

# A survey article on some subgroup embeddings and local properties for soluble PST-groups

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**ABSTRACT.** Let  $G$  be a group and  $p$  a prime number.  $G$  is said to be a  $Y_p$ -group if whenever  $K$  is a  $p$ -subgroup of  $G$  then every subgroup of  $K$  is an S-permutable subgroup in  $N_G(K)$ . The group  $G$  is a soluble PST-group if and only if  $G$  is a  $Y_p$ -group for all primes  $p$ .

One of our purposes here is to define a number of local properties related to  $Y_p$  which lead to several new characterizations of soluble PST-groups. Another purpose is to define several embedding subgroup properties which yield some new classes of soluble PST-groups. Such properties include weakly S-permutable subgroup, weakly semipermutable subgroup, and weakly seminormal subgroup.

## 1. Introduction and statement of results

All groups considered in this survey are finite.

A subgroup  $H$  of a group  $G$  is said to permute with a subgroup  $K$  of  $G$  if  $HK$  is a subgroup of  $G$ .  $H$  is said to be permutable (S-permutable) if it permutes with all the subgroups (Sylow subgroups, respectively) of  $G$ . Examples of permutable subgroups include the normal subgroups of  $G$ . However, if  $G$  is a modular, non-Dedekind  $p$ -group,  $p$  a prime, we see permutability is quite different from normality. For instance, letting  $C_n$  denote the cyclic group of order  $n$ , we see that  $C_2$  is permutable but not

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normal in the group  $C_8 \rtimes C_2$  where the generator for  $C_2$  maps a generator of  $C_8$  to its fifth power. Kegel [12] proved that an S-permutable subgroup is always subnormal. In particular, a permutable subgroup of a group is subnormal.

A group  $G$  is called a PST-group (PT-group) if S-permutability (permutability, respectively) is a transitive relation. By Kegel's result, a group  $G$  is a PST-group (PT-group) if every subnormal subgroup of  $G$  is S-permutable (permutable, respectively) in  $G$ .

A number of research papers have been written on these groups. See, for example, [1–10, 16].

Another class of groups is the so called T-groups. A group  $G$  is a T-group if normality in  $G$  is transitive, that is, if  $H \trianglelefteq K \trianglelefteq G$  then  $H \trianglelefteq G$ . There are several nice characterizations of T-groups in [15].

Soluble PST-, PT- and T-groups were characterized by Agrawal [1], Zacher [18] and Gaschütz [11] respectively.

**Theorem 1.** 1) *A soluble group  $G$  is a PST-group if and only if the nilpotent residual  $L$  of  $G$  is an abelian Hall subgroup of  $G$  on which  $G$  acts by conjugation as a group of power automorphisms.*  
 2) *A soluble PST-group  $G$  is a PT-group (T-group) if and only if  $G/L$  is a modular (Dedekind, respectively) group.*

Note that if  $G$  is a soluble T-, PT- or PST-group then every subgroup and every quotient of  $G$  inherits the same properties.

We mention that in [6, Chapter 2] many of the beautiful results on these classes of groups are presented.

Subgroup embedding properties closely related to permutability and S-permutability are semipermutability and S-semipermutability. A subgroup  $X$  of a group  $G$  is said to be *semipermutable* (*S-semipermutable*) in  $G$  provided that it permutes with every subgroup (Sylow subgroup, respectively)  $K$  of  $G$  such that  $\gcd(|X|, |K|) = 1$ . A semipermutable subgroup of a group need not be subnormal. For example, a 2-Sylow subgroup of the non-abelian group of order 6 is semipermutable but not subnormal.

Note that a subnormal semipermutable (S-semipermutable) subgroup  $X$  of a group  $G$  must be normalized by every subgroup (Sylow subgroup, respectively)  $P$  of  $G$  such that  $\gcd(|X|, |P|) = 1$ . This observation was the basis for Beidleman and Ragland [10] to introduce the following subgroup embedding properties.

A subgroup  $X$  of a group  $G$  is said to be *seminormal* ( $S$ -*seminormal*)<sup>1</sup> in  $G$  if it is normalized by every subgroup (Sylow subgroup, respectively)  $K$  of  $G$  such that  $\gcd(|X|, |K|) = 1$ .

By [10, Theorem 1.2], a subgroup of a group is seminormal if and only if it is S-seminormal. Furthermore, seminormal subgroups are not necessarily subnormal; it is enough to consider a non-subnormal subgroup  $H$  of a group  $G$  such that  $\pi(H) = \pi(G)$ . To see some of the properties of these subgroups see Examples 1, 2 and 3 in Section 2. However, a  $p$ -subgroup of a group  $G$ ,  $p$  a prime, which is also seminormal is subnormal ([10, Theorem 1.3]).

Semipermutable, S-semipermutable and seminormal subgroups have been investigated in [10, 17, 19, 20].

The following result gives some embedding properties on subnormal subgroups of a group which yields several characterizations of soluble PST-groups.

**Theorem 2** ([10, Theorem 1.5]). *Let  $G$  be a soluble group. Then the following statements are pairwise equivalent:*

- 1)  $G$  is a PST-group;
- 2) all the subnormal subgroups of  $G$  are seminormal in  $G$ ;
- 3) all the subnormal subgroups of  $G$  are semipermutable in  $G$ ;
- 4) all the subnormal subgroups of  $G$  are  $S$ -semipermutable in  $G$ .

The following is a beautiful result of H. Wielandt which seems to have inspired the authors of [5] to introduce the concept of weakly S-permutable subgroups of a subgroup  $H$  of a group  $G$ .

**Theorem 3** ([13, Theorem 7.3.3]). *Let  $H$  be a subgroup of a group  $G$ . Then the following statements are equivalent:*

- 1)  $H$  is subnormal in  $G$ ;
- 2)  $H$  is subnormal in  $\langle H, H^g \rangle$  for all  $g \in G$ ;
- 3)  $H$  is subnormal in  $\langle H, g \rangle$  for all  $g \in G$ .

This embedding property led to several new characterizations of soluble PST-groups which are presented in the following theorem from [5].

**Theorem 4** ([5]). *Let  $G$  be a group. The following statements are pairwise equivalent:*

- 1)  $G$  is a soluble PST-group.
- 2) Every subgroup of  $G$  is weakly  $S$ -permutable in  $G$ .

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<sup>1</sup>Note that the term *seminormal* has several different meanings in the literature.

- 3) For every prime number  $p$ , every  $p$ -subgroup of  $G$  is weakly  $S$ -permutable in  $G$ .

Theorems 3 and 4 motivate the following definition.

**Definition 1.** Let  $H$  be a subgroup of a group  $G$ .

- 1)  $H$  is said to be weakly  $S$ -permutable in  $G$  if whenever  $g \in G$  and  $H$  is  $S$ -permutable in  $\langle H, H^g \rangle$ , then  $H$  is  $S$ -permutable in  $\langle H, g \rangle$ .
- 2)  $H$  is said to be weakly semipermutable in  $G$  if whenever  $g \in G$  and  $H$  is semipermutable in  $\langle H, H^g \rangle$ , then  $H$  is semipermutable in  $\langle H, g \rangle$ .
- 3)  $H$  is said to be weakly  $S$ -semipermutable in  $G$  if whenever  $g \in G$  and  $H$  is  $S$ -semipermutable in  $\langle H, H^g \rangle$ , then  $H$  is  $S$ -semipermutable in  $\langle H, g \rangle$ .
- 4)  $H$  is said to be weakly seminormal in  $G$  if whenever  $g \in G$  and  $H$  is seminormal in  $\langle H, H^g \rangle$ , then  $H$  is seminormal in  $\langle H, g \rangle$ .

The next theorem relates the concept of  $S$ -permutable subgroups of a group  $G$  with weakly  $S$ -permutable subgroups of  $G$ .

**Theorem 5** ([5]). *A subgroup  $H$  of a group  $G$  is  $S$ -permutable in  $G$  if and only if  $H$  is  $S$ -permutable in  $\langle H, g \rangle$  for every  $g \in G$ .*

Theorem 5 and its proof are used to establish Theorem 6 in [7].

**Theorem 6** ([7]). *Let  $H$  be a subnormal subgroup of a group  $G$ . Then*

- 1)  $H$  is  $S$ -semipermutable in  $G$  if and only if  $H$  is  $S$ -semipermutable in  $\langle H, g \rangle$  for every  $g \in G$ .
- 2)  $H$  is seminormal in  $G$  if and only if  $H$  is seminormal in  $\langle H, g \rangle$  for every  $g \in G$ .

A class of groups  $G$  is a PST-group if and only if Sylow permutability is a transitive relation in  $G$ .

We next define several local properties which provide a number of new local characterizations of soluble PST-groups.

**Definition 2.** Let  $G$  be a group and  $p$  be a prime. Then

- 1)  $G$  is a  $Y_p$ -group if for every  $p$ -subgroup  $K$  of  $G$  every subgroup of  $K$  is  $S$ -permutable in  $N_G(K)$ .
- 2)  $G$  is a  $\hat{Y}_p$ -group if for every  $p$ -subgroup  $K$  of  $G$  every subgroup of  $K$  is semipermutable in  $N_G(K)$ .
- 3)  $G$  is a  $\tilde{Y}_p$ -group if for every  $p$ -subgroup  $K$  of  $G$  every subgroup of  $K$  is  $S$ -semipermutable in  $N_G(K)$ .

- 4)  $G$  is a  $\tilde{Y}_p$ -group if for every  $p$ -subgroup  $K$  of  $G$  every subgroup of  $K$  is seminormal in  $N_G(K)$ .
- 5)  $G$  is a  $\underline{Y}_p$ -group if for every  $p$ -subgroup  $K$  of  $G$  every subgroup of  $K$  is weakly S-permutable in  $N_G(K)$ .
- 6)  $G$  is a  $\tilde{Y}_p$ -group if for every  $p$ -subgroup  $K$  of  $G$  every subgroup of  $K$  is weakly S-semipermutable in  $N_G(K)$ .
- 7)  $G$  is a  $\tilde{\underline{Y}}_p$ -group if for every  $p$ -subgroup  $K$  of  $G$  every subgroup of  $K$  is weakly seminormal in  $N_G(K)$ .

The following result is a very nice local characterization of soluble PST-groups.

**Theorem 7** ([6, Theorem 2.2.9] and [4, Theorem 4]). *A group  $G$  is a soluble PST-group if and only if it satisfies  $Y_p$  for all primes  $p$ .*

Our next result shows how some of the classes in Definition 2 are related to the class  $Y_p$ .

**Theorem 8** ([10, Theorem 1.8]). *Let  $p$  be a prime and  $G$  a group. Then  $Y_p = \hat{Y}_p = \tilde{Y}_p = \tilde{\tilde{Y}}_p$ .*

Using Theorems 7 and 8 we note that the next result shows all of the classes  $\underline{Y}_p$ ,  $\tilde{\underline{Y}}_p$  and  $\tilde{\tilde{Y}}_p$  are just  $Y_p$ .

**Theorem 9** ([7]). *Let  $p$  be a prime and  $G$  a group. Then*

- 1)  $G \in Y_p$  if and only if  $G \in \underline{Y}_p$ .
- 2)  $G \in \tilde{Y}_p$  if and only if  $G \in \tilde{\underline{Y}}_p$ .
- 3)  $G \in \tilde{\tilde{Y}}_p$  if and only if  $G \in \tilde{\tilde{\underline{Y}}}_p$ .

From Theorems 8 and 9 we obtain several results that yield new local characterizations of soluble PST-groups.

**Corollary 1.** *Let  $p$  be a prime. Then*

$$Y_p = \underline{Y}_p = \hat{Y}_p = \tilde{Y}_p = \tilde{\underline{Y}}_p = \tilde{\tilde{Y}}_p = \tilde{\tilde{\underline{Y}}}_p.$$

Using Theorem 9 and Corollary 1 we obtain one of the main results of this survey paper.

**Theorem 10** ([7]). *Let  $G$  be a group and  $p$  a prime. Then the following statements are pairwise equivalent:*

- 1)  $G$  is a soluble PST-group.
- 2)  $G$  is a  $Y_p$ -group for all primes  $p$ .

- 3)  $G$  is a  $\underline{Y}_p$ -group for all primes  $p$ .
- 4)  $G$  is a  $\hat{Y}_p$ -group for all primes  $p$ .
- 5)  $G$  is a  $\check{Y}_p$ -group for all primes  $p$ .
- 6)  $G$  is a  $\tilde{Y}_p$ -group for all primes  $p$ .
- 7)  $G$  is a  $\tilde{\tilde{Y}}_p$ -group for all primes  $p$ .
- 8)  $G$  is a  $\underline{\tilde{Y}}_p$ -group for all primes  $p$ .

## 2. Examples

**Example 1.** Let  $S_4$ ,  $A_4$  and  $K_4$  denote, respectively, the symmetric group of order 4, the alternating group of order 4, and the Klein 4-group. Let  $G = S_4$  and let  $H = \langle (123) \rangle$ . The  $H$  is S-semipermutable in  $G$  but it is not semipermutable in  $G$  since it does not permute with an element of order 2 in  $K_4$ , the Sylow 2-subgroup of  $A_4$ .

An S-permutable subgroup of a group is subnormal. That this is not the case with S-semipermutable subgroups can be seen in the subgroup  $H$  in  $S_4$ . Notice that  $H$  is not seminormal in  $S_4$ .

**Example 2.** Let  $D_{10} = \langle x, y \mid x^5 = y^2 = 1, x^y = x^{-1} \rangle$ , the dihedral group of order 10, and  $C_{15} = \langle t, s \mid t^5 = s^3 = 1, ts = st \rangle$ , the cyclic group of order 15. Let  $G = D_{10} \times C_{15}$  and let  $K = \langle y \rangle \times \langle t \rangle$ . Since  $\langle s \rangle$  centralizes  $K$  it follows that  $K$  is seminormal in  $G$ . Note that  $K$  is not subnormal in  $G$ .

**Example 3.** Let  $H = \langle x \rangle \times \langle y \rangle$  be a semidirect product of a cyclic group,  $\langle x \rangle$ , of order 11 by a cyclic group,  $\langle y \rangle$ , of order 5. Let  $G = H \times S_4$ . Set  $K = \langle x \rangle \times S_3$  where  $S_3$  is a copy of the symmetric group on three elements in  $S_4$ . Then  $K$  is a seminormal subgroup of  $G$  which is not subnormal.

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