УДК 517.958:519.63

Numerical analysis of a multiscale model of the elastic body with the thin cover

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A model, that describes the stress-strain state of 2D heterogeneous elastic body with thin cover, is considered. For the numerical solution of such problem we propose an algorithm, based on the coupling of finite element method/boundary element method (FEM/BEM) using domain decomposition. Numerical experiments show that this approach is useful in modelling problems of elasticity theory for bodies with thin covers.

Key words: boundary element method, finite element method, domain decomposition, plane strain.

Introduction. A lot of structures, occurring in nature or engineering, are inhomogeneous and contain thin covers or thin inclusions. Such bodies often arise in mechanics, medicine, ecology, geophysics, biology, etc.

As a result, it is of a great importance to study the influence of the thin objects on the stress-strain state of the bodies.

The difficulty in modelling and numerical simulations for such processes arises because of the existence of several subdomains with different physical properties and having different scales [1].

The analysis of different types of problems for the bodies with covers or inclusions is carried out by several approaches. For the bodies with canonical shape of inclusion analytical techniques are possible to use; near the singularities asymptotic analysis can be performed. Topics related to building adequate mathematical models and their numerical solution for the bodies with covers and inclusions are considered in [2-5].

In the present work we formulate a model that describes such phenomena. For the description of this process the equations of elasticity theory and Timoshenko shell theory [2] are used. We illustrate this approach with an example of 2D problem for the body with a cover or an inclusion. Numerical solution is based on the coupling of finite element method/boundary element method (FEM/BEM) using domain decomposition algorithm.

1. Boundary integral equations formulation for the plain stress problem

Consider an elasticity problem for a cylinder's cross-section occupying domain Ω_1 with a cover in Ω_2. The cylinder's cross-section is contained in $\Omega \in R^2$, $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$ with the Lipschitz boundary Γ (Fig. 1).

Displacements inside the body in Ω_1 can by characterized by [6]

$$\frac{1}{2}u_j(x_0) = \int_\Gamma \left[t_i(x) G_{ij}(x, x_0) - F_{ij}(x, x_0) u_i(x) \right] d\Gamma(x), \qquad (1)$$

which holds for $x_0 \in \Gamma$. Here $u(x) = u_1(x), u_2(x)$ and $t(x) = t_1(x), t_2(x)$ with $x = x_1, x_2$ are the vectors of displacements and tractions for Ω_1 along x_1 and x_2 axis; $G_{ij}(x,\xi) = C_1\left(C_2\delta_{ij}\ln r - y_i y_j / r^2\right)$ is the Green's function, $F_{ij} = \left(C_3/r^2\right)\times$ $\times\left[C_4\left(n_j y_i - n_i y_j\right) + \left(C_4\delta_{ij} + 2y_i y_j/r^2\right) y_k n_k\right]$ is the co-normal derivative of Green's function, where

$$r^2 = y_i y_i, \quad y_i = x_i - \xi_i,$$
$$C_1 = -1/\left[8\pi\mu\left(1-\upsilon_1\right)\right], \quad C_2 = 3 - 4\upsilon_1,$$
$$C_3 = -1/\left[4\pi\left(1-\upsilon_1\right)\right], \quad C_4 = 1 - 2\upsilon_1,$$

n_i are the components of the outer normal vector for Ω_1, μ is the shear modulus, which is characterized by $\mu = E_1/\left[2\left(1+\nu_1\right)\right]$, E_1 is the Young's modulus, ν_1 is the Poisson's ratio. Let us denote by σ_{ij}, $i, j = 1, 2$ the components of stress tensor for the body in Ω_1.

On the outer boundary of the body in Ω_1 we impose the following boundary conditions:

$$u = u_D \quad \text{for} \quad x \in \Gamma_D,$$
$$t = t_N \quad \text{for} \quad x \in \Gamma_N.$$

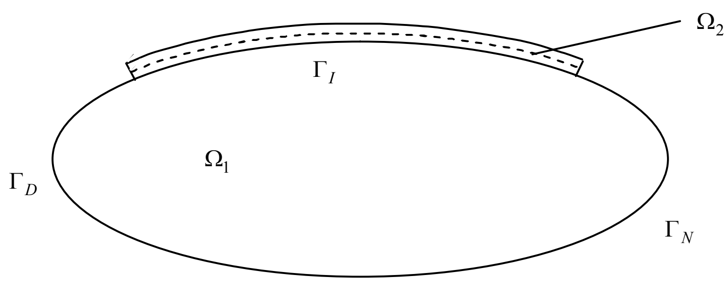

Fig. 1. Body with cover

2. Formulation of the equations for the cover

For the description of the cover in Ω_2 we use the equations of Timoshenko shell theory [2] of the form

$$-\frac{1}{A_1}\frac{dT_{11}}{d\xi_1}-k_1T_{13}=p_1\,,\quad -\frac{1}{A_1}\frac{dT_{13}}{d\xi_1}+k_1T_{11}=p_3\,,$$
$$-\frac{1}{A_1}\frac{dM_{11}}{d\xi_1}+T_{13}=m_1\,,\quad -1\le\xi_1\le 0\,, \tag{2}$$

where T_{11}, T_{13} and M_{11} are the forces and moments in the shell; $A_1=A_1(\xi_1)$, $k_1=k_1(\xi_1)$ correspond to Lame parameter and median surface curvature parameter; p_1, p_3, m_1 are given functions; it holds

$$T_{11}=\frac{E_2h}{1-\nu_2^2}\varepsilon_{11}\,,\quad T_{13}=k'G'h\varepsilon_{13}\,,\quad M_{11}=\frac{E_2h^3}{12\left(1-\nu_2^2\right)}\chi_{11}\,,$$
$$\varepsilon_{11}=\frac{1}{A_1}\frac{dv_1}{d\xi_1}+k_1w\,,\quad \varepsilon_{13}=\frac{1}{A_1}\frac{dw}{d\xi_1}+\gamma_1-k_1v_1\,,\quad \chi_{11}=\frac{1}{A_1}\frac{d\gamma_1}{d\xi_1}\,,$$
$$p_1=\left(1+k_1\frac{h}{2}\right)\sigma_{13}^{+}-\left(1-k_1\frac{h}{2}\right)\sigma_{13}^{-}\,,\quad p_3=\left(1+k_1\frac{h}{2}\right)\sigma_{33}^{+}-\left(1-k_1\frac{h}{2}\right)\sigma_{33}^{-}\,,$$
$$m_1=\frac{h}{2}\left[\left(1+k_1\frac{h}{2}\right)\sigma_{13}^{+}+\left(1-k_1\frac{h}{2}\right)\sigma_{13}^{-}\right].$$

Here E_2 is the Young's modulus for the shell, ν_2 is the Poisson's ratio; g_1, g_3 are the components of the volume forces vector, that act on the shell (without the loss of generality we assume these forces equal to zero); σ_{ij}^{+}, σ_{ij}^{-}, $i,j=1,3$, are the components of the stress tensor on the outer $(\xi_3=h/2)$ and inner $(\xi_3=-h/2)$ surfaces of the shell. It is known, that in case of isotropic bodies we have $k'=5/6$, $G'=E_2/\left[2\left(1+\nu_2\right)\right]$.

At each end of the thin cover we impose boundary conditions either on the displacements v_1, w and γ_1 (if the end is fixed) or on the forces T_{11}, T_{13} and moment M_{11} in the shell (if the end is subjected to load or free). At the outer surface of the shell we prescribe to σ_{13}^{+} and σ_{33}^{+} some given stresses.

Remark. The choice of 2D curvilinear coordinate system for the shell as ξ_1,ξ_3 (instead of ξ_1,ξ_2) is based on the fact, that 2D problem is obtained from the 3D case by assuming the cylinder being infinite in the direction of ξ_2.

3. Coupling conditions

On the boundary Γ_I common to both Ω_1 and Ω_2, we impose the following coupling conditions [2]:

$$u_\nu = w\,, \quad u_\tau = v_1 - \frac{h}{2}\gamma_1\,; \tag{3}$$

$$\sigma_{\nu\nu} = \sigma_{33}^{-}\,; \quad \sigma_{\nu\tau} = \sigma_{13}^{-}\,. \tag{4}$$

Here

$$u_\nu = u_1 n_1 + u_2 n_2\,, \quad u_\tau = u_1\tau_1 + u_2\tau_2\,,$$

$\sigma_{\nu\nu}$ and $\sigma_{\nu\tau}$ are the components of stress tensor in the (n,τ) coordinate system, τ is the vector, which is tangent to Γ.

4. Numerical approximation of the model

For the numerical solution of the model (1)-(4) with the appropriate boundary conditions domain decomposition algorithm is used.

Boundary integral equation in Ω_1 is solved by the boundary element method (BEM) [6] and Galerkin method [7].

To apply BEM we divide the boundary Γ_1 of Ω_1 into the elements and then choose the shape functions defined on Γ_1. As a result, the solution is in the form

$$u_i(\xi) = \sum_{j=1}^{m} u_{ij}\phi_j(\xi)\,, \quad i = 1,2\,, \tag{5}$$

$$t_i(\xi) = \sum_{j=1}^{m} t_{ij}\phi_j(\xi)\,, \quad i = 1,2\,, \tag{6}$$

where u_{ij}, t_{ij} are the unknown coefficients, $\xi \in \Gamma$.

Let us substitute (5), (6) into the integral equation (1) and apply Galerkin method. Taking into account boundary conditions on the outer boundary of Ω_1, we obtain system of linear equations with respect to u_{ij} and t_{ij}.

As the basis system, bubble functions are chosen [1].

As for the solution of the system of equations (2) in Ω_2 finite element method with bubble basis functions, is used. On each element these basis functions are of the form

$$\Phi_0(\xi) = \frac{1-\xi}{2}\,, \quad \Phi_1(\xi) = \frac{1+\xi}{2}\,,$$

$$\Phi_j(\xi) = \sqrt{\frac{2j-1}{2}} \int_{-1}^{\xi} P_{j-1}(t)\,dt\,, \quad j = 2,3,...$$

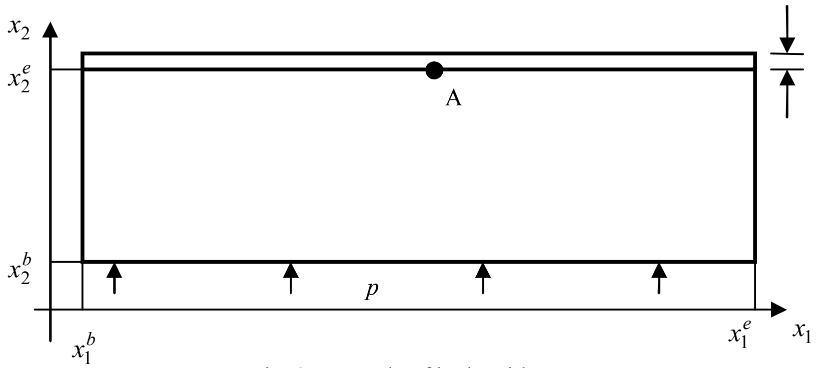

Fig. 2. Example of body with cover

where $\xi \in [-1,1]$ is the local coordinate, obtained by mapping each element onto the interval $[-1,1]$; $P_j(t)$ are the Legendre polynomials.

The approximate solutions in Ω_1 and Ω_2 are combined via iterative domain decomposition method, Dirichlet-Neumann algorithm [8], where the unknown displacement on the interface is given by the relation

$$u^{k+1} = u^k + \theta S_2^{-1}\left(G - Su^k\right),$$

where $S = S_1 + S_2$ is the Steklov-Poincare operator; S_1 and S_2 are Steklov-Poincare operators corresponding to the domains Ω_1 and Ω_2; G is the right side of Steklov-Poincare equation $Su = G$; $0 < \theta < \theta_{\max}$.

5. Numerical experiments

Let Ω_1 be a rectangle with $x_1^b = 0,05$; $x_1^e = 1,05$; $x_2^b = 0,05$; $x_2^e = 0,55$. To the top of the concrete body contained in Ω_1 a thin steel cover of thickness $h = 0,05$ is attached that occupies Ω_2 (Fig. 2). For the main body we take the following physical parameters: $\nu = 0,33$; $E = 25\,000$ MPa. They correspond to concrete, for the shell we change the material (see Table). Three materials are taken for the shell: concrete, aluminium ($E = 70\,000$ MPa, $\nu = 0,33$) and steel ($E = 200\,000$ MPa, $\nu = 0,33$). The body with cover is fixed from both sides, from the bottom it is subjected to the load of $p = 1$ MPa, the top boundary is traction-free.

Table

Displacements of the middle of the interface (point A) along x_2 axis, E-05

	Concrete	Aluminium	Steel
$h = 0,05$	3,10	2,81	2,57
$h = 0,02$	3,29	3,12	2,97

We use 96 equal linear boundary elements (for the main body) and 32 finite elements (for the shell) in our simulations.

In Table we show the displacements in the middle of the interface (at point A (see Fig. 2)) along x_2 direction. It is clear, that at point A the displacement in x_2 direction achieves its maximum among the whole interface. From Table we conclude that as the shell gets thinner, the displacements along the interface increase. On the other hand, the increase in Young's modulus of the shell results in the drop of the displacement.

Parameter θ in the domain decomposition is chosen empirically. If we fix the parameters of the main body and vary the thickness and Young's modulus of the shell E_2, then with the increase of E_2 θ should be increased. On the other hand, when h gets smaller, θ should be decreased. In the case of $h = 0{,}02$ and steel shell we take $\theta = 0{,}00225$.

The above example was analyzed using domain decomposition approach and the results on the boundary common to both bodies were compared to COMSOL and FEMAP packages for modelling PDE's. All three solutions for both of the displacements are very close to each other, but COMSOL package produces a non-smooth solution for the stresses, for example for von-Mises stress in the case of significantly different physical parameters for the main body and the shell (for example with steel cover).

We remark, that actually for the domain decomposition algorithm, it is completely enough to know only displacements on the boundary, since the problem afterwards can easily be split into two independent problems that can be solved by any suitable method.

Conclusions. The Domain Decomposition algorithm presented in this article gives good approximation for the solution on the interface of the problem for the body with covers. Moreover, it allows us to use completely different models as well as approximations in the main body and its cover.

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ISSN 1816-1545 Фізико-математичне моделювання та інформаційні технології
2012, вип. 15, 49-55

Числовий аналіз різномасштабної задачі для пружного тіла з тонким покриттям

Ігор Макар, Ярема Савула, Андрій Стягар

У статті розглянуто модель, яка описує напружено-деформований стан двовимірного неоднорідного пружного тіла з тонким покриттям. Для числового розв'язання задачі запропоновано метод, який полягає у поєднанні методу граничних елементів і методу скінченних елементів за допомогою декомпозиції областей. Числові результати показують, що цей підхід ефективний для знаходження напружено-деформованого стану тіл із тонким покриттям.

Численный анализ разномасштабной задачи для упругого тела с тонким покрытием

Игорь Макар, Ярема Савула, Андрий Стягар

В статье рассмотрена модель, описывающая напряженно-деформированное состояние двумерного неоднородного упругого тела с тонким покрытием. Для численного решения задачи предложено использовать метод, суть которого состоит в объединении методов граничных и конечных элементов с помощью декомпозиции областей. Численные результаты показывают эффективность применения данного метода для решения задач нахождения напряженно-деформируемого состояния тел с тонким покрытием.

Отримано 11.04.12