_k) = e(y_1)$ and zero otherwise. If $e \in Q_0$ and $x \in Q_1$ then $ex = x$ if $e(x) = e$ and zero otherwise; similarly $xe = x$ if $s(x) = e$ and zero otherwise.

Now we return to original associative algebra A. Let Λ be a basic algebra which is Morita equivalent to A and $Q(\Lambda)$ be its quiver, then there exists an epimorphism

$$\pi : \mathbf{k}[Q(\Lambda)] \twoheadrightarrow \Lambda \tag{4}$$

(see [5]). Let $I \subset \ker\pi$ be the set of generators of the ideal $\ker\pi$, then the pair (Q, I) is called the *quiver with relations* corresponding to A. It follows from (4) that $\mathbf{k}[Q(\Lambda)]/\langle I\rangle \simeq \Lambda$. The main use of quivers with relations is that the category A-mod is equivalent to the category of representations of the corresponding quiver with relations. Once again, we reduce the problem of describing modules over associative algebra to studying modules over its quiver. There are plenty techniques in order to determine the representation type of a quiver, as well as number of results for different classes of finite dimensional associative algebras (for example, for local algebras [15], for algebras with the radical squared zero [6], for special biserial algebras [7]).

3. Representation type of Jordan basic algebras

Recall that for any $a, b \in J$ we write $a \cdot b$ for their product in J, while by ab we denote their product in $U(J)$. Any finite-dimensional $\mathbf{k}$-algebra J may be written as a direct sum of a semi-simple Jordan algebra $S(J)$ and the radical $\mathrm{Rad}\, J$ of J (i.e. the unique maximal nilpotent ideal of J), $J = S(J) \oplus \mathrm{Rad}\, J$. Denote by $QJ(J) := Q(U(J))$ and by $Q_iJ(J) := Q_i(U(J))$, $i = 0, 1$.

Theorem 3.1 ([11, Teorem 2.3]). *Let $J^{\#} = J + \mathbf{k}1$ be the algebra obtained by the formal adjoining of the identity element to J then the category J-*bimod *is isomorphic to the category of unital modules $J^{\#}$-*bimod$_1$ *of $J^{\#}$.*

Thus, without loss of generality we will suppose that J has an identity element c.

Next we recall the Peirce decomposition of Jordan algebra relative to idempotent or system of idempotents.

Theorem 3.2 ([9, III.1]). *Let e be an idempotent in J then we have the following decomposition into a direct sum of subspaces*

$$J = J_1 \oplus J_{\frac{1}{2}} \oplus J_0,$$

where $J_i = \{x \in J \mid x \cdot e = ix\}$, for $i = 0,\ \frac{1}{2},\ 1$.

This decomposition is called the *Peirce decomposition of J relative to idempotent e*. The multiplication table for the Peirce components J_i is:

$$\begin{array}{ccc} J_1^2 \subseteq J_1, & J_1 \cdot J_0 = 0, & J_0^2 \subseteq J_0, \\ J_0 \cdot J_{\frac{1}{2}} \subseteq J_{\frac{1}{2}}, & J_1 \cdot J_{\frac{1}{2}} \subseteq J_{\frac{1}{2}}, & J_{\frac{1}{2}}^2 \subseteq J_0 \oplus J_1. \end{array} \tag{5}$$

Furthermore, we have the following generalization: if J is a Jordan algebra with an identity element c which is a sum of pairwise orthogonal idempotents e_i, i.e. $c = \sum_{i=1}^n e_i$, we have the refined *Peirce decomposition of J relative to idempotents* $\{e_1, \ldots, e_n\}$:

$$J = \bigoplus_{1 \leqslant i \leqslant j \leqslant n} J_{ij} \tag{6}$$

where $J_{ii} = \{x \in J \mid x \cdot e_i = x\}$ and $J_{ij} = \left\{x \in J \mid x \cdot e_i = x \cdot e_j = \frac{1}{2}x\right\}$. The multiplication table for the Peirce components is:

$$\begin{array}{ccc} J_{ii}^2 \subseteq J_{ii}, & J_{ij} \cdot J_{ii} \subseteq J_{ij}, & J_{ij}^2 \subseteq J_{ii} \oplus J_{jj}, \\ J_{ij} \cdot J_{jk} \subseteq J_{ik}, & J_{ii} \cdot J_{jj} = J_{ii} \cdot J_{jk} = J_{ij} \cdot J_{kl} = 0, \end{array} \tag{7}$$

where the indices i, j, k, l are all different.

In the following proposition we recall the Peirce decomposition of $U(J)$ inherited from the Peirce decomposition of J.

Proposition 3.3 ([12, Prop 5.1]). *Let J be a Jordan algebra and $U = U(J)$ be the universal multiplicative envelope of J and let c be the identity element in J.*

(i) Put $C_0 = (c-1)(2c-1)$, $C_{1/2} = 4c(1-c)$, $C_1 = c(2c-1)$, *where* 1 *denotes the identity element in* $U(J)$*. Then the* C_i, $i = 0, \frac{1}{2}, 1$ *are central orthogonal idempotents in* U*, moreover* $C_0 + C_{\frac{1}{2}} + C_1 = 1$ *which implies the following decomposition of* $U(J)$ *into the direct sum of ideals* $U = UC_0 \oplus UC_1 \oplus UC_{1/2}$.

(ii) Let $e_1, \ldots, e_n$ *be orthogonal idempotents in* J *such that* $\sum e_i = c$. *Then the elements*

$$C_{ij} = 4e_ie_j,\ C_{ii} = e_i(2e_i - 1),\ C_{0i} = 4e_i(1-c)$$

are pairwise orthogonal idempotents in U *such that*

$$\sum_{i \leqslant j} C_{ij} = C_1,\quad \sum_i C_{0i} = C_{1/2}.$$

We put also $C_{00} = C_0$ *to have an orthogonal sum* $1 = \sum_{i,j=0}^n C_{ij}$.

(iii) If the idempotents e_i *are central then* C_{ij} *are central in* U *and* U *is a direct sum of ideals* $C_{ij}U$.

The proof follows from (2)-(3), the second relation for e_i, e_i, e_j , $i \neq j$ provides that e_i and e_j commute in $U(J)$. They also imply the following relations between C_{ij} and e_k in $U(J)$

$$\begin{array}{lll} C_{0i}e_i = \frac{1}{2}C_{0i}, & C_{ii}e_i = C_{ii}, & C_{ij}e_i = \frac{1}{2}C_{ij}, \\ C_{0i}e_k = 0, & C_{ii}e_k = 0, & C_{ij}e_k = 0, \end{array} \tag{8}$$

here all i, j, k are different.

Now suppose that J is a basic Jordan algebra, i.e. is a direct sum of basic field $\mathbf{k}$. Write $S(J) \simeq \mathbf{k}^m = \mathbf{k}e_1 + \cdots + \mathbf{k}e_m$ Then by Lemma 5.3 [11] we know that $S(U(J)) = U(S(J))$, i.e. the semi-simple part of $U(S(J))$ is isomorphic to the subalgebra generated by the images of elements of semi-simple part $S(J)$ of J together with $1 \in U(J)$, thus $U(J)$ is basic associative algebra and $\{C_{ij} | 0 \leqslant i \leqslant j \leqslant m\}$ is the complete set of primitive orthogonal idempotents. In particular it follows that any basic Jordan semi-simple algebras is of finite representation type.

Suppose further that $(\operatorname{Rad} J)^2 = 0$, then $\operatorname{Rad} J$ is completely determined by $S(J)$-bimodule structure of $\operatorname{Rad} J$, or equivalently by its Peirce decomposition relative to $\{e_i | 1 \leqslant i \leqslant m\}$.

We start with some technical lemma, following two crucial examples.

Lemma 3.4. *Suppose that* $J = \mathbf{k}e_1 + \cdots + \mathbf{k}e_m + R_{ij}$, $R_{ij} = (\operatorname{Rad} J)_{ij}$.

(i) *Let* $a \in R_{ij}$, $b \in R_{kl}$ *with* $(i,j) \neq (k,l)$*, then* a *and* b *commute in* $U(J)$.

(ii) *Let* $a \in R_{ii}$ *then* a *and* e_k, $1 \leqslant k \leqslant m$ *commute in* $U(J)$.
(iii) *Let* $a \in R_{ij}$ *then* a *and* e_k, $k \neq i, j$ *commute in* $U(J)$.
(iv) *Let* $a \in R_{ii}$, *then we have the following decomposition of* a *in* $U(J)$

$$a = C_{0i}a + C_{ii}a + \sum_{\substack{1 \leqslant j \leqslant m, \\ i \neq j}} C_{ij}a \tag{9}$$

(v) *Let* $b \in R_{ij}$, $i \neq j$, *then we have the following decomposition of* b *in* $U(J)$

$$b = C_{ii}bC_{ij} + C_{ij}bC_{jj} + C_{jj}bC_{ij} + C_{ij}bC_{ii} + C_{0i}bC_{0j} + C_{0j}bC_{0i} \tag{10}$$

Proof. Since $(i, j) \neq (k, l)$ there exists t such that e_t acts differently on a and b. Then equation (3) for e_t, a, b guarantees that $ab = ba$. If $a \in R_{ii}$ the same relation for e_k, e_k, a gives $e_ka - ae_k + 2\delta_{i,k}ae_k - 2\delta_{i,k}e_ka = 0$, thus we obtain (ii). The argument for (iii) is analogous.

If $a \in R_{ii}$, by (ii) $[e_k, a] = 0$ for any $1 \leqslant k \leqslant m$, therefore $C_{ij}aC_{kl} = 0$ unless $i = k$, $j = l$. Relation (2) gives

$$(2c-1)(c-1)a = C_{00}a = 0, \qquad (2e_k - 1)e_ka = C_{kk}a = 0,$$
$$2e_kca - 2e_ka = -\frac{1}{2}C_{0k} = 0, \quad k \neq i.$$

Thus (9) follows.

Finally suppose that $b \in R_{ij}$, analogously to (iv), $C_{00}b = 0$. Further by (iii) we have that $[e_k, b] = 0$ for $k \neq i, j$, therefore from (8) it follows that $C_{lk}bC_{st} = C_{st}bC_{lk} = 0$ for any $(s,t) \neq (l,k)$. When $k \neq i, j$ relation (2) for e_k, e_k and b provides $C_{kk}b = 0$, while for e_k, e_i and b we obtain

$$4e_ke_ib + 4be_ie_k - 2be_k = C_{ik}b + bC_{ik} - 2be_k = 0.$$

It follows that $C_{tk}bC_{tk} = 0$, $0 \leqslant k \leqslant m$. We continue to eliminate summands in the Peirce decomposition of b. For e_i, b, e_j the relation (2) results in

$$2e_ibe_j + 2e_jbe_i - be_i - be_j = 0,$$

then $C_{ii}bC_{ii} = C_{0i}bC_{0i} = C_{ii}bC_{0i} = C_{0i}bC_{ii} = 0$, the same equations hold if we substitute j instead of i. Moreover we also obtain that $C_{ii}bC_{jj} = C_{jj}bC_{ii} = 0$. Changing the order of the elements in (2) we get

$$2e_ie_jb + 2be_je_i - be_i - be_j = 0$$

and consequently $C_{0i}bC_{jj} = C_{jj}bC_{0i} = C_{0j}bC_{ii} = C_{ii}bC_{0j} = 0$ and (10) follows. □

Now we will define the representation type of basic Jordan algebras with $\dim \operatorname{Rad} J = 1$. There are only two non-isomorphic algebras $J^{ii} = \mathbf{k}e_1 + \cdots + \mathbf{k}e_m + R_{ii}$, and $J^{ij} = \mathbf{k}e_1 + \cdots + \mathbf{k}e_m + R_{ij}$, for $i \neq j$.

Example 3.5 (Representation type of J^{ii})**.** Let $\langle a \rangle = \operatorname{Rad} J^{ii}$, then from Lemma 3.4, (iv) it follows that the arrows of $Q_1J(J^{ii})$ are $a_i := C_{ij}a$, $b := C_{0i}a$ and $c := C_{ii}a$. To calculate the relations, write (2) for a, a, e_j if $j \neq i$: $a^2 e_j = 0$, while $a^2(e_i - 1) = 0$. Then $a_i^2 = C_{ij}a^2 = 0$ and $b^2 = C_{0i}a^2 = 0$, while (2) for a, a, a gives $a^3 = 0$ therefore $c^3 = 0$. The quiver $QJ(J^{ii})$ is the union of isolated points and the following quivers corresponding to idempotents C_{ij}, C_{0i} and C_{ii} respectively,

$$\begin{tikzcd} \bullet \arrow[loop above, "a_i"] & \bullet \arrow[loop above, "b"] & \bullet \arrow[loop above, "c"] \end{tikzcd} \tag{11}$$

with the relations $a_i^2 = b^2 = c^3 = 0$. By Proposition 4.1(vi) Jordan algebra J^{ii} is of finite representation type.

Example 3.6 (Representation type of J^{ij})**.** Let $\langle b \rangle = \operatorname{Rad} J^{ij}$, then from Lemma 3.4, (v) it follows that the arrows of $Q_1J(J^{ij})$ are

$$\begin{array}{lll} f := C_{0i}bC_{0j}, & g = C_{0j}bC_{0i}, & h = C_{ij}bC_{jj}, \\ k = C_{jj}bC_{ij}, & l = C_{ii}bC_{ij}, & m = C_{ij}bC_{ii}. \end{array}$$

To calculate the relations, we again write (2) for e_i, b, b and obtain $b^2 = e_ib^2 + b^2e_i$, which yields $C_{0i}b^2C_{0i} = C_{0j}b^2C_{0j} = C_{ii}b^2C_{ii} = C_{jj}b^2C_{jj} = 0$. Therefore, we have

$$gf = fg = hk = ml = 0.$$

Now consider

$$C_{ij}b^2C_{ij} = C_{ij}b(C_{ii} + C_{jj})bC_{ij} = lm + kh.$$

On the other hand, (2) for b, e_i, b gives $2be_ib = b^2$. Substituting (10) for b and multiplying it by C_{ij} we obtain

$$\begin{aligned} C_{ij}b^2C_{ij} &= 2C_{ij}be_ibC_{ij} = 2C_{ij}(bC_{jj} + bC_{ii})e_i(C_{ii}b + C_{jj}b)C_{ij} \\ &= 2C_{ij}bC_{jj}bC_{ij} = 2kh. \end{aligned}$$

and we have the relation $kh = lm$. The quiver $QJ(J^{ij})$ is the union of isolated points and the following quivers corresponding to idempotents C_{0i}, C_{0j}, C_{ii}, C_{ij} and C_{jj} correspondingly

$$\begin{tikzcd} \bullet \arrow[r, bend left, "f"] & \bullet \arrow[l, bend left, "g"] \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, bend left, "l"] & \bullet \arrow[l, bend left, "m"] \arrow[r, bend left, "h"] & \bullet \arrow[l, bend left, "k"] \end{tikzcd} \qquad I = \langle fg, gf, hk, ml, kh - lm \rangle \tag{12}$$

By Proposition 4.1 (vi) $\mathbf{k}[Q]/\langle I\rangle$ is of tame representation type and so is J^{ij}.

Theorem 3.7 (Representation type of basic Jordan algebras J with $(\operatorname{Rad} J)^2=0$)**.** *Let* $J=\mathbf{k}e_1+\cdots+\mathbf{k}e_m+\operatorname{Rad} J$ *be basic Jordan algebra with* $(\operatorname{Rad} J)^2=0$ *and let* $\operatorname{Rad} J=\sum_{1\leqslant i\leqslant j\leqslant m} R_{ij}$ *be the Peirce decomposition of* $\operatorname{Rad} J$ *relative to* $e_1,\ldots,e_m$*. Then*

i) *J has finite representation type iff J is semi-simple Jordan algebra.*

ii) *J has tame representation type iff* $\operatorname{Rad} J\neq 0$*, for any i,j* $\dim R_{ij}\leqslant 1$*; if $i\neq j$ and* $\dim R_{ij}=1$ *then* $\dim R_{ii}=\dim R_{jj}=0$ *and, finally, for any j*

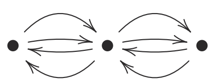
$$\sum_{i\neq j}\dim R_{ij}\leqslant 2.$$

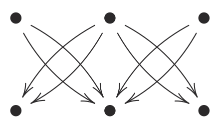
Proof. By [9] any semi-simple Jordan algebra has finite representation type. Both algebras J^{ii} and J^{ij} are of tame representation type, then from Proposition 4.1, (i) if $\operatorname{Rad} J\neq 0$, algebra J is of wild or tame representation type.

Next from Remark 5.3, [11], it follows that if $J_1=\mathbf{k}e_1+\cdots+\mathbf{k}e_m+\operatorname{Rad} J_1$, $J_2=\mathbf{k}e_1+\cdots+\mathbf{k}e_m+\operatorname{Rad} J_2$ and $J=\mathbf{k}e_1+\cdots+\mathbf{k}e_m+\operatorname{Rad} J_1+\operatorname{Rad} J_2$ then

$$Q_1J(J)=Q_1J(J_1)\sqcup Q_1J(J_2). \tag{13}$$

First we will prove that if J if of tame type then $\dim\operatorname{Rad} R_{ij}\leqslant 1$ for any $i,j\in\{1,\ldots,m\}$. Suppose $\dim\operatorname{Rad} R_{ij}\geqslant 2$, then if $i\neq j$ from (13) it follows that $QJ(J)$ contains the following subquiver Q' corresponding to the vertices C_{ii}, C_{ij} and C_{ii}

Q' : 

$D(Q')$: 

By Theorem 4.2 the quotient algebra $\mathbf{k}[Q']/\operatorname{Rad}^2(\mathbf{k}[Q'])$ is of wild representation type thus, by Proposition 4.1(i), J has wild type as well. If $\dim\operatorname{Rad} R_{ii}\geqslant 2$, by (13), $C_{ii}U(J)C_{ii}$ contains a wild subalgebra $\mathbf{k}\langle x,y,z\rangle/\langle x,y,z\rangle^2$, see Proposition 4.1 (iv). As a consequence of Proposition 4.1 (i) J is wild.

Further, suppose that there exist $i\neq j$ such that $\dim\operatorname{Rad} R_{ij}=\dim\operatorname{Rad} R_{ii}=1$. Then by (13) $QJ(J)$ contains the following subquiver corresponding to the vertices C_{ii}, C_{ij} and C_{jj}.

$$Q'' : \begin{tikzcd} \bullet \arrow[r, shift left] & \bullet \arrow[l, shift left] \arrow[r, shift left] \arrow[loop above] & \bullet \arrow[l, shift left] \arrow[loop above] \end{tikzcd} \qquad D(Q'') : \begin{tikzcd} \bullet \arrow[dr] & \bullet \arrow[dl] \arrow[d] \arrow[dr] & \bullet \arrow[dl] \arrow[d] \\ \bullet & \bullet & \bullet \end{tikzcd}$$

By Theorem 4.2, the quotient algebra $\mathbf{k}[Q'']/\operatorname{Rad}^2(\mathbf{k}[Q''])$ is of wild representation type, therefore J has wild type as well. This implies that for any tame algebra J whenever $i \neq j$ and $\dim R_{ij} = 1$ both $\dim R_{ii} = \dim R_{jj} = 0$.

Finally, suppose there exist i, j, k, l such that $\dim R_{ij} = \dim R_{kj} = \dim R_{lj} = 1$ then $Q(J)$ contains the following subquiver Q'''

$$\begin{tikzcd} C_{ii} \arrow[r, shift left] & C_{ij} \arrow[l, shift left] \arrow[r, shift left] & C_{jj} \arrow[l, shift left] \arrow[r, shift left] \arrow[d, shift left] & C_{kj} \arrow[l, shift left] \arrow[r, shift left] & C_{kk} \arrow[l, shift left] \\ & & C_{lj} \arrow[u, shift left] \arrow[r, shift left] & C_{ll} \arrow[l, shift left] & \end{tikzcd}$$

By the double quiver arguments J has wild representation type.

Next we will show that all the remaining non-semi-simple Jordan algebras are of tame representation type. By (13)

$$Q_1 J(S(J) + \oplus_{1 \leqslant i,j \leqslant m} R_{ij}) = \sqcup_{1 \leqslant i,j \leqslant m} Q_1 J(J^{ij}),$$

recall that $Q_1 J(J^{ij})$ are constructed in Example 3.5 for $i = j$ and in Example 3.6 for $i \neq j$. Moreover, if we denote by I_{ij} the relations corresponding to the quiver of algebra J^{ij} then $\sqcup_{1 \leqslant i,j \leqslant m} I_{ij}$ belongs to the set of generators I of J. The remaining relations we have to check correspond to the following two cases. First case, suppose there exist i, j, k such that $\dim R_{ij} = \dim R_{kj} = 1$. Let $R_{ij} = \langle b \rangle$ and $R_{kj} = \langle d \rangle$, then denote as $C_{ij} b C_{jj} = f$, $C_{jj} b C_{ij} = h$, $C_{kj} d C_{jj} = g$, $C_{jj} d C_{kj} = l$. Then we have the following subquiver

$$\begin{tikzcd} C_{ij} \arrow[r, shift left, "h"] & C_{jj} \arrow[l, shift left, "f"] \arrow[r, shift left, "g"] & C_{kj} \arrow[l, shift left, "l"] \end{tikzcd} \tag{14}$$

and by Example 3.6 $hf = 0$ and $lg = 0$. Further from (2) for e_i, b, c it follows that $bc = 2e_i cb + cbe_i$, moreover by Lemma 3.4 (i) b and c commute, thus $fl = gh = 0$. The second case to check for new relations is when there exist i, j such that $\dim R_{ii} = \dim R_{jj}$. Let $R_{ii} = \langle a \rangle$ and

$R_{jj} = \langle c \rangle$ and denote as $C_{ij} a C_{ij} = f$ and $C_{ij} c C_{ij} = g$. Then we have the following subquiver in the vertex corresponding to C_{ij}

$$\begin{tikzcd} \bullet \arrow[loop above, "f"] \arrow[loop below, "g"'] \end{tikzcd} \tag{15}$$

and by Example 3.5 $f^2 = g^2 = 0$. By Lemma 3.4 (i) b, c and e_i commute then (2) for b, c, e_i provides that $bc = 2e_i bc$, this gives the relation $gf = fg$. By Proposition 4.1 (ii) the quiver in (15) has tame representation type.

Further, it remains to prove that if J satisfies the condition $ii)$ of the theorem it is of tame representation type. Observe that such algebras may be written and a direct sum of local algebras J^{ii} and algebras $J^{i_1,i_2} + J^{i_2,i_3} + \cdots + J^{i_{k-1},i_k}$, then $QJ(J)$ also can be viewed as a disjoint union of quivers from Example 3.5 and (15) (both are tame) and two quivers

$$\begin{tikzcd} C_{i_1 i_1} \arrow[r, shift left, "h"] & C_{i_1 i_2} \arrow[l, shift left, "f"] \arrow[r, shift left, "g"] & C_{i_2 i_2} \arrow[l, shift left, "m"] \arrow[r, shift left, "k"] & C_{i_2 i_3} \arrow[l, shift left, "l"] \arrow[r, dotted, no head] & C_{i_{k-1} i_k} \arrow[r, shift left] & C_{i_k i_k} \arrow[l, shift left] \end{tikzcd}$$

$$I_1 = \langle hf - mg, fh, gm, lk, gh, fm, ml, kg \rangle,$$

$$\begin{tikzcd} C_{0 i_1} \arrow[r, shift left, "a"] & C_{0 i_2} \arrow[l, shift left, "c"] \arrow[r, shift left, "b"] & C_{0 i_3} \arrow[l, shift left, "d"] \end{tikzcd}$$

$$I_2 = \langle ac, ca, bd, db, ba, cd \rangle.$$

These quivers correspond to special biserial algebras, check Proposition 4.1, (vi), thus are of tame representation type. $\square$

4. Appendix: quiver type glossary

In the proposition below we summarize some known facts from the associative theory that we will need to determine the representation type of algebras.

Proposition 4.1. *Let A be a finite dimensional associative* $\mathbf{k}$*-algebra and* $\mathbf{k}\langle x_1, \dots, x_n \rangle$ *be the free associative algebra in n generators.*

(i) *If a homomorphic image of an algebra A is of wild type then so is A. A quotient algebra of tame algebra is either tame or finite. If A contains subalgebra of wild type then A is wild. In particular if $Q(A)$ contains subquiver which corresponds to wild algebra then A is wild.*

(ii) *If A is a local algebra and $A \simeq \mathbf{k}\langle x, y\rangle/I$, where $I = \langle yx, xy\rangle$ or $I = \langle x^2, y^2\rangle$, then A is tame.*
(iii) *If A is a local algebra which has a homomorphic image isomorphic to $\mathbf{k}\langle x, y\rangle/I$, where $I = \langle x^2, xy - yx, y^2x, y^3\rangle$, then A is wild.*
(iv) *The algebras $\mathbf{k}\langle x, y, z\rangle/\langle x, y, z\rangle^2$ and $\mathbf{k}\langle x, y\rangle/\langle x, y\rangle^3$ are wild.*
(v) *For any positive integer n, the algebra $\mathbf{k}\langle x\rangle/\langle x^n\rangle$ is of finite type.*
(vi) *An algebra A is called* special biserial *if A is isomorphic to $\mathbf{k}[Q]/R$ for some quiver Q and admissible ideal R, such that*
   1) *any vertex of Q is the starting point of at most two arrows;*
   2) *any vertex of Q is the end point of at most two arrows;*
   3) *if b is an arrow in Q_1, then there is at most one arrow a with $ab \in R$;*
   4) *if b is an arrow in Q_1, then there is at most one arrow c with $bc \in R$.*

   Any special biserial algebra is of tame or finite representation type.

Proof. For (i) see [5], (ii)–(v) are proved in [15] and (vi) can be found in [3, Cap. VI]. □

The following theorem contains the stunning result for an associative algebra A with $\mathrm{Rad}^2 A = 0$.

Theorem 4.2 ([6], [14])**.** *An algebra A with $\mathrm{Rad}^2 A = 0$ is of finite (tame) representation type if and only if its double quiver $D(Q(A))$ is a disjoint finite union of simply laced Dynkin diagrams (correspondingly simply laced extended Dynkin diagrams).*

The simply laced Dynkin diagrams (simple laced extended Dynkin diagrams) are A_n, $n \geqslant 1$, D_n, $n \geqslant 4$ and E_n $n = 6, 7, 8$ (correspondingly $\tilde{A}_n$, $n \geqslant 1$, $\tilde{D}_n$, $n \geqslant 4$ and $\tilde{E}_n$ $n = 6, 7, 8$). Recall that for a quiver $Q = (Q_0, Q_1)$ with $Q_0 = \{1, \ldots, n\}$, its *double quiver* is defined as $D(Q) = (D(Q_0), D(Q_1))$ where $D(Q_0) = \{1', \ldots, n'\} \cup \{1'', \ldots, n''\}$, $D(Q_1)$ has the same cardinality as Q_1, and to every $a \in Q_1$ with $e(a) = m$, $s(a) = k$ corresponds a unique $d(a) \in D(Q_1)$ with $e(d(a)) = m''$ and $s(d(a)) = k'$.

References

[1] A. Albert, *A structure theory of Jordan algebras*, Ann. of Math. (2) 48, (1947), 546–567.

[2] S. Eilenberg, *Extension of general algebras,* Ann. Soc. Polon. Math., **21**, (1948), 125-134.

[3] K. Erdmann, *Blocks of Tame Representation Type and Related Algebras,* Lecture Notes in Mathematics, **1428**, 1990.

[4] Yu. A. Drozd, *Tame and wild matrix problems,* in: "Representations and Quadratic Forms", Kiev, (1979), 39-74 (AMS Translations **128**).

[5] Yu. Drozd, V. Kirichenko, *Finite-Dimensional Algebras,* Springer-Verlag, Berlin, 1994.

[6] P. Gabriel, *Indecomposable Representations II,* Symposia Mathematica, vol. XI Academic Press, London (1973), 81-104.

[7] M. Gel'fand, V. A. Ponomarev, *Indecomposable representation of the Lorentz group,*, Russian Mathematical Surveys, **23**, 1968, 1-58.

[8] N. Jacobson, *Structure of alternative and Jordan bimodules,* Osaka Math. J., **6**, (1954), 1-71.

[9] N. Jacobson, *Structure and representations of Jordan algebras,* AMS Colloq. Publ., **39**, AMS, Providence 1968.

[10] P. Jordan, J. von Neumann, E. Wigner, *On an algebraic generalization of quantum mechanical formalism*, Ann. of Math. (2) 36 (1934), 29–64.

[11] I. Kashuba, S. Ovsienko, I. Shestakov, *Representation type of Jordan algebras,* Adv. Math. **226**, (2011), no. 1, 385-418.

[12] I. Kashuba, I. Shestakov, *Jordan Algebras of Dimension Three: Geometric Classification and Representation Type,* Actas del XVI Coloquio Latinoamericano de Álgebra, Biblioteca de la Revista Matemática Iberoamericana, (2007), 295-315.

[13] I. Kashuba, V. Serganova, *On the Tits-Kantor-Koecher construction of unital Jordan bimodules,* J. Algebra **481**, (2017), 420-463..

[14] L. Nazarova, *Representations of quivers of infinite type,* Izv. Akad. Nauk SSSR Ser. Mat. **37**, (1973), 752-791.

[15] C. M. Ringel, *The Representation Type of Local Algebras,* Springer Lecture Notes **488**, (1975), 282-305.

Contact information

Iryna Kashuba, Ivan Shestakov

Instituto de Matemática e Estatística, Universidade de São Paulo, R. do Matão 1010, São Paulo 05311-970, Brazil
E-Mail(s): ikashuba@gmail.com,
ivan.shestakov@gmail.com

Serge Ovsienko

Faculty of Mechanics and Mathematics, Kiev Taras Shevchenko University, Volodymyrska St, 60, Kyiv, 01033 Ukraine

Received by the editors: 28.03.2017.