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Forced oscillations of liquid in a reservoir of paraboloidal shape

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Peculiarities of behavior of the mechanical system «reservoir – liquid with a free surface» under reservoir motion disturbance by horizontal harmonic force are under investigation. In particular, we consider the example of small depth filling of the paraboloidal reservoir for different kinds of harmonic force. We investigate the problem of transition to the steady mode of motion on the basis of multimodal nonlinear discrete model of dynamics of combined motion of bounded liquid volume with a free surface and the reservoir, which performs translational motion in horizontal plane.

Keywords: nonlinear oscillations, parabolic reservoir, free surface of liquid, transition to steady mode of motion.

Introduction. The problem of studying of wave motion of bounded liquid volume with a free surface in immovable reservoirs of non-cylindrical shape can be represented as

$$\Delta\varphi = 0 \quad \text{in} \quad \tau, \qquad \frac{\partial\varphi}{\partial n} = 0 \quad \text{on} \quad \Sigma,$$

$$\frac{\partial\varphi}{\partial n} = -\frac{\partial\eta/\partial t}{\left\|\vec{\nabla}\eta\right\|} \quad \text{on} \quad S, \qquad \frac{\partial\varphi}{\partial t} + \frac{1}{2}\left(\vec{\nabla}\varphi\right)^2 + U = 0 \quad \text{on} \quad S.$$

Motion is described in the Cartesian reference frame *Oxyz,* fixed with reservoir. For description of oscillations of bounded liquid volume in reservoir we introduce the following denotation: τ is domain, occupied by liquid, S is a free surface of liquid, Σ is moisten wall of the reservoir, $\eta(x, y, z, t) = 0$ is the equation of a free surface of liquid, U is potential energy of external forces applied to liquid. Since description of non-vortex motion of ideal homogeneous incompressible liquid is reduced to motion of its boundaries, mathematical formulation of the problem of dynamics of the system «reservoir – liquid with a free surface» represents an aggregate of kinematic and dynamic boundary conditions. Kinematic conditions are considered as mechanical constraints, which superimpose restrictions on variations of unknowns, on statement of the problem of motion of the mechanical system on the basis of the Hamilton-Ostrogradskiy variational principle. Here the dynamic boundary condition follows from the Hamilton-Ostrogradskiy variational principle as natural.

Similar to publication [1] for description of liquid motion we introduce non-Cartesian parameterization of the domain τ, occupied by liquid

$$\alpha = r/f(z); \quad \beta = z/H.$$

Center of the coordinate system in the center of undisturbed free surface of liquid, the axis Oz is directed upward, we denote by $r = f(z)$ the generatrix of the reservoir in the cylindrical coordinate system, H is depth of liquid in the reservoir, $z = 0$ coincides with undisturbed free surface of liquid. The system of cylindrical coordinates (r, θ, z) is substituted for the new non-Caertesian one (α, θ, β). In the accepted non-Cartesian system of coordinates the domain of liquid obtains cylindrical shape, but metric of this domain is non-Euclidean. It is impossible to represent the equation of a free surface in old variable (r, θ, z) as $z = \xi(r, \theta, t)$. However, because of cylindrical shape of liquid domain in new system of variables new parameterization makes it possible to represent the equation of a free surface of liquid as

$$\beta = \frac{1}{H}\xi(\alpha, \theta, t) \quad \text{or} \quad H\beta - \xi(\alpha, \theta, t) = 0 .$$

Further this enables usage of the perturbation techniques and the Kantorovich method for construction of nonlinear model of dynamics of the system «reservoir – liquid».

Construction of the discrete model is done on the basis of problem statement in the form of the Hamilton-Ostrogradskiy variational principle with preliminary satisfying of kinematic boundary conditions and solvability conditions. Transition from continuum structure of the system reservoir – liquid to its discrete model is done on the basis of the Kantorovich method and on the whole is similar to the case, when reservoir has cylindrical shape. However, here there are a number of fundamental differences, caused by non-cylindrical shape of the domain, occupied by liquid, and by the necessity of holding the solvability conditions of the problem for perturbed state of liquid. In particular, there are the following new properties, which differ the considered case from the case of cylindrical reservoir:

- the system of coordinate functions, by which decomposition of the velocity potential is performed, hold the non-flowing condition on the moisten border approximately;
- this system of coordinate functions supplementary holds the non-flowing conditions on certain prolongation of the moisten boundary of liquid, where wave crests can reach, which are consequences of the solvability conditions;
- aggregate of geometrical nonlinearities enters the solution, which complicates nonlinear interaction between normal modes of oscillations in the system and liquid with reservoir.

1. Object of investigation

In the considered problem we investigate dynamic peculiarities of combined motion of the mechanical system «paraboloidal reservoir – liquid with a free surface» in the case of low depth filling. At the initial time reservoir and liquid are at rest state. System motion is disturbed by harmonic force. We assume that reservoir performs its motion only in the horizontal plane.

In publication [2] on the basis of the method of the article [1] discrete model of the system «paraboloidal reservoir – liquid with a free surface» was constructed in the form of ordinary differential equations of the second order, namely, equations of combined motion of the system «reservoir – liquid» in amplitude parameters a_i and parameters of motion of the carrying body $\vec{\varepsilon}$

$$\sum_i \ddot{a}_i \left(V_{ir}^1 + \sum_j a_j V_{irj}^2 + \sum_{j,k} a_j a_k V_{irjk}^3 \right) +$$

$$+\ddot{\vec{\varepsilon}} \left(\vec{U}_r^1 + \sum_i a_i \vec{U}_{ri}^2 + \sum_{i,j} a_i a_j \vec{U}_{rij}^3 + \sum_{i,j,k} a_i a_j a_k \vec{U}_{rijk}^4 \right) =$$

$$= \sum_{i,j} \dot{a}_i \dot{a}_j V_{ijr}^{2*} + \sum_{i,j,k} \dot{a}_i \dot{a}_j a_k V_{ijkr}^{3*} + \dot{\vec{\varepsilon}} \left(\sum_i \dot{a}_i \vec{U}_{ir}^{2*} + \sum_{i,j} \dot{a}_i a_j \vec{U}_{irj}^{3*} + \sum_{i,j,k} \dot{a}_i a_j a_k \vec{U}_{ijkr}^{4*} \right) -$$

$$-g \left(\sum_i a_i W_{ir}^2 + \frac{3}{2} \sum_{i,j} a_i a_j W_{ijr}^3 + 2 \sum_{i,j,k} a_i a_j a_k W_{ijkr}^4 \right), \quad r = \overline{1,N}, \tag{1}$$

$$\frac{\rho}{\left(M_{rez} + M_{liq}\right)} \left[\sum_i \ddot{a}_i \left(\vec{U}_i^{\,1} + \sum_j a_j \vec{U}_{ij}^{\,2} + \sum_j a_j a_k \vec{U}_{ijk}^{\,3} \right) \right] + \ddot{\vec{\varepsilon}} =$$

$$\frac{F}{\left(M_{res} + M_{liq}\right)} - g\vec{z}_0 - \frac{\rho}{\left(M_{res} + M_{liq}\right)} \sum_j \dot{a}_j \dot{a}_j \left(\vec{U}_{ij}^{\,2} + 2 \sum_k a_k \vec{U}_{ijk}^{\,3} \right). \tag{2}$$

(ρ is liquid density, g is free falling acceleration, M_{res} and M_{liq} are masses of reservoir and liquid correspondingly). The equations (1) describe dynamics of amplitudes of oscillations of a free surface of liquid, and the equations (2) describe dynamics of reservoir, which performs horizontal translational motion. Combined motion of the reservoir with liquid is completely characterized by independent generalized coordinates a_i and $\vec{\varepsilon}$. The number of equations is equal to the number of system degrees of freedom, so the suggested model (1), (2) is of minimal dimension.

According to technique of publication [1] let us represent the equation of a free surface of liquid ξ in the form

$$\xi = \overline{\xi}(t) + \sum_I a_i \overline{\psi}_i(\alpha) T_i(\theta).$$

Taking into account character of variation of frequency parameters we accepted the following system of coordinate functions and their arrangement for decomposition of elevation of a free surface

$$\psi_1 = \overset{*}{\psi}_{11} \sin\theta,\ \ \psi_2 = \overset{*}{\psi}_{11} \cos\theta,\ \ \psi_3 = \overset{*}{\psi}_{01},\ \ \psi_4 = \overset{*}{\psi}_{21} \sin 2\theta,\ \ \psi_5 = \overset{*}{\psi}_{21} \cos 2\theta,$$

$$\psi_6 = \overset{*}{\psi}_{02},\ \ \psi_7 = \overset{*}{\psi}_{11} \sin 3\theta,\ \ \psi_8 = \overset{*}{\psi}_{31} \cos 3\theta,\ \ \psi_9 = \overset{*}{\psi}_{12} \sin\theta,\ \ \psi_{10} = \overset{*}{\psi}_{12} \cos\theta,$$

where $\overset{*}{\psi}_{mk}$ is solution of the problem about refined (with satisfying boundary conditions on crests of waves on reservoir walls) determination of normal modes of oscillations of a free surface with angular number m, which is associated with the k-th eigenvalue (arrangement of coordinate functions was accepted in ascending order of eigenvalues).

For solving nonlinear problem of combined motion of reservoir and free surfaced liquid used $n_1 = 10$ coordinate functions (amplitude parameters), from which the first $n_2 = 6$ were considered accurate to values of the second order, the first $n_3 = 3$ coordinate functions of them were studied accurate to values of the third order. We accepted perturbation of a liquid free surface with respect to the first antisymmetric normal mode. Here $\overline{\xi}(t)$ is term of decomposition of free surface elevation, which is determined from the requirement of conservation of liquid volume in its perturbed motion. Coefficients of the dynamical model (1), (2) are determined as quadratures from coordinate functions over undisturbed liquid free surface.

For investigation of nonlinear dynamics of combined motion of the system «reservoir – liquid» series of numerical experiments were performed.

2. Results of numerical modeling of the problem of transition of the system «reservoir – liquid» to steady mode of motion

We consider the paraboloidal reservoir $r = \sqrt{2}\sqrt{z+H}$ with vertical axis of symmetry Oz, which performs translational motion in horizontal plane. Reservoit of radius R with mass M_{res} is partially filled by water with mass M_{lig} with small depth (below we adduce results for the filling level $H = 0{,}3$).

The system of equations (1), (2) is linear relative to the second order derivatives of variables, this makes it possible to reduce the equations on every step of numerical solving to the Cauchy normal form with numerical integration of the system by means of the Runge-Kutta method on practical implementation of the model. Here on the stage of transformation to the Cauchy normal form order of derivatives was reduced by means of introduction of generalized velocities $\dot{a}_i$ as equal right independent variables (together with a_i). Step of numerical integration was accepted as $\Delta t = 0{,}02$ s. On solving the problem of determination of coordinate functions we used decomposition of solution with respect to $N = 22$ harmonic polynomials.

Peculiarities of transition of the system on steady motion was considered in the case, when ratio of masses of reservoir and liquid is $M_{res} = 10M_{lig}$, and amplitude of the external force is equal to 0,06. The oscillation process is investigated on time interval equal to 40 periods of oscillations with respect to the first normal mode. System motion is disturbed by the force, which varies with frequency close to frequency of the first normal mode.

The graph of variation in time of amplitude of elevation of a free surface of liquid with respect to the first normal mode for the external force $F_x = A\sin\varpi t$ is shown in Fig. 1*a*, and for $F_x = A\cos\varpi t$ it is shown in Fig. 1*b*. As it is seen from the graphs, for the specified ratio of masses of reservoir and liquid law of variation of amplitude of the first normal mode is similar. Here transition to steady mode of oscillations in classical sense is not observed. Process is developed for clearly manifested modulation with practically constant mean value. For certain time instants phenomenon of antiresonance is manifested, when during several period of oscillations variations of amplitudes are in a vicinity of zero. Graphs of variation of elevation near reservoir wall and law of variation of the main vector of pressure on reservoir walls have also periodic character with

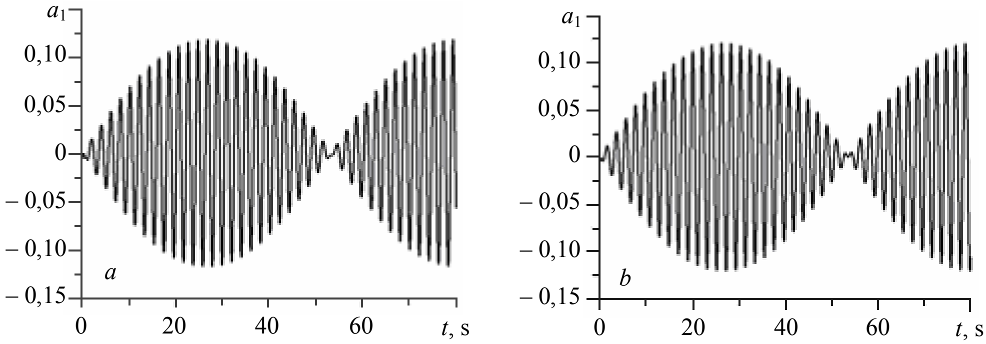

Fig. 1. Amplitude of excitation of the first normal mode for $M_{res} = 10M_{lig}$

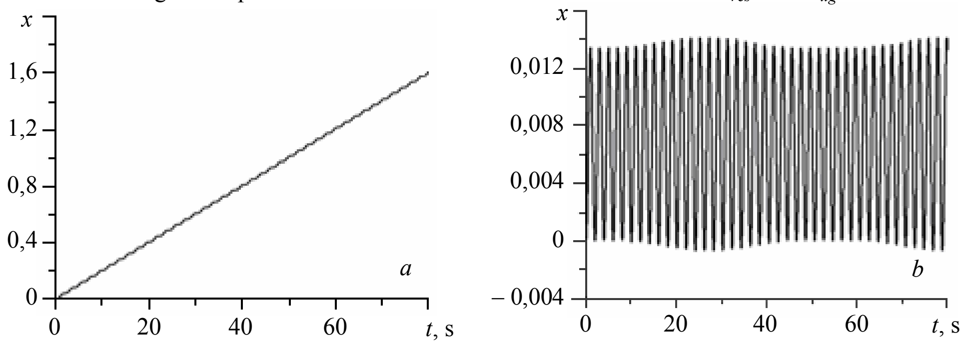

Fig. 2. Trajectory of motion of the reservoir for $M_{res} = 10M_{lig}$

manifested modulation for both cases of harmonic external force. Moreover, period of modulation is practically constant ant it is about 25 periods of oscillations with respect to the first normal mode. Fig. 2*a* shows variation in time of reservoir displacement in the horizontal direction in the case of the applied force $F_x = A\sin\varpi t$, and Fig. 2*b* corresponds to $F_x = A\cos\varpi t$. In the first case reservoir moves with small oscillations of mean velocity in the direction of *Ox* axis. In the second case reservoir performs oscillations near certain constant value of *x*. Direction of motion at certain time instants can be opposite to the direction of the applied force. Such inverse motion is caused by influence of internal sloshing of liquid, which results in considerable displacement of liquid mass center takes place.

For other ratio of masses of reservoir and liquid, for example, in the case $M_{res} = 0{,}1M_{lig}$, shown in Fig. 3 (then mass of reservoir is considerably lesser), transition to steady mode is not manifested at all and graphs of variation of amplitude of a_1 for both cases of external loading even have no clear form of modulation. Moreover, drift of mean value of amplitudes of oscillations takes place. Graphs of variation of the main vector of pressure on reservoirs walls and elevation of liquid near reservoir walls have similar character. However, character of displacement of reservoir in the horizontal plane for this ratio of masses of liquid and reservoir does not change considerably (Fig. 4). Behavior of the system in two cases of variation of external force $F_x = A\sin\varpi t$ and $F_x = A\cos\varpi t$ is similar and graphs of variation of velocity of reservoir have similar modulation (Fig. 5). In the case 5*a* variation of velocity has mean value, which differs from zero. This indicates that

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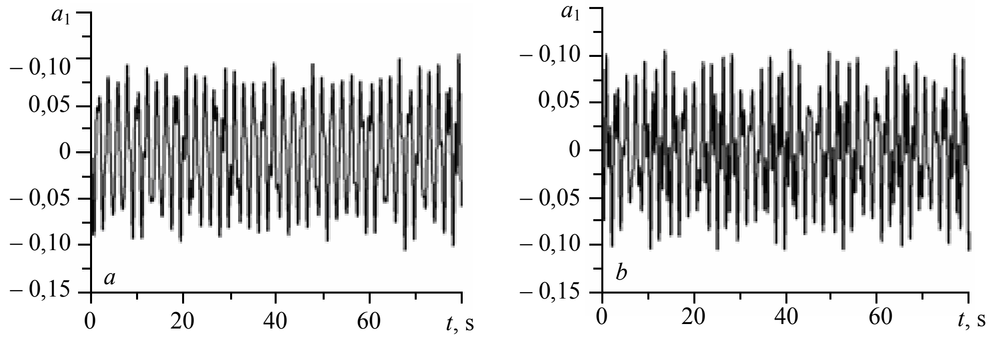

Fig. 3. Amplitude of excitation of the first normal mode for $M_{res} = 0{,}1M_{lig}$

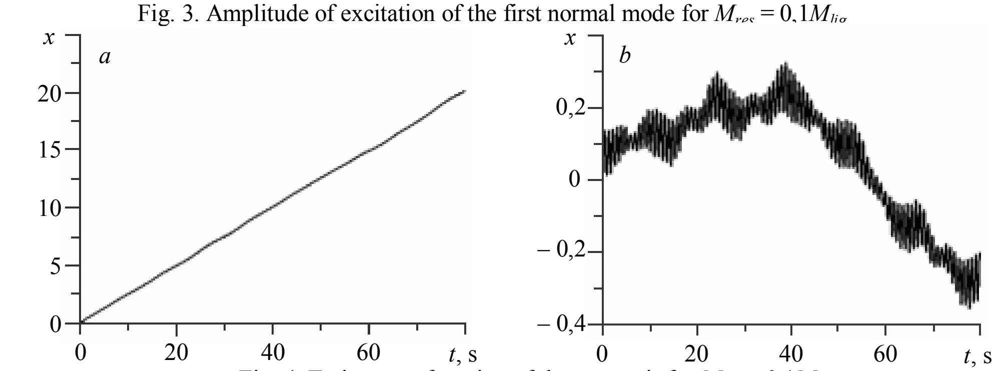

Fig. 4. Trajectory of motion of the reservoir for $M_{res} = 0{,}1M_{lig}$

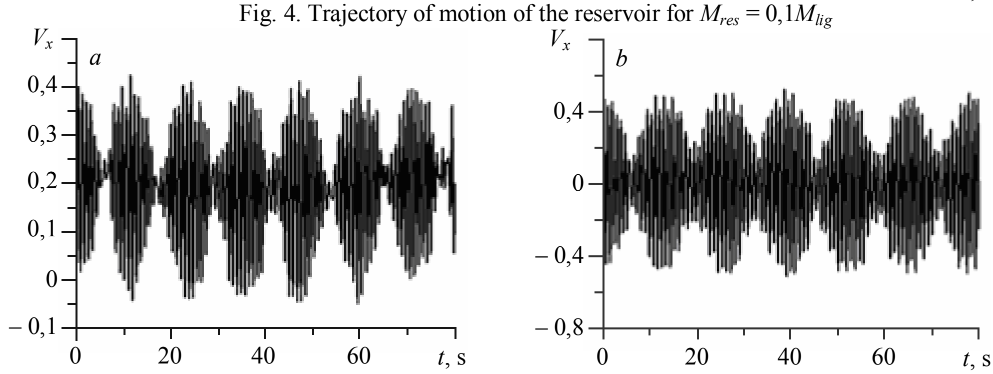

Fig. 5. Velocity of motion of the reservoir for $M_{res} = 0{,}1M_{lig}$

the reservoir performs systematic motion. In the case 5*b* mean value has zero magnitude and reservoir performs oscillation in a vicinity of starting position.

Numerical experiments for different ratio of masses of reservoir and liquid, filling depths and frequencies and laws of harmonic external loading in a vicinity of main resonance showed, that transition to steady mode of oscillations with considerable modulation (this mode does not coincide with classical understanding of resonance) depends mostly on ratio of masses of reservoir and liquid. Moreover, modulation period also depends on this ratio.

In the case of small filling of the reservoir ($H = 0{,}3$) liquid motion with iteration of oscillations cycles with strongly expressed modulation occurs starting from ratio factor of masses $M_{res} = 5M_{lig}$, although for greater higher filling level ($H = 1$) process attains to steady mode at values $M_{res} = M_{lig}$. On increase of amplitude of the external force parameters of reservoir motion differs, but law of amplitudes variation conserves.

Conclusion. We consider problem of modeling of nonlinear forced motion of liquid with a free surface in movable paraboloidal reservoir. Motion was disturbed by harmonic force in the horizontal direction. In particular, we consider the case of small filling of reservoir by liquid was considered. It was shown that transition to steady mode of motion in the considered nonlinear multifrequency systems of «reservoir – liquid» type is not manifested at all. Modulation periodicity of graph of variation in time of amplitudes of perturbation of a free surface occurs on increase of ratio of mass of reservoir with respect to mass of liquid. It was shown also that law of external harmonic force effects only parameters of reservoir motion, but not on liquid.

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Вимушені коливання рідини в резервуарі параболоїдальної форми

Олег Лимарченко, Ірина Семенова

У роботі досліджено особливості поведінки механічної системи «резервуар – рідина з вільною поверхнею» у разі збурення руху резервуара горизонтальною гармонічною силою. Розглядається приклад малого заповнення параболоїдального резервуара для різних варіантів гармонічної сили. Задача виходу на усталений режим вивчається на основі багатомодової нелінійної дискретної моделі динаміки сумісного руху обмеженого об'єму рідини з вільною поверхнею та резервуара, який здійснює горизонтальний поступальний рух.

Вынужденные колебания жидкости в резервуаре параболоидальной формы

Олег Лимарченко, Ирина Семенова

В работе исследованы особенности поведения механической системы «резервуар – жидкость со свободной поверхностью» при возбуждении движения параболоидального резервуара горизонтальной гармонической силой. Рассматривается пример малого заполнения параболического резервуара для разных законов гармонической силы. Задача выхода на установившийся режим изучается на основе многомодовой нелинейной динамики совместного движения ограниченного объема жидкости со свободной поверхностью и резервуара, который выполняет горизонтальное движение.

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