Algebra and Discrete Mathematics
Volume **23** (2017). Number 2, pp. 279–284

RESEARCH ARTICLE

On recurrence in G-spaces

Igor Protasov and Ksenia Protasova

To the memory of Vitaly Sushchansky

Abstract. We introduce and analyze the following general concept of recurrence. Let G be a group and let X be a G-space with the action $G \times X \longrightarrow X$, $(g, x) \longmapsto gx$. For a family $\mathfrak{F}$ of subset of X and $A \in \mathfrak{F}$, we denote $\Delta_{\mathfrak{F}}(A) = \{g \in G : gB \subseteq A$ for some $B \in \mathfrak{F}$, $B \subseteq A\}$, and say that a subset R of G is $\mathfrak{F}$-recurrent if $R \bigcap \Delta_{\mathfrak{F}}(A) \neq \varnothing$ for each $A \in \mathfrak{F}$.

Let G be a group with the identity e and let X be a G-space, a set with the action $G \times X \longrightarrow X$, $(g, x) \longmapsto gx$. If $X = G$ and gx is the product of g and x then X is called a left regular G-space.

Given a G-space X, a family $\mathfrak{F}$ of subset of X and $A \in \mathfrak{F}$, we denote

$$\Delta_{\mathfrak{F}}(A) = \{g \in G : gB \subseteq A \text{ for some } B \in \mathfrak{F}, B \subseteq A\}.$$

Clearly, $e \in \Delta_{\mathfrak{F}}(A)$ and if $\mathfrak{F}$ is upward directed ($A \in \mathfrak{F}$, $A \subseteq C$ imply $C \in \mathfrak{F}$) and if $\mathfrak{F}$ is G-invariant ($A \in \mathfrak{F}$, $g \in G$ imply $gA \in \mathfrak{F}$) then

$$\Delta_{\mathfrak{F}}(A) = \{g \in G : gA \cap A \in \mathfrak{F}\}, \qquad \Delta_{\mathfrak{F}}(A) = (\Delta_{\mathfrak{F}}(A))^{-1}.$$

If X is a left regular G-space and $\varnothing \notin \mathfrak{F}$ then $\Delta_{\mathfrak{F}}(A) \subseteq AA^{-1}$.

For a G-space X and a family $\mathfrak{F}$ of subsets of X, we say that a subset R of G is $\mathfrak{F}$-*recurrent* if $\Delta_{\mathfrak{F}}(A) \cap R \neq \varnothing$ for every $A \in \mathfrak{F}$. We denote by $\mathfrak{R}_{\mathfrak{F}}$ the filter on G with the base $\cap\{\Delta_{\mathfrak{F}}(A) : A \in \mathfrak{F}'\}$, where $\mathfrak{F}'$ is a finite subfamily of $\mathfrak{F}$, and note that, for an ultrafilter p on G, $\mathfrak{R}_{\mathfrak{F}} \in p$ if and only if each member of p is $\mathfrak{F}$-recurrent.

2010 MSC: 37A05, 22A15, 03E05.

Key words and phrases: G-space, recurrent subset, ultrafilters, Stone-Čech compactification.

The notion of an $\mathfrak{F}$-recurrent subset is well-known in the case in which G is an amenable group, X is a left regular G-space and $\mathfrak{F} = \{A \subseteq X : \mu(A) > 0$ for some left invariant Banach measure μ on $X\}$. See [1] and [2] for historical background.

Now we endow G with the discrete topology and identity the Stone-Čech compactification βG of G with the set of all ultrafilters on G. Then the family $\{\overline{A} : A \subseteq G\}$, where $\overline{A} = \{p \in \beta G : A \in p\}$, forms a base for the topology of βG. Given a filter φ on G, we denote $\overline{\varphi} = \cap\{\overline{A} : A \in \varphi\}$.

We use the standard extension [3] of the multiplication on G to the semigroup multiplication on βG. We take two ultrafilters $p, q \in \beta G$, choose $P \in p$ and, for each $x \in P$, pick $Q_x \in q$. Then $\cup_{x \in P} x Q_x \in pq$ and the family of these subsets forms a base of the ultrafilter pq.

We recall [4] that a filter φ on a group G is *left topological* if φ is a base at the identity e for some (uniquely at defined) left translation invariant (each left shift $x \longmapsto gx$ is continuous) topology on G. If φ is left topological then $\overline{\varphi}$ is a subsemigroup of βG [4]. If $G = X$ and a filter φ is left topological then $\varphi = \mathfrak{R}_{\varphi}$.

Proposition 1. *For every G-space X and any family $\mathfrak{F}$ of subsets of X, the filter $\mathfrak{R}_{\mathfrak{F}}$ is left topological.*

Proof. By [4], a filter φ on a group G is left topological if and only if, for every $\Phi \in \varphi$, there is $H \in \varphi$, $H \subseteq \Phi$ such that, for every $x \in H$, $xH_x \subseteq \Phi$ for some $H_x \in \varphi$.

We take an arbitrary $A \in \mathfrak{F}$, put $\Phi = \triangle_{\mathfrak{F}}(A)$ and, for each $g \in \triangle_{\mathfrak{F}}(A)$, choose $B_g \in \mathfrak{F}$ such that $gB_g \in A$. Then $g\triangle_{\mathfrak{F}}(B_g) \subseteq \triangle_{\mathfrak{F}}(A)$ so put $H = \Phi$.

To conclude the proof, let $A_1, \dots, A_n \in \mathfrak{F}$. We denote

$$\Phi_1 = \triangle_{\mathfrak{F}}(A_1), \quad \dots, \quad \Phi_n = \triangle_{\mathfrak{F}}(A_n), \quad \Phi = \Phi_1 \cap \dots \cap \Phi_n.$$

We use the above paragraph, to choose $H_1, \dots, H_n$ corresponding to $\Phi_1, \dots, \Phi_n$ and put $H = H_1 \cap \dots \cap H_n$. □

Let X be a G-space and let $\mathfrak{F}$ be a family of subsets of X. We say that a family $\mathfrak{F}'$ of subsets of X is $\mathfrak{F}$-*disjoint* if $A \cap B \notin \mathfrak{F}$ for any distinct $A, B \in \mathfrak{F}'$.

A family $\mathfrak{F}'$ of subsets of X is called $\mathfrak{F}$-*packing large* if, for each $A \in \mathfrak{F}'$, any $\mathfrak{F}$-disjoint family of subsets of X of the form gA, $g \in G$ is finite.

We say that a subset S of a group G is a $\triangle_\omega$-*set* if $e \in A$ and every infinite subset Y of G contains two distinct elements x, y such that $x^{-1}y \in S$ and $y^{-1}x \in S$.

Proposition 2. *Let X be a G-space and let $\mathfrak{F}$ be a G-invariant upward directed family of subsets of X. Then $\mathfrak{F}$ is $\mathfrak{F}$-packing large if and only if, for each $A \in \mathfrak{F}$, the subset $\triangle_{\mathfrak{F}}(A)$ of G is a $\triangle_\omega$-set.*

Proof. We assume that $\mathfrak{F}$ is $\mathfrak{F}$-packing large and take an arbitrary infinite subset Y of G. Then we choose distinct $g, h \in Y$ such that $gA \cap hA \in \mathfrak{F}$, so $g^{-1}h \in \triangle_{\mathfrak{F}}(A)$, $hg \in \triangle_{\mathfrak{F}}(A)$ and $\triangle_{\mathfrak{F}}(A)$ is a $\triangle_\omega$-set.

Now we suppose that $\triangle_{\mathfrak{F}}(A)$ is a $\triangle_\omega$-set and take an arbitrary infinite subset Y of G. Then there are distinct $g, h \in Y$ such that $g^{-1}h \in \triangle_{\mathfrak{F}}(A)$ so $g^{-1}hA \cap A \in \mathfrak{F}$ and $gA \cap hA \in \mathfrak{F}$. It follows that the family $\{gA : g \in Y\}$ is not $\mathfrak{F}$-disjoint. $\square$

Proposition 3. *For every infinite group G, the following statements hold*

(i) *a subset $A \subseteq G$ is a $\triangle_\omega$-set if and only if $e \in A$ and every infinite subset Y of G contains an infinite subset Z such that $x^{-1}y \in A$, $y^{-1}x \in A$ for any distinct $x, y \in Z$;*

(ii) *the family φ of all $\triangle_\omega$-sets of G is a filter;*

(iii) *if $A \in \varphi$ then $G = FA$ for some finite subset F of G.*

Proof. (i) We assume that A is a $\triangle_\omega$-set and define a coloring χ of $[Y]^2$, $\chi : [Y]^2 \longrightarrow \{0, 1\}$ by the rule: $\chi(\{x, y\}) = 1$ if and only if $x^{-1}y \in A$, $y^{-1}x \in A$. By the Ramsey theorem, there is an infinite subset Z of Y such that χ is monochrome on $[Z]^2$. Since A is a $\triangle_\omega$-set $\chi(\{x, y\}) = 1$ for all $\{x, y\} \in [Z]^2$.

(ii) follows from (i).

(iii) We assume the contrary and choose an injective sequence $(x_n)_{n\in\omega}$ in G such that $x_{n+1} \notin x_i A$ for each $i \in \{0, \dots, n\}$, and denote $Y = \{x_n : n \in \omega\}$. Then $x_m^{-1}x_n \in A$ for every m, n, $m < n$, so A is not a $\triangle_\omega$-set. $\square$

Proposition 4. *Let G be a infinite group and let φ denotes the filter of all $\triangle_\omega$-sets of G. Then $\overline{\varphi}$ is the smallest closed subset of βG containing all ultrafilters on G of the form $q^{-1}q$, $q \in \beta G$, $g^{-1} = \{A^{-1} : A \in q\}$.*

Proof. We denote by Q the smallest closed subset of βG containing all $q^{-1}q$, $q \in \beta G$. It follows directly from the definition of the multiplication in βG that $p \in Q$ if and only if either p is principal and $p = e$ or, for each $P \in p$, there is an injective sequence $(x_n)_{n\in\omega}$ in G such that $x_m^{-1}x_n \in P$ for all $m < n$.

Applying Proposition 3(i), we conclude that $q^{-1}q \in \overline{\varphi}$ for each $q \in \beta G$ so $Q \subseteq \overline{\varphi}$. On the other hand, if $p \notin \overline{\varphi}$ then there is $P \in p$ such that $G \setminus P$ is a $\triangle_\omega$-set. By above paragraph, $p \notin Q$ so $\overline{\varphi} \subseteq Q$. □

Now let G be an amenable group, X be a left regular G-space and $\mathfrak{F} = \{A \in X : \mu(A) > 0$ for some left invariant Banach measure μ on $G\}$. For combinatorial characterization of $\mathfrak{F}$ see [6]. Clearly, $\mathfrak{F}$ is upward directed G-invariant and $\mathfrak{F}$-packing large. By Proposition 2, $\overline{\varphi} \subseteq \overline{\mathfrak{R}_{\mathfrak{F}}}$. By Proposition 4, $\overline{\mathfrak{R}_{\mathfrak{F}}}$ contains all ultrafilters of the form $q^{-1}q$, $q \in \beta G$, so we get Theorem 3.14 from [1].

We suppose that a G-space X is endowed with a G-invariant probability measure μ defined on some ring of subsets of X. Then the family $\mathfrak{F}\{A \subseteq X : \mu(B) > 0$ for some $B \subseteq A\}$ is $\mathfrak{F}$-packing large.

In particular, we can take a compact group X, endow X with the Haar measure, choose an arbitrary subgroup G of X and endow G with the discrete topology.

Another example: let a discrete group G acts on a topological space X so that, for each $g \in G$, the mapping $X \longrightarrow X$, $(g, x) \longmapsto gx$ is continuous. We take a point $x \in X$, denote by $\mathfrak{F}$ the filter of all neighborhoods of x, and recall that x is *recurrent* if, for every $U \in \mathfrak{F}$, there exists $g \in G \setminus \{e\}$ such that $gx \in U$. Clearly, x is a recurrent point if and only if $G \setminus \{e\}$ if a set of $\mathfrak{F}$-recurrence, so by Proposition 1, x defines some non-discrete left translation invariant topology on G.

Proposition 5. *Let G be a infinite group, A be a $\triangle_\omega$-set of G and let τ be a left translation invariant topology on G with continuous inversion $x \longmapsto x^{-1}$ at the identity e. Then the closure $cl_\tau A$ is a neighborhood of e in τ.*

Proof. On the contrary, we suppose that $cl_\tau A$ is not a neighborhood of e, put $U = G \setminus cl_\tau A$. Then U is open and $e \in cl_\tau U$.

We take an arbitrary $x_0 \in U$ and choose an open neighborhood U_0 of the identity such that $x_0 U_0^{-1} \subseteq U$. Then we take $x_1 \in U_0 \cap U$ and choose an open neighborhood U_1 of e such that $U_1 \subseteq U_0$ and $x_1 U_1^{-1} \subseteq U$. We take $x_2 \in U_1 \cap U$ and choose an open neighborhood U_0 of e such that $U_2 \subseteq U_1$ and $x_2 U_2^{-1} \subseteq U$ and so on. After ω steps, we get a sequence $(x_n)_{n \in \omega}$ in G such that $x_n x_m^{-1} \in U$ for all $n < m$. We denote $Y = \{x_n^{-1} : n \in \omega\}$. Then $(x_n^{-1})^{-1} x_m^{-1} \in A$ for all $n < m$, so A is not a $\triangle_\omega$-set. □

A subset A of an infinite group G is called a $\triangle_{<\omega}$*-set* if $e \in A$ and there exists a natural number n such that every subset Y of G, $\mid Y \mid = n$

contains two distinct $x, y \in Y$ such that $x^{-1}y \in A$, $y^{-1}x \in A$. These subsets were introduced in [5] under name thick subsets, but thick subsets are well-known in combinatorics with another meaning [3]: A is thick if, for every finite subset F of, there is $g \in A$ such that $Fg \subseteq A$. The family ψ of all $\triangle_{<\omega}$-sets of G is a filter [5], clearly, $\psi \subseteq \varphi$. Every infinite group G has a $\triangle_\omega$-set but not $\triangle_{<\omega}$-set A: it suffices to choose inductively a sequence $(X_n)_{n\in\omega}$ of subsets of G, $\mid X_n \mid= n$ such that $\bigcup_{n\in\omega} X_n^{-1}X_n$ has no infinite subsets of the form $Y^{-1}Y$ and put

$$A = \{e\} \cup (G \setminus \bigcup_{n\in\omega} X_n^{-1}X_n),$$

so $\psi \subset \varphi$.

By analogy with Propositions 3 and 4, we can prove

Proposition 6. *Let G be an infinite group and let ψ be the filter of all $\triangle_{<\omega}$-subsets of G. Then $p \in \overline{\psi}$ if and only if either p is principal and $p = e$ or, for every $A \in p$, there exists a sequence $(X_n)_{n\in\omega}$ of subsets of G, $|X_n| = n + 1$, $X_n = \{x_{n0}, \dots, x_{nn}\}$ such that $x_{ni}^{-1}x_nj \in A$ for all $i < j \leqslant n$.*

Let A be a subset of a group G such that $e \in A$, $A = A^{-1}$. We consider the Cayley graph Γ_A with the set of vertices G and the set of edges $\{\{x, y\} : x^{-1}y \in A, x \neq y\}$. We recall that a subset S of vertices of a graph is *independent* if any two distinct vertices from S are not incident. Clearly, A is a $\triangle_\omega$-set if and only if any independent set in Γ_A is finite, and A is $\triangle_\omega$-set if and only if there exists a natural number n such that any independent set S is of size $|S| < n$.

Problem 1. Characterize all infinite graphs with only finite independent set of vertices.

Problem 2. Given a natural number n, characterize all infinite graphs such that any independent set S of vertices is of size $|S| < n$.

In the context of this note, above problems are especially interesting in the case of Cayley graphs of groups.

References

[1] V. Bergelson, N. Hindman,*Quotient sets and density recurrent sets*, Trans. Amer. Math. Soc. **364** (2012), 4495-4531.

[2] H. Furstenberg, *Poincare recurrence and number theory*, Bulletin Amer. Math. Soc. **5.3** (1981),211-234.

[3] N. Hindman, D. Strauss, Algebra in the Stone-Čech compactification, de Gruyter, Berlin, 1998.

[4] I. Protasov, *Filters and topologies on groups*, Mat. Stud. **3** (1994),15-28.

[5] E. Reznichenko, O. Sipacheva, *Discrete subsets in topologicaln groups and countable extremally disconnected groups*, preprint (arxiv: 1608, 03546v2) 2016.

[6] P. Zakrzewski, *On the complexity of the ideal of absolute null sets*, Ukr. Math. J. **64** (2012),306-308.

Contact information

I. Protasov,
K. Protasova

Kyiv University, Department of Cybernetics, Kyiv National University, Volodimirska 64, Kyiv 01033, Ukraine
E-Mail(s): i.v.protasov@gmail.com, k.d.ushakova@gmail.com

Received by the editors: 04.02.2017.