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Mathematical model for interference of elastic waves in a geological medium with a stressed layer. The case of uniform compression

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A layered geological medium with a compressed layer is considered. The medium is modeled by a horizontally layered heterogeneous elastic structure consisting of several finite depth layers covering a half-space. Elastic waves are excited in the structure by a deep point impulse source localized in an incumbent layer. The impact of the compression on the acoustic properties of the structure is accounted by the dependences of the phase velocities of longitudinal and transversal waves on the compression value. The matrix method is used to solve the problem of the wave interference in the structure. The obtained numerical solution is used to study the time structure of the wave field at a point on the free surface of the medium. The analysis enables to conclude that time dependences of the displacement components, measured at the point, contain information about the value of compression acting in the stressed layer. It was shows that this value can be determined, in particular, by the arrival times of the interference peaks, formed by the waves reflected from the upper and bottom boundaries of the compressed layer.

Key words: layered geological medium, acoustic waves in stressed media, matrix method for wave interference

Introduction. Parameters of elastic waves depend on the structure and mechanical properties of the propagation medium. So, if to excite the elastic waves in the object and to measure the displacements produced by them at certain points on the object free surface, one can obtain a posteriori information about the internal structure of the object. Such an approach is used particularly in applied geophysics for examination of the internal structure of geological structures [1].

Mechanical stresses change the phase velocities of elastic waves in geological mediums [2]. Hence, the stresses can influence on the wave field pattern. Under certain conditions this can be used for determination of mechanical stresses acting in the geological structure layers. So, the evaluation of the influence of initial stresses on wave field interference in stressed geological structures is an important problem.

A horizontally layered geological medium is considered. One of the layers is stressed — static homogeneous and isotropic stresses (uniform compression) act in it. Acoustic waves are excited in the medium by a point impulse seismic source localized in the other layer. The objective of the paper is to evaluate the influence of the compression on interference of the elastic waves excited by the seismic source. The used approach is based on: consideration of the medium as a layered structure consisting of several finite depth layers and a half-space; application of the matrix method [1, 3] for numerical solving of the direct problem for waves propagation in the structures; and analyzing the wave field pattern on the structure free surface.

1. The model for dynamics of small elastic disturbances in stressed medium

Each layer of the structure is considered as an isotropic and homogeneous elastic medium. The initial static stress-strained state of such medium is defined by Cauchy stress tensor $\boldsymbol{\sigma} = \{\sigma_{ij}\}\left(i, j = \overline{1,3}\right)$ and linear strain tensor $\boldsymbol{\varepsilon} = \{\varepsilon_{ij}\}$ .The last one can be expressed in terms of the displacement vector $\mathbf{u} = \{u_i\}$ as

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{1}$$

Here σ_{ij} and ε_{ij} are Cartesian components of the stress and strain tensors, x_i stand for Cartesian coordinates.

We start from the Murnaghan elastic potential represented in the form [5]

$$\Phi = \frac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + \frac{1}{6}\Gamma_{ijklmn}\varepsilon_{ij}\varepsilon_{kl}\varepsilon_{mn}, \tag{2}$$

where $i, j, k, l = \overline{1,3}$, C_{ijkl} and Γ_{ijklmn} are the second and the third order elastic modules, which is the case of isotropic medium look like

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}\right), \tag{3}$$

$$\Gamma_{ijklmn} = \Gamma'_{(ijklmn)},$$

$$\Gamma'_{ijklmn} = 2(l+2m)\delta_{ij}\delta_{kl}\delta_{mn} - 6m\delta_{ij}\left(\delta_{kl}\delta_{mn} - \delta_{kn}\delta_{lm}\right) + n \in_{ikm}\in_{jln}. \tag{4}$$

Here λ, μ and l, m, n stand for Lame and Murnagan coefficients. The parentheses in the denotation $\Gamma'_{(ijklmn)}$ indicate symmetrization with respect to permutation of the indices in the pairs ij, kl and mn and to cyclic permutation of the pair; δ_{ij} and $\in_{ijk}$ are Kronecker delta and Levi-Civita symbols respectively.

The elasticity relationships follow from the potential (2) with account (3), (4)

$$\sigma_{ij} = \left(C_{ijkl} + \frac{1}{2}\Gamma_{ijklmn}\varepsilon_{mn}\right)\varepsilon_{kl}. \tag{5}$$

Let a small elastic disturbance be superimposed on the initial stress-strained state. The strain tensor $\mathbf{e} = \{e_{ij}\}$ of the disturbance is defined by the displacement vector $\mathbf{w} = \{w_i\}$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right). \tag{6}$$

The displacement vector $\tilde{\mathbf{u}}$ in the disturbed state is defined as $\tilde{\mathbf{u}} = \mathbf{u} + \mathbf{w}$. As we restrict the consideration by geometrically linear approximation, due to relationships (1) the strains are additive: $\tilde{\varepsilon}_{ij} = \varepsilon_{ij} + e_{ij}$. In this case, taking into account the smallness of the strain components e_{ij}, we will obtain from (5) the following relationships for stress components $\tilde{\sigma}_{ij}$ in the disturbed state

$$\tilde{\sigma}_{ij} = \sigma_{ij} + C^{\varepsilon}_{ijkl} e_{kl}, \text{ where } C^{\varepsilon}_{ijkl} = C_{ijkl} + \frac{1}{2}\Gamma_{ijklmn}\varepsilon_{mn}. \tag{7}$$

As the initial stressed state is static, we obtain from relationships (6) and (7) the equations for dynamics of small elastic disturbances in the stressed medium

$$\rho \frac{\partial^2 w_i}{\partial t^2} = C^{\varepsilon}_{ijkl} \frac{\partial^2 w_l}{\partial x_j \partial x_k} + f_i, \tag{8}$$

where $f_i = f_i(x_1, x_2, x_3, t)$ stands for Cartesian components of the volume force density.

2. Formulation of the problem and the method for its solving

Define the geometric model of the horizontally layered geological medium as an infinite layered structure

$$\boldsymbol{\mathcal{S}} = \bigcup_{o=1}^{n+1} \boldsymbol{\mathcal{S}}^{(o)},$$

consisting of $n \in \mathbb{N}$ isotropic elastic layers $\boldsymbol{\mathcal{S}}^{(o)}, o \in \overline{1,n}$, of finite depths h_o and an elastic half-space $\boldsymbol{\mathcal{S}}^{(n+1)}$

$$\boldsymbol{\mathcal{S}}^{(o)} = (-\infty < x_1 < \infty) \otimes (-\infty < x_2 < \infty) \otimes (z_{o-1} < x_3 < z_o)$$
$$\boldsymbol{\mathcal{S}}^{(n+1)} = (-\infty < x_1 < \infty) \otimes (-\infty < x_2 < \infty) \otimes (z_n < x_3 < \infty),$$

where symbol $\otimes$ denotes Cartesian product; $z_0 = 0, z_p = \sum_{o=1}^{p} h_o, p = \overline{1,n}$; x_1, x_2, x_3 are Cartesian coordinates.

The layers $\boldsymbol{\mathcal{S}}^{(o)}$ differ by the values of the elastic constants λ, μ and l, m, n.

The surface $x_3 = 0$ is free of traction, and the conditions of continuity for displacement and stress vectors are valid on the boundaries $x_3 = z_p$ of adjacent layers.

Let a layer $\boldsymbol{\mathcal{S}}^{(p)}, 1 < p \le n$ be prestressed: the static isotropic homogeneous stresses $\sigma_{ij} = P\delta_{ij}$ act in it, where P is a real constant.

Elastic waves are excited in the structure by non-stationary volume force $f_i(x_1, x_2, x_3, t)$. We restrict the consideration by the case of isotropic impulse point seismic source, localized in a layer $\mathcal{S}^{(s)}, 1 < s < p$ at a point $S = (0, 0, z_S)$, $z_S \in (z_s, z_{s+1})$, taking f_i in the form

$$f_1 = \rho M \varphi(t) \delta'(x_1) \delta(x_2) \delta(x_3 - z_S), \quad f_2 = \rho M \varphi(t) \delta(x_1) \delta'(x_2) \delta(x_3 - z_S),$$
$$f_3 = \rho M \varphi(t) \delta(x_1) \delta(x_2) \delta'(x_3 - z_S), \tag{9}$$

where $\varphi(t)$ is a pulse of finite duration, $\delta(\ldots)$ and $\delta'(\ldots)$ stand for Dirac delta-function and its derivative.

Let us study the influence of the compression value P on the wave field produced by the source (9) in the structure $\mathcal{S}$.

As the isotropic stresses $\sigma_{ij} = P\delta_{ij}$ do not change the material symmetry of the layer $\mathcal{S}^{(p)}$, the influence of the stresses σ_{ij} on acoustic property of the layer can be taken into account in terms of dependences of the Lame constants on parameter P. We obtain from relations (5) and (7)

$$\lambda = \lambda_0 + \frac{6l - 2m + n}{3K} P, \quad \mu = \mu_0 + \frac{6m - n}{6K} P, \quad \rho = \rho_0 \left(1 + \frac{P}{K} \right), \tag{10}$$

where $K = \lambda_0 + \frac{2}{3}\mu_0$ stands for bulk modulus; «zero» subscript at symbols λ, μ and ρ denotes the value of the corresponding parameter at zero stresses.

With account formulas (10) the velocities of longitudinal $C_L = \sqrt{(\lambda + 2\mu)/\rho}$ and transversal $C_T = \sqrt{\mu/\rho}$ plane waves for layer $\mathcal{S}^{(s)}$ become dependent on compression P

$$C_L{}^2 = C_L^{02} \frac{1 + \dfrac{2l + (4/3)m}{\lambda_0 + 2\mu_0} \dfrac{P}{K}}{1 + P/K}, \quad C_T{}^2 = C_T^{02} \frac{1 + \dfrac{m - n/6}{\mu_0} \dfrac{P}{K}}{1 + P/K}. \tag{11}$$

Taking into account the material isotropy of the layers we reduce the system (8) for each $\mathcal{S}^{(o)}, o = \overline{1, n+1}$, to two independent wave equations: one with respect to the scalar φ and another for vector $\boldsymbol{\psi}$ potentials, such that $\mathbf{w} = \nabla\varphi + \nabla \times \boldsymbol{\psi}$

$$\frac{\partial^2 \varphi(\mathbf{r}, t)}{\partial t^2} = C_L{}^2 \Delta\varphi(\mathbf{r}, t) + \varphi_0(\mathbf{r}, t), \qquad \frac{\partial^2 \boldsymbol{\psi}(\mathbf{r}, t)}{\partial t^2} = C_T{}^2 \Delta\boldsymbol{\psi}(\mathbf{r}, t), \tag{12}$$

where the source $\varphi_0(\mathbf{r}, t)$ is non-zero just for layer $\mathcal{S}^{(s)}$ and has the form

$$\varphi_0(\mathbf{r}, t) = M\varphi(t)\delta(\mathbf{r} - \mathbf{r}_S). \tag{13}$$

The coefficients C_L and C_T in equations (12) are determined for the stressed layer $\mathcal{S}^{(p)}$ by formulas (11), whereas for other layers they are equal to their values for unstressed materials.

Solutions $\varphi(\mathbf{r},t)$ and $\boldsymbol{\psi}(\mathbf{r},t)$ of equations (12) should be subordinated to the conditions of zero traction on the structure free surface

$$\sigma_{i3}\big|_{x_3=0} = 0 . \tag{14}$$

of ideal mechanical contact on the boundaries dividing the neighboring layers

$$\left[w_i\right]\Big|_{x_3=z_o}, \qquad \left[\sigma_{i3}\right]\Big|_{x_3=z_o}, \qquad o=\overline{1,n}, \tag{15}$$

and of zero initial conditions

$$\mathbf{w}\big|_{t=0} = \frac{\partial \mathbf{w}}{\partial t}\bigg|_{t=0} = 0 . \tag{16}$$

To solve the problem (12)-(16) the matrix method was used. This method enables to represent the exact solution in a closed integral form [1, 3, 4]. To evaluate the influence of the layer compression on wave interference on the daylight surface of the structure a series of numerical experiments was conducted with the use of the obtained solution.

3. Results of the numerical experiments and conclusion

In the experiments the layered structure consisting of four layers $\mathcal{S}^{(o)}$, $o=\overline{1,4}$, covering the half-space $\mathcal{S}^{(5)}$ was considered. The layer $\mathcal{S}^{(3)}$ was assumed as the prestressed one. The geometrical and acoustical parameters of the unstressed structure are given in the Table. The acoustical properties of the compressed layer were calculated by formulas (11) using the values of elastic modules: λ_0 = 1,67 GPa, μ_0 = 18,2 GPa, l = – 3371 GPa, m = – 6742 GPa, n = – 6600 GPa [2]. The point source (9) was supposed to be localized in the layer $\mathcal{S}^{(2)}$ at the distance z_s = 7000 m from the day surface. The function $\varphi(t)$ was taken in the form of symmetric triangular pulse with duration 0,1s.

Table

Geometrical and acoustic parameters of the structure

C_L^0, m/s	C_T^0, m/s	ρ_0, 10^3 kg/m^3	h, m
2500	2000	2,00	4000
3100	2400	2,30	3000
3100	2400	2,30	3000
3790	2621	2,65	4000
4100	2750	2,75	3000
4200	2800	3,80	—

The influence of compression P on interference of the waves at the point of observation A, situated at the distance r_A = 5000 m from the source epicenter, was studied.

The problem (12)-(16) was solved in the time period 0...20 s for various compression values. The time diagrams of u_x, u_y, and u_z components of the displacement vector for various time periods are presented in Fig. 1-3. Traces 1-4 in all figures correspond the compression values P = 0, 4, 8 and 15 MPa.

Analyzing the problem and the results of calculation we can make some conclusions.

The isotropic point seismic source (9) causes in the layer $\mathcal{S}^{(2)}$ only the wave of dilatation. Transversal waves are secondary ones; they are generated in the structure by the transformation of longitudinal waves at the layers boundaries.

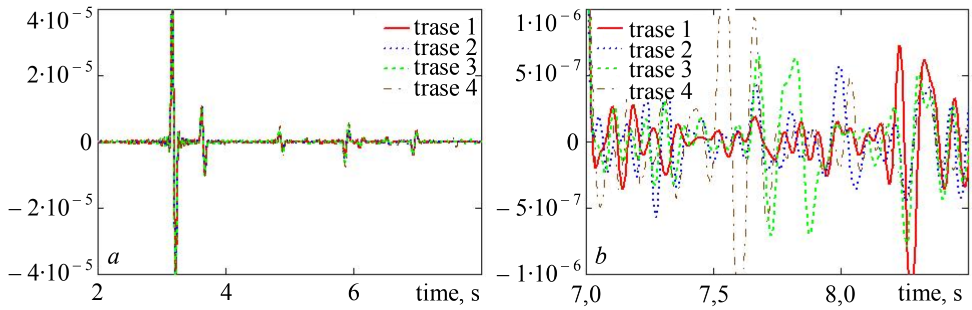

Fig. 1. Time diagrams for u_x component in the periods: 2…8 s — a; 7…8,5 s — b

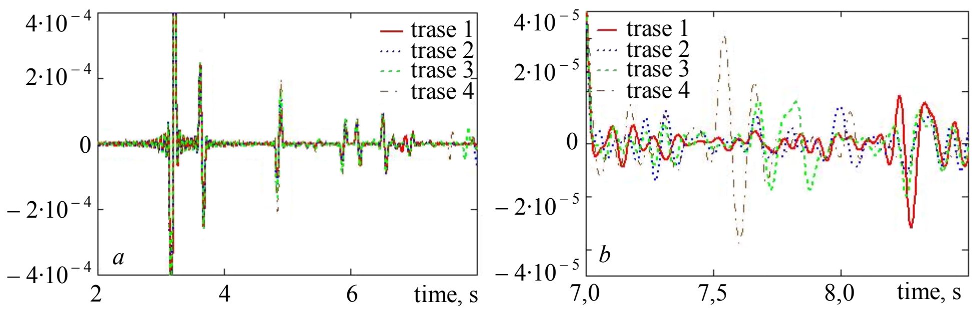

Fig. 2. Time diagrams for u_y component in the periods: 2…8 s — a; 7…8,5 s — b

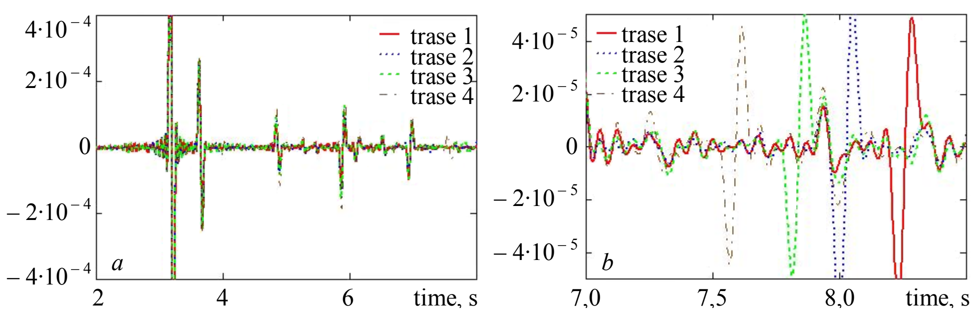

Fig. 3. Time diagrams for u_z component in the periods: 2…8 s — a; 7…8,5 s — b

We can recognize in the timing charts the interference peaks corresponding to the waves passing different paths in the structure. The first peaks in the charts presented in Fig. 1, 2 and 3 (*a*) are caused by the longitudinal wave passing directly from the source *S* through the layer $\mathcal{S}^{(2)}$ and then, after refraction, at the boundary $\mathcal{S}^{(2)}/\mathcal{S}^{(1)}$, through the layer $\mathcal{S}^{(1)}$. Other peaks on the diagrams should be associated with the converted transversal wave, generated at this boundary by the incident longitudinal wave; with longitudinal waves coming to the point *A* after repeated reflection and refraction at the boundary $\mathcal{S}^{(2)}/\mathcal{S}^{(1)}$ and free surface, etc.

The impact of the compression on the field pattern can be observed just for the waves reflected from the boundary $\mathcal{S}^{(2)}/\mathcal{S}^{(3)}$ and for those passed through the layer $\mathcal{S}^{(3)}$. We can see distinctions in displacements calculated for different values of the compression *P* in the graphs in Fig. 1, 2 and 3, *b* which present the timing diagrams for u_x, u_y and u_z components in the period 7 ... 8,5*s*. In particular, the shifts of some peaks along the time axis caused by the compression can be noticed. This is the most visible in the graphs for u_z components Fig. 3*b*. For any compression value we can clearly find a bipolar pulse of duration of about 0,1*s*. The pulse is shifted in time for a period depending on the compression: for P = 4, 8 and 15 MPa the time shift of the pulse relatively to its position for uncompressed state equals about 0,24; 0,42 and 0,67 *s* respectively. Similar effect can be observed in the diagrams for u_x and u_y components (Fig. 1*b* and Fig. 2*b*).

The considered simple model problem enables us to conclude that the layer compression can significantly change the space-time structure of the wave field excited in a geological medium by the deep impulse point source. Information about the value of compression can be obtained from the time dependences of the displacement components, measured at a point on the free surface. The value can be estimated by the arrival times of the interference peaks, formed by the waves reflected from the upper and bottom boundaries of the compressed layer. These effects can be useful for remote determination of the stresses, acting in geological structures, for earthquakes prediction. To develop such methods for remote layer characterization the problem should be formulated more specifically, i. e. the real geometry of the considered geological structure and anisotropy of its stressed state should be taken into consideration, elastic property of the layers should also be precisely determined.

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Математична модель інтерференції пружних хвиль у геологічному середовищі з напруженим шаром. Однорідний стиск

Василь Чекурін, Дмитро Малицький

У статті розглянули шарувате геологічне середовище, один із шарів якого перебуває під дією напружень ізотропного однорідного стискання. Середовище моделювали горизонтально-шаруватою пружною структурою, яку утворюють декілька шарів скінченної товщини, що покривають пружний півпростір. Імпульсне точкове сейсмічне джерело, яке локалізоване в одному із верхніх шарів, збуджує в середовищі пружні хвилі. Вплив напружень на акустичні властивості стисненого шару враховували залежністю густини мас і фазових швидкостей поздовжніх й поперечних плоских хвиль від величини стиску. Сформульовано задачу поширення хвиль у шаруватому середовищі, яку розв'язували матричним методом. Числові розв'язки задачі, отримані для різних значень напружень стиску, застосували для дослідження інтерференції хвиль у точці спостереження, розташованій на вільній поверхні на відстані від епіцентру джерела. Проведений кількісний аналіз дозволив зробити висновок, що часові залежності компонент вектора переміщень, визначені у точці спостереження, містять інформацію про величину напружень стиснення шару. Показано, що ці напруження можна визначити за часом досягнення точки спостереження інтерференційними піками, утвореними хвилями, відбитими від верхньої та нижньої меж напруженого шару.

Математическая модель интерференции упругих волн в геологической среде с напряженным слоем. Однородное сжатие

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Рассматривается слоистая геологическая среда, один из слоев которой находится под действием напряжений всестороннего однородного сжатия. Среда моделируется горизонтально слоистой структурой, образованной несколькими слоями конечной толщины, покрывающими упругое полупространство. Импульсный точечный сейсмический источник, локализированный в одном из верхних слоев, возбуждает в среде упругие волны. Влияние напряжений на акустические свойства среды учитывались зависимостью плотности масс и фазовых скоростей продольных и поперечных плоских волн напряженного слоя от величины сжатия. Сформулирована задача распространения волн в слоистой среде, которую решали матричным способом. Численные решения задачи, полученные для различных значений напряжений всестороннего сжатия, использовались для исследования интерференции волн в точке наблюдения, находящейся на свободной поверхности на некотором удалении от эпицентра сейсмического источника. Выполненный количественный анализ позволяет заключить, что временные зависимости компонент вектора перемещений, определенные в точке наблюдения, содержат информацию о величине сжатия напряженного слоя. Показано, что эти напряжения можно определить, в частности, по времени прибытия в точку наблюдения интерференционных пиков, образованных волнами, отраженными от верхней и нижней поверхностей напряженного слоя.

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