УДК 539.3

The investigation of the deformations of the elastic bodies with thin coating using D-adaptive finite element model

Olha Kossak[1], Yarema Savula[2]

[1] Doctor of Sciences, I. Franko National University of Lviv, 1, Universytetska Str., Lviv, e-mail: olhakossak@yahoo.com
[2] Doctor of Sciences, Professor, I. Franko National University of Lviv, 1, Universytetska Str., Lviv, e-mail: savula@franko.lviv.ua

A D-adaptive mathematical model that combines elastic structures with a thin coating is analysed. The main idea of this approach is based on the formulation of the combined model, which allows the use of 3-D elasticity theory model over one part of the domain, and 2-D Timoshenko's shell model over the other part. The boundary and variation formulation of the D-adaptive model problem are presented. The numerical results are compared with available exact solutions. The numerical results demonstrate the applicability of this approach.

Keywords: D-adaptive model, thin coating, finite element method, Timoshenko's shell model, 3-D elasticity theory model.

Introduction. The purpose of this paper is to present a new approach for the formulation of the D-adaptive [1] mathematical models (combined models) of the structures elastic deformation with thin surface coating. In papers [2, 3] the model of a thin coating model is obtained from 3-D elasticity theory equations by passing to the limit with thin coating thickness tending to zero.

The main idea of our approach is based on the formulation of the combined model, which permits the use of 3-D elasticity theory model over one part of the domain, and 2-D Timoshenko's shell model over the other part. The differential equations of the system are interconnected by special boundary conditions — junction conditions. The numerical investigations of the problems, which are described by combined mathematical models, are performed by means of finite element method (FEM) [4, 5]. The numerical results from this method are checked with the available exact solutions reported by other authors.

1. Theory

Let us consider an elastic continuum within a domain $\Omega \in R^3$, which consists of two parts $\Omega = \Omega_1 \cup \Omega_2^*$. We have regarded Ω_1 as an arbitrary 3-D domain with a Lipschitzian boundary $G = G_1 \cup G_2 \cup G_3$. Let Ω_2^* be a 3-D domain limited by two surfaces Ω_2^-, Ω_2^+. Denote by h the distance between Ω_2^-, Ω_2^+ (thickness), which is much less than other measurements of Ω_2^*, and by Ω_2^* — the middle surface. The surface G_3 coincides with Ω_2^-. Suppose that curvilinear orthogonal coordinates $\xi = (\xi_1, \xi_2, \xi_3)$ are determined in the domain Ω_1. Let us determine an orthogonal basis $\overline{\nu} = (\overline{\nu}_1, \overline{\nu}_2, \overline{\nu}_3)$ on its boundary *G*, where $\overline{\nu}_3$ is the unit normal to the surface. The radius-vector, which describes the points of the domain Ω_1, can be given in the form $\overline{R} = \overline{R}(\xi_1, \xi_2, \xi_3) \in \Omega_1$.

The Ω_2^* domain is described by the curvilinear orthogonal coordinates $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. The radius vector can be given in the form

$$\overline{r}(\alpha_1, \alpha_2, \alpha_3) = \overline{r}_0(\alpha_1, \alpha_2) + \alpha_3 \overline{n}(\alpha_1, \alpha_2) \in \Omega_2^*, (\alpha_1, \alpha_2) \in \Omega_2, \alpha_3 \in \left] -\frac{h}{2}, \frac{h}{2} \right[,$$

where $\overline{r}_0$ is radius-vector which determines surface Ω_2^*, $\overline{n}$ is the unit normal to this surface. At each point of the Ω_2^* the main orthogonal basis $(\overline{e}_1, \overline{e}_2, \overline{n})$ is determined. Here $\overline{e}_1$ and $\overline{e}_2$ are vectors which are directed along the α_1, α_2 directions respectively. The unit vectors $\overline{\nu}_1, \overline{\nu}_2$ coincide with vectors $\overline{e}_1, \overline{e}_2$. The normal to the middle surface coincides with $\overline{\nu}_3$ and forms α_3 axis. The Ω_2^* has a Lipschitzian boundary $\Gamma = \Gamma_1 \cup \Gamma_2$. The orthogonal pair of the unit vectors $\overline{t} = (\overline{t}_1, \overline{t}_2)$ on the middle surface boundary is determined. Here $\overline{t}_1$ is the outward normal to the boundary; $\overline{t}_2$ is the tangent vector.

The stress-strain state of the continuum within the Ω_1 domain can be written in terms of the 3-D linear elasticity theory. Three equations of stress equilibrium are of the following form for $i = \overline{1,3}$; $k \neq i$; $\xi \in \Omega$ [6, 7]

$$\frac{1}{H_1 H_2 H_3} \sum_{\beta=1}^{3} \frac{\partial}{\partial \xi_\beta} \left(\frac{H_1 H_2 H_3}{H_\beta} \sigma_{\beta i} \right) + \sum_{k=1}^{3} \frac{1}{H_i H_k} \frac{\partial H_i}{\partial \xi_k} \sigma_{ik} - \sum_{k=1}^{3} \frac{1}{H_i H_k} \frac{\partial H_k}{\partial \xi_i} \sigma_{kk} = 0. \quad (1)$$

Here H_i are the Lame's coefficients of the curvilinear coordinate system, σ_{ij} are the components of the stress tensor.

Constitutive equations that relate stresses and strains $e_{ij}\left(i, j = \overline{1,3}\right)$ for a linear elastic material are described by the generalized Hook's law

$$\sigma_{ij} = \sum_{k=1}^{3} \sum_{l=1}^{3} C_{ijkl} e_{kl}, \quad i, j = \overline{1,3}; \quad (2)$$

here C_{ijkl} are nonzero elastic constants for the isotropic homogeneous materials, where E_1, ν_1 are Young's modulus and Poisson's ratio of the elastic body.

The components of displacement along the directions ξ_1, ξ_2, ξ_3 are $U_i\left(i = \overline{1,3}\right)$. One can write down the strain - displacement relations, $\xi \in \Omega_1$, $k \neq i$

$$e_{ii} = \frac{1}{H_i}\frac{\partial U_i}{\partial \xi_i} + \sum_{k=1}^{3} \frac{1}{H_i H_k}\frac{\partial H_i}{\partial \xi_k} U_k, \; e_{ij} = \frac{H_j}{2H_i}\frac{\partial}{\partial \xi_i}\frac{U_j}{H_j} + \frac{H_i}{2H_j}\frac{\partial}{\partial \xi_j}\frac{U_i}{H_i}, \; i,j=\overline{1,3}. \quad (3)$$

The stress-strain state of the medium within the Ω_2^* domain can be written in terms of the Timoshenko's shell theory. Five equations of stress equilibrium are of the following form [8] for $i \neq j;\ i,j=1,2$

$$\frac{\partial\left(A_j T_i\right)}{A_i A_j \partial\alpha_i} - \frac{\partial\left(A_j\right)}{A_i A_j \partial\alpha_i} T_j + \frac{\partial\left(A_i^2 S\right)}{A_i^2 A_j \partial\alpha_j} + k_i Q_i + \frac{\partial\left(A_i k_i H\right)}{A_i A_j \partial\alpha_j} + \frac{k_j H \partial\left(A_i\right)}{A_i A_j \partial\alpha_j} + p_i = 0;$$

$$-k_1 T_1 - k_2 T_2 + \frac{\partial\left(A_2 Q_1\right)}{A_1 A_2 \partial\alpha_1} + \frac{\partial\left(A_1 Q_2\right)}{A_1 A_2 \partial\alpha_2} + p_3 = 0,$$

$$-Q_i + \frac{\partial\left(A_j M_i\right)}{A_i A_j \partial\alpha_i} - \frac{\partial\left(A_j\right)}{A_i A_j \partial\alpha_i} M_j + \frac{\partial\left(A_i^2 H\right)}{A_i^2 A_j \partial\alpha_j} + m_i = 0. \quad (4)$$

Here A_1, A_2 are the Lame's coefficients, k_1, k_2 are the principal curvatures of the Ω_2^* surface; $T_k, S, Q_k, M_k, H\left(k=1,2\right)$ denote the stresses and moments. Let us denoted by $p_i, m_k\left(i=\overline{1,3}\right)$ the surface loads; $p_i = p_i^+ + p_i^-$, $m_k = \frac{h}{2}\left(p_k^+ - p_k^-\right)$, where p_i^+, p_i^- are the components of the surface loads on Ω_2^-, Ω_2^+ surfaces respectively, which are related to the α_i coordinates. The stresses-moments and strains $\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23}, \chi_{11}, \chi_{22}, \chi_{12}$ relationship for the case of isotropic materials can be expressed in the terms of the Timoshenko's shells theory

$$T_k = B\left(\varepsilon_{kk} + \nu_2 \varepsilon_{ll}\right), \; S = B\frac{1-\nu_2}{2}\varepsilon_{12}, \; Q_k = G\varepsilon_{k3}, \; M_k = D\left(\chi_{kk} + \nu_2 \chi_{ll}\right),$$

$$H = D\frac{1-\nu_2}{2}\chi_{12}, \; B = \frac{E_2 h}{1-\nu_2^2}, \; D = \frac{E_2 h^3}{1-\nu_2^2}, \; G = \frac{5}{12}E_2 h\left(1+\nu_2\right), \; k \neq l; \; k,l = 1,2. \quad (5)$$

Let u_1, u_2, w are the displacement components of the middle-surface points in the $\alpha_1, \alpha_2, \alpha_3$ directions and γ_1, γ_2 are rotation angulars of a normal vector to the middle-surface in α_1, α_2 directions. The strain components are expressed below in terms of the middle surface displacements for $i,j=1,2;\ i \neq j$

$$\varepsilon_{ii} = \frac{\partial\left(u_i\right)}{A_i \partial\alpha_i} + \frac{\partial\left(A_i\right)}{A_i A_j \partial\alpha_j} u_j + k_i w; \quad \varepsilon_{12} = \frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\frac{u_1}{A_1} + \frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\frac{u_2}{A_2};$$

$$\varepsilon_{i3} = -k_i u_i + \frac{\partial(w)}{A_i \partial\alpha_i} + \gamma_i; \quad \chi_{ii} = \frac{\partial\left(\gamma_i\right)}{A_i \partial\alpha_i} + \frac{\partial\left(A_i\right)}{A_i A_j \partial\alpha_j}\gamma_j;$$

$$2\chi_{12}=\frac{k_1}{A_2}\frac{\partial u_1}{\partial\alpha_2}-\frac{k_2}{A_1A_2}\frac{\partial(A_1)}{\partial\alpha_2}u_1+\frac{k_2}{A_1}\frac{\partial u_2}{\partial\alpha_1}-\frac{k_1}{A_1A_2}\frac{\partial(A_2)}{\partial\alpha_1}u_2+$$
$$+\frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\frac{\gamma_1}{A_1}+\frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\frac{\gamma_2}{A_2}. \quad (6)$$

The formulas for stresses $\bar{\sigma}_{ij}\left(i,j=\overline{1,3}\right)$ can be found in [9].

On the boundaries of the Ω_1,Ω_2^* domains we can write down the kinematic and static boundary conditions respectively [9]

$$U_{\nu_i}=0;\quad i=\overline{1,3},\quad \xi\in G_1; \quad (7)$$
$$\sigma_{\nu_i\nu_3}=q_{\nu_i};\quad i=\overline{1,3};\quad \xi\in G_2; \quad (8)$$
$$u_{t_k}=0;\quad w=0;\quad \gamma_{t_k}=0;\quad k=1,2;\quad \alpha\in\Gamma_1; \quad (9)$$
$$T_{t_k}=0;\quad Q=0;\quad M_{t_k}=0;\quad k=1,2;\quad \alpha\in\Gamma_2. \quad (10)$$

The junction conditions on the surface G_3 express the continuity of the displacements of the medium and static equilibrium conditions

$$U_{\nu_i}=u_i-\frac{h}{2}\gamma_i,\quad U_3=w,\quad i=1,2,\quad G_3=\Omega_2^-; \quad (11)$$
$$\sigma_{\nu_j\nu_3}(U_1,U_2,U_3)=-p_j^-;\quad j=\overline{1,3},\quad G_3=\Omega_2^-. \quad (12)$$

Let's bring together all those relations (1)-(6), boundary conditions (7)-(10) and junction conditions (11)-(12) that will be needed for a D-adaptive model problem.

2. The variation statement of the problem

Let us denote and define

$$\bar{V}=\left(V\left(V_1(\xi),V_2(\xi),V_3(\xi)\right),\quad v\left(v_1(\alpha),v_2(\alpha),v_3(\alpha)\right),v_4(\alpha),v_5(\alpha)\right),$$
$$D_1=\left\{V:\quad V\in\left[W_2^{(1)}(\Omega_1)\right]^3,\quad V_i(\xi)=0,\quad i=\overline{1,3},\quad \xi\in G_1\right\};$$
$$D_2=\left\{v:\quad v\in\left[W_2^{(1)}(\Omega_2)\right]^5,\quad v_j(\alpha)=0,\quad j=\overline{1,5},\quad \alpha\in\Gamma_1\right\};$$
$$D=\left\{(V,v):V\in D_1,\ v\in D_2,\ V_{\nu_j}=v_j-\frac{h}{2}\nu_{j+3},\ V_3=v_3;\ j=1,2;\ \xi\in G_3;\ \alpha\in\Omega_2^*\right\}.$$

Let us examine the following variation boundary value problem. The functions $F=(F_1,F_2,F_3)\in H_1, f=(f_1,f_2,f_3,f_4,f_5)\in H_2, H_1=\left[L_2(\Omega_1)\right]^3, H_2=\left[L_2\left(\Omega_2^*\right)\right]^5$ are given. It is required to find the displacement vector $\bar{U}=(U,u)\in D$ (a weak solution of the D-adaptive problem) which satisfies correlation

$$A(\bar{U},\bar{V})=L(\bar{V})+B(\bar{U},\bar{V});\quad A(\bar{U},\bar{V})=\sum_{i=1}^{3}A_i(U,V_i)+\sum_{j=1}^{5}a_j(u,v_j); \tag{13}$$

$$L(\bar{V})=\sum_{i=1}^{3}L_i(V_i)+\sum_{j=1}^{5}l_j(v_j);\ B(\bar{U},\bar{V})=\sum_{i=1}^{3}B_i(U,V_i)+\sum_{j=1}^{5}b_j(u,v_j);$$

where

$$A_i(U,V_i)=\int\limits_{\Omega_1}\left(\sum_{\beta=1}^{3}\frac{\partial V_i}{H_\beta\partial\xi_\beta}\sigma_{\beta i}-\sum_{k=1}^{3}\frac{V_i}{H_iH_k}\frac{\partial H_i}{\partial\xi_k}\sigma_{ik}+\sum_{k=1}^{3}\frac{V_i}{H_iH_k}\frac{\partial H_k}{\partial\xi_i}\sigma_{kk}\right)d\Omega,$$

$$L_i(V_i)=\int\limits_{G_2}V_ip_{\nu_i}dG,\quad B_i(U,V_i)=\int\limits_{G_3}V_{\nu_i}\sigma_{\nu_i\nu_3}dG_3,\quad i=\overline{1,3},\quad k\neq i;$$

$$a_i(u,v_i)=\int\limits_{\Omega_2}\left(\frac{T_i\partial v_i}{A_i\partial\alpha_j}+\frac{v_jT_j\partial A_j}{A_iA_j\partial\alpha_i}+\frac{SA_i\partial}{A_j\partial\alpha_j}\left(\frac{v_i}{A_i}\right)-\right.$$

$$\left.-v_ik_iQ_i+\frac{k_iH\partial v_i}{A_j\partial\alpha_j}-\frac{k_jHv_i\partial A_i}{A_iA_j\partial\alpha_j}\right)d\Omega,$$

$$a_{i+3}(u,v_{i+3})=\int\limits_{\Omega_2}\left(v_{i+3}Q_i+\frac{M_i\partial(v_{i+3})}{A_i\partial\alpha_i}+\frac{v_{i+3}M_j\partial(A_j)}{A_iA_j\partial\alpha_i}+\frac{A_iH\partial}{A_j\partial\alpha_j}\left(\frac{v_{i+3}}{A_i}\right)\right)d\Omega,$$

$$a_3(u,v_3)=\int\limits_{\Omega_2}\left(\sum_{l=1}^{2}k_lT_lv_3+\frac{Q_2\partial(v_3)}{A_l\partial\alpha_l}\right)d\Omega,\quad i\neq j,\quad i,j=1,2;$$

$$l_k(v_k)=\int\limits_{\Omega_2}v_kf_kd\Omega,\quad b_k(u,v_k)=\int\limits_{\Omega_2}v_kf_k^-d\Omega,\quad k=\overline{1,5};$$

$$f_1=p_1^+,\quad f_2=p_2^+,\quad f_3=p_3^+,\quad f_4=\frac{h}{2}p_1^+,\quad f_5=\frac{h}{2}p_2^+,$$

$$f_1^-=p_1^-,\quad f_2^-=p_2^-,\quad f_3^-=p_3^-,\quad f_4^-=-\frac{h}{2}p_1^-,\quad f_5^-=-\frac{h}{2}p_2^-,$$

for each function $\bar{V}=(V,v)\in D$.

Let us consider now the sum

$$\sum_{i=1}^{3}B_i(U,V_i)+\sum_{j=1}^{5}b_j(u,v_j). \tag{14}$$

Taking into account the fact that $\bar{V}\in D$ and (11)-(12), we can now easily show that this sum equals zero. Hence, we get

$$\sum_{i=1}^{3} A_i\left(U,V_i\right)+\sum_{j=1}^{5} a_j\left(u,v_j\right)=\sum_{i=1}^{3} L_i\left(U,V_i\right)+\sum_{j=1}^{5} l_j\left(u,v_j\right). \tag{15}$$

Note that the conditions (8), (10), (12) are natural boundary conditions.

3. An application of the finite element method to the combined model

Let us consider the static combined problem, which deals with rotation bodies with thin coating. Cylindrical coordinates $\left(r,\phi,z\right)$ are naturally suited to such problems, with the z-axis being the axis of rotational symmetry and ϕ-axis being the circular axis. Put the shell (coating) domain into $\left(\alpha_1,\phi\right)$ coordinates. For the combined problem the displacement, strain and stress vectors can be written in the form [10]

$$U=\left(U_r,U_z,U_\phi,u_1,u_2,w,\gamma_1,\gamma_2\right),$$
$$\varepsilon=\left(e_{rr},e_{\phi\phi},e_{zz},e_{r\phi},e_{rz},e_{\phi z},\varepsilon_{11},\varepsilon_{22},\varepsilon_{12},\varepsilon_{13},\varepsilon_{23},\kappa_{11},\kappa_{22},\kappa_{12}\right),$$
$$\sigma=\left(\sigma_{rr},\sigma_{\phi\phi},\sigma_{zz},\sigma_{r\phi},\sigma_{rz},\sigma_{\phi z},\bar{\sigma}_{11},\bar{\sigma}_{22},\bar{\sigma}_{12},\bar{\sigma}_{13},\bar{\sigma}_{23}\right).$$

Seek the problem solution by means of a semi-analytical FEM. According to this technique the trigonometric functions $\sin m\phi, \cos m\phi$ defined at [0, 2π], are selected as basic functions along the circular coordinate ϕ. These functions constitute an orthogonal system in the energy metric of the operators of the elasticity theory and Timoshenko's shell theory. As to other variables, quadratic approximations of the finite element method are used. Due to the orthogonality of the basic functions, the problem decomposes into L problems $\left(m=\overline{0,L}\right)$.

Numerical example. Let us consider the stress-strain equilibrium of a two-layer cylinder. Let us compare the numerical solution with the exact solution, using the elasticity theory. Consider the axisymmetric (m = 0) strain problem for a two-layer cylinder under normal pressure on the inner and outer surfaces.

Denote by $U_r^{(i)}$ and $\sigma_{rr}^{(i)},\sigma_{\phi\phi}^{(i)},\sigma_{zz}^{(i)},\sigma_{r\phi}^{(i)},\sigma_{rz}^{(i)}\left(\sigma_{r\phi}^{(i)}=\sigma_{rz}^{(i)}=0\right)$ the exact solution within the domain Ω_i (i = 1, 2): a_i, b_i are outer and inner radiuses of the Ω_i domain; and $x=r/a_1$, $y=r/a_2$, $x_1=b_1/a_1$, $y_1=b_2/a_2$, $a_1=b_2$.

Boundary conditions

$$x=1:\quad \sigma_{rr}^{(1)}=-q_0,\quad \sigma_{rz}^{(1)}=0,\quad \sigma_{r\phi}^{(1)}=0;$$
$$x=x_1:\quad \sigma_{rr}^{(1)}=-q_1,\quad \sigma_{rz}^{(1)}=0,\quad \sigma_{r\phi}^{(1)}=0; \tag{16}$$
$$y=1:\quad \sigma_{rr}^{(2)}=-p_0,\quad \sigma_{rz}^{(2)}=0,\quad \sigma_{r\phi}^{(2)}=0;$$
$$y=y_1:\quad \sigma_{rr}^{(2)}=-p_1,\quad \sigma_{rz}^{(2)}=0,\quad \sigma_{r\phi}^{(2)}=0; \tag{17}$$
$$\phi=0,\quad \phi=\phi_0:\quad \sigma_{r\phi}^{(1)}=\sigma_{r\phi}^{(2)}=0,\quad u_\phi^{(1)}=u_\phi^{(2)}=0. \tag{18}$$

As the solution is not depending on z and ϕ, it will be enough to calculate only a part of the construction using conditions of symmetry on the boundaries G_1, G_2. Then we can write down the formulas

$$\sigma_{rr}^{(1)} = \frac{1}{\left(1-x_1^2\right)x^2}\left[\left(x^2-1\right)q_1x_1^2-\left(x^2-x_1^2\right)q_0\right]; \quad \sigma_{rz}^{(1)} = \sigma_{r\phi}^{(1)} = 0;$$

$$\sigma_{\phi\phi}^{(1)} = \frac{1}{\left(1-x_1^2\right)x^2}\left[\left(x^2+1\right)q_1x_1^2-\left(x^2+x_1^2\right)q_0\right]; \quad \sigma_{zz}^{(1)} = \frac{2\nu_1\left(q_1x_1^2-q_0\right)}{\left(1-x_1^2\right)}; \tag{19}$$

$$\sigma_{rr}^{(2)} = \frac{1}{\left(1-y_1^2\right)y^2}\left[\left(y^2-1\right)p_1y_1^2-\left(y^2-y_1^2\right)p_0\right]; \quad \sigma_{rz}^{(2)} = \sigma_{r\phi}^{(2)} = 0;$$

$$\sigma_{\phi\phi}^{(2)} = \frac{1}{\left(1-y_1^2\right)y^2}\left[\left(y^2+1\right)p_1y_1^2-\left(y^2+y_1^2\right)p_0\right]; \quad \sigma_{zz}^{(2)} = \frac{2\nu_2\left(p_1y_1^2-p_0\right)}{\left(1-y_1^2\right)}. \tag{20}$$

Here stresses values q_1, p_0 are given, stresses q_{01}, p_1 are unknown. We can determine them from the junction conditions

$$q_0 = p_1, \quad u_r^{(1)} = u_r^{(2)}, \quad \text{when} \quad x = 1, \quad y = y_1; \tag{21}$$

$$u_r^{(1)} = \frac{a_1\left(1+\nu_1\right)}{E_1\left(1-x_1^2\right)}\left[\left(1-2\nu_1\right)\left(q_1x_1^2-q_0\right)x+\left(q_1-q_0\right)\frac{x_1^2}{x}\right];$$

$$u_r^{(2)} = \frac{a_2\left(1+\nu_2\right)}{E_2\left(1-y_1^2\right)}\left[\left(1-2\nu_2\right)\left(p_1y_1^2-p_0\right)y+\left(p_1-p_0\right)\frac{y_1^2}{y}\right]. \tag{22}$$

The conditions (16)-(18) are satisfied. Let us substitute (22) to (21) and define

$$p_1 = q_0 = \frac{2p_0\beta y_1\left(1-\nu_2\right)+2\left(1-\nu_1\right)q_1x_1^2}{y_1\beta\left[\left(1-2\nu_2\right)y_1^2+1\right]+\left(1-2\nu_1\right)+x_1^2}; \quad \beta = \frac{a_2E_1\left(1-x_1^2\right)/\left[2\left(1+\nu_1\right)\right]}{a_1E_2\left(1-y_1^2\right)/\left[2\left(1+\nu_2\right)\right]}.$$

The stresses and displacement are defined from (19), (20), (22). All variables (radial displacement and stresses) are the functions of only one coordinate r [6].

Let us compare the exact solution with numerical. For the construction of the combined model numerical solution we can use the Timoshenko's shell theory equations within the Ω_2^* domain.

Let us consider: (*a*) the body is under uniform internal pressures, $f=1$, $p_3^+=0$; (*b*) the body is under uniform external pressures, $f=0$, $p_3^+=1$.

The material properties data of the cylinder and of the coating are given below: $E_1 = 6\cdot 10^{10}(N/m^2)$, $\nu_1 = 0.31$ (ceramic), $E_2 = 1.08\cdot 10^{11}(N/m^2)$, $\nu_2 = 0.31$ (copper).

Values of geometric parameters of the body are selected as: shell thickness, $h = a_2 - a_1$; radius (to the centre of the thickness), $R = a_1 + h/2$; inner radius of the body, $b_1 = 4m$; outer radius of the body, $a_2 = 4{,}4m$; distance to the junction surface for various a. A series of experiments have been performed for various values of $h/R \approx (1/40;1/100;1/400;1/1000)$ and $a_1 = (4{,}3;\, 4{,}36;\, 4{,}39;\, 4{,}396;\, 4{,}399)$, respectively. The finite element grid consists of 16 divisions along r and only 1 along z since solution is independent of z. Compare the numerical $(U_r, \sigma_{rr}, \sigma_{\phi\phi}, \sigma_{zz})$ and exact $(U_r^{(1)}, \sigma_{rr}^{(1)}, \sigma_{\phi\phi}^{(1)}, \sigma_{zz}^{(1)})$ solutions at the nodes with the same coordinates in the Ω_1 domain, also the numerical $(w, \sigma_{11}, \sigma_{22})$ and exact $(U_r^{(2)}, \sigma_{rr}^{(2)}, \sigma_{\phi\phi}^{(2)})$ solutions on the middle surface Ω_2. The body is under uniform internal (case a) and external (case b) pressures

$$(a)\ f = 1, \quad p_3^+ = 0; \qquad (b)\ f = 0, \quad p_3^+ = 1.$$

Let us denote magnitudes of displacement and stress errors

$$\Delta U_r = \frac{\left(U_r^{(1)} - U_r\right)}{U_r^{(1)}} 100\%; \quad \Delta w = \frac{\left(U_r^{(2)} - w\right)}{U_r^{(2)}} 100\%; \quad \Delta\sigma_{rr} = \frac{\left(\sigma_{rr}^{(1)} - \sigma_{rr}\right)}{\sigma_{rr}^{(1)}} 100\%;$$

$$\Delta\sigma_{\phi\phi} = \frac{\left(\sigma_{\phi\phi}^{(1)} - \sigma_{\phi\phi}\right)}{\sigma_{\phi\phi}^{(1)}} 100\%; \quad \Delta\sigma_{zz} = \frac{\left(\sigma_{zz}^{(1)} - \sigma_{zz}\right)}{\sigma_{zz}^{(1)}} 100\%;$$

$$\Delta\sigma_{11} = \frac{\left(\sigma_{zz}^{(2)} - \sigma_{11}\right)}{\sigma_{zz}^{(2)}} 100\%; \quad \Delta\sigma_{22} = \frac{\left(\sigma_{\phi\phi}^{(2)} - \sigma_{22}\right)}{\sigma_{\phi\phi}^{(2)}} 100\%.$$

Consider firstly the case (a) (cylinder is under internal pressure). Numerical results are listed in Tables 1 and 2.

Table 1

The displacement and stress errors in the Ω_1 domain (case (a))

h/R	1/40	1/100	1/400	1/1000
ΔU_r(%)	0,36935	0,06381	0,00483	0,00062
$\Delta\sigma_{rr}$(%)	0,65133	0,27347	0,24917	0,85055
$\Delta\sigma_{\phi\phi}$(%)	0,38840	0,06713	0,00363	0,00101
$\Delta\sigma_{zz}$(%)	0,43375	0,07476	0,00262	0,00177

Table 2

The displacement and stress errors on the middle surface Ω_2 (case (a))

h/R	1/40	1/100	1/400	1/1000
Δw(%)	1,65250	0,14438	0,04726	0,01984
$\Delta\sigma_{11}$(%)	1,84340	0,80862	0,21228	0,08579
$\Delta\sigma_{22}$(%)	0,67954	0,34936	0,09833	0,04028

When studying the above results, one can observe that the errors become smaller as the shell thickness becomes smaller. It is therefore reasonable to ask whether the natural conditions (12) are satisfied. For this purpose let us recall the third equilibrium equation (4) and regard the formula

$$p_3 = \left(1 + \frac{h}{R}\right) p_3^+ + \left(1 - \frac{h}{R}\right) p_3^- ,$$

and finally obtain $p_3^- = \left[T_2 h/R - \left(1 - (h/R)^2\right)\right] \Big/ (1 - h/R)$. The magnitudes of errors $\delta = \frac{\left(\sigma_{rr}^{(1)} + p_3^-\right)}{\sigma_{rr}^{(1)}} 100\%$ on the G_3 for each $\frac{h}{R}$ are listed in Table 3.

Table 3

The satisfaction of the natural boundary conditions (case (*a*))

h/R	1/40	1/100	1/400	1/1000
δ(%)	0,02144	0,07473	0,34601	0,89375

It can be seen that the natural condition (12) are satisfied. The error increases slightly as the shell thickness becomes smaller.

Now let us consider the (*b*)-case (coating is under external pressure). The tables 4-6 are similarly to Tables 1-3.

Table 4

The displacement and stress errors in the Ω_1 domain (case (*b*))

h/R	1/40	1/100	1/400	1/1000
ΔU_r(%)	0,45802	0,10572	0,01532	0.00516
$\Delta\sigma_{rr}$(%)	0,44693	0,09281	0,00156	0.00875
$\Delta\sigma_{\phi\phi}$(%)	0,45866	0,10522	0,01472	0,00457
$\Delta\sigma_{zz}$(%)	0,45856	0,10424	0,01363	0,00343

Table 5

The displacement and stress errors on the middle surface Ω_2 (case (*b*))

h/R	1/40	1/100	1/400	1/1000
Δw(%)	0,16534	0,80859	0,03088	0,01329
$\Delta\sigma_{11}$(%)	7,17860	7,24020	7,35020	7,37950
$\Delta\sigma_{22}$(%)	2,33040	2,30930	2,38000	2,35970

Table 6

The satisfaction of the natural boundary conditions (case (*b*))

h/R	1/40	1/100	1/400	1/1000
δ(%)	1,77550	0,52757	0,10480	0,03231

From a visual examination of these tables one can see that the numerical solution yields results which agree well with existing exact results. The results from Table 6 indicate that the natural conditions are satisfied.

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Дослідження деформування пружних тіл із тонким покриттям на основі D-адаптивної скінченноелементної моделі

Ольга Коссак, Ярема Савула

У статті наведено аналіз D-адаптивної математичної моделі, яка поєднує пружні конструкції з тонким покриттям. Цей підхід базується на формулюванні комбінованої математичної моделі, яка дозволяє використовувати одночасно тривимірну лінійну модель теорії пружності в області масивних фрагментів конструкції та двовимірну модель теорії оболонок типу Тимошенка в області покриття. Записано граничне та варіаційне формулювання задачі. Для окремих випадків проведено порівняння отриманого числового розв'язку з аналітичним.

Исследование деформирования упругих тел с тонким покрытием, базирующихся на D-адаптивной конечноэлементной модели

Ольга Коссак, Ярема Савула

В статье анализируется D-адаптивная математическая модель, которая объединяет упругие конструкции с тонким покрытием. Этот подход базируется на формулировании комбинированной математической модели, которая использует одновременно трехмерную линейную модель теории упругости в области массивных фрагментов конструкции и двухмерную модель теории оболочек типа Тимошенко в области покрытия. Записана граничная и вариационная постановка задачи. Для отдельных случаев проведено сравнение полученного численного решения с аналитическим.

Отримано 13.04.10