УДК 532.595

Nonlinear forced liquid sloshing in a hyperboloid reservoir

Iryna Semenova

Ph. D., Taras Shevchenko National University of Kyiv, Academician Glushkov Avenue, 4e, Kyiv,
e-mail: is25@bigmir.net

Peculiarities of behaviour of mechanical system «reservoir – liquid with a free surface» under excitation by the given horizontal harmonic force are investigated. The problem of liquid forced oscillations is studied on the basis of multimodal nonlinear model of dynamics of combined motion of limited volume of liquid with a free surface and reservoir of a hyperboloid shape. It is shown that finally the system of oscillations does not transit to steady state.

Keywords: nonlinear oscillations, hyperboloid reservoir, free surface, transition to steady mode of motion.

Introduction. We investigate translational motion in the horizontal plane of absolutely rigid hyperboloid reservoir, partially filled with liquid. The problem of oscillations of ideal incompressible liquid with a free surface in a cavity has been studied by many authors [1-3]. Mathematical formulation of the problem of dynamics of the system «reservoir – liquid with a free surface» represents a system of kinematic and dynamic requirements (boundary conditions):

$$\Delta\varphi = 0 \quad \text{in} \quad \tau,$$

$$\frac{\partial\varphi}{\partial n} = 0 \quad \text{on} \quad \Sigma,$$

$$\frac{\partial\varphi}{\partial n} = -\frac{\partial\eta/\partial t}{\left\|\vec{\nabla}\eta\right\|} \quad \text{on} \quad S,$$

$$\frac{\partial\varphi}{\partial t} + \frac{1}{2}\left(\vec{\nabla}\varphi\right)^2 + U = 0 \quad \text{on} \quad S.$$

The motion is described in the Cartesian reference frame *Oxyz,* fixed with reservoir. For description of oscillations of bounded liquid volume in reservoir, we introduce the following denotations: φ is the velocity potential of liquid, τ is the domain, occupied by liquid, $\frac{\partial}{\partial n}$ is external normal derivative to a surface, *S* is a free surface of liquid in its perturbed motion, Σ is the boundary of the contact of liquid with reservoir walls in perturbed motion (for convenience, we also introduce Σ_0, which corresponds to boundary of contact of liquid with tank walls in unperturbed motion and $\Delta\Sigma$ variations of the contact boundary caused by liquid perturbation ($\Sigma = \Sigma_0 + \Delta\Sigma$), $\eta(x, y, z, t) = 0$ is the equation of a free surface of liquid, U is the function of potential energy of liquid, t is time. Kinematic conditions are considered as mechanical constraints, which superimpose restrictions on variations of unknowns, for the statement of the problem of motion of the mechanical system on the basis of the Hamilton-Ostrogradskiy variation principle. Dynamic boundary conditions are naturally obtained from the Hamilton-Ostrogradskiy principle. Following the publication [4, 5], for description of liquid motion we introduce non-Cartesian parameterization of the domain τ, occupied by liquid

$$\alpha = \frac{r}{f(z)}; \quad \beta = \frac{z}{H}.$$

In the accepted non-Cartesian system of coordinates (α, θ, β), the domain of liquid takes cylindrical shape. New parameterization makes it possible to represent the equation of a free surface of liquid as $\beta = \frac{1}{H}\xi(\alpha, \theta, t)$ or $H\beta - \xi(\alpha, \theta, t) = 0$ because of cylindrical shape of liquid domain in new system of variables. The study of liquid sloshing for arbitrary tank geometry is based on variational formulation. Construction of the motion equations of the system is done on the basis the Hamilton-Ostrogradskiy variation principle, used to the system of volume of bounded liquid and a rigid body with cavity of revolution with preliminary satisfying of kinematic boundary conditions and solvability conditions. For transition from continuum structure of the initial model of the system rigid body – liquid to its discrete model we make use of the Kantorovich method. This way is similar to the case, when reservoir has cylindrical shape. Main distinction consists in the following two properties: decomposition of the velocity potential with respect to the coordinate functions holds approximately the non-flowing conditions; the group of geometrical nonlinearities caused by non-cylindrical shape of the reservoir predetermines additional dependences of all natural modes of oscillations.

1. Object of investigation

We investigate dynamic peculiarities of combined motion of the mechanical system «hyperboloid reservoir – liquid with a free surface». The reservoir performs translational motion in the horizontal plane under action of active periodic force. In publication [5] on the basis of the method of the article [4], a discrete model of the system «hyperboloid reservoir – liquid with a free surface» was constructed. The motion equations of the system reservoir – liquid in amplitude parameters a_i and parameters of translational motion of the carrying body $\vec{\varepsilon}$ take the form

$$\sum_i \ddot{a}_i \left(V_{ir}^1 + \sum_j a_j V_{irj}^2 + \sum_{j,k} a_j a_k V_{irjk}^3 \right) +$$

$$+\ddot{\vec{\varepsilon}} \cdot \left(\vec{U}_r^1 + \sum_i a_i \vec{U}_{ri}^2 + \sum_{i,j} a_i a_j \vec{U}_{rij}^3 + \sum_{i,j,k} a_i a_j a_k \vec{U}_{rijk}^4 \right) =$$

$$
\begin{aligned}
&=\sum_{i,j}\dot{a}_i\dot{a}_j V_{ijr}^{2*}+\sum_{i,j,k}\dot{a}_i\dot{a}_j a_k V_{ijkr}^{3*}+\\
&+\ddot{\vec{\varepsilon}}\cdot\left(\sum_i \dot{a}_i\vec{U}_{ir}^{2*}+\sum_{i,j}\dot{a}_i a_j\vec{U}_{irj}^{3*}+\sum_{i,j,k}\dot{a}_i a_j a_k\vec{U}_{ijkr}^{4*}\right)-\\
&-g\left(\sum_i a_i W_{ir}^2+\frac{3}{2}\sum_{i,j}a_i a_j W_{ijr}^3+2\sum_{i,j,k}a_i a_j a_k W_{ijkr}^4\right),\quad r=\overline{1,N}, \qquad (1)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\rho}{\left(M_{res}+M_{liq}\right)}\left[\sum_i \ddot{a}_i\left(\vec{U}_i^{\,1}+\sum_j a_j\vec{U}_{ij}^{\,2}+\sum_j a_j a_k\vec{U}_{ijk}^{\,3}\right)\right]+\ddot{\vec{\varepsilon}}=\\
&+\frac{F}{M_{res}+M_{liq}}-g\vec{z}_0-\frac{\rho}{M_{res}+M_{liq}}\sum_j\dot{a}_j\dot{a}_j\left(\vec{U}_{ij}^{\,2}+2\sum_k a_k\vec{U}_{ijk}^{\,3}\right) \qquad (2)
\end{aligned}
$$

(ρ is the liquid density, g is the free falling acceleration, M_{res} and M_{liq} are masses of reservoir and liquid). The equations (1) describe dynamics of amplitudes of normal modes of oscillations of a free surface of liquid, and the equations (2) describe dynamics of translational motion of reservoir.

According to the technique of publication [5], let us represent the equation of a free surface of liquid ξ as follows

$$\xi=\xi(t)+\sum_I a_i\overline{\psi}_i(\alpha)T_i(\theta).$$

Taking into account the character of variation of frequency parameters, we have accepted the following system of coordinate functions and their arrangement for decomposition of elevation of a free surface:

$$\psi_1=\overset{*}{\psi}_{11}\sin\theta;\ \psi_2=\overset{*}{\psi}_{11}\cos\theta;\ \psi_3=\overset{*}{\psi}_{01};\ \psi_4=\overset{*}{\psi}_{21}\sin(2\theta);\ \psi_5=\overset{*}{\psi}_{21}\cos(2\theta);$$
$$\psi_6=\overset{*}{\psi}_{02};\ \psi_7=\overset{*}{\psi}_{31}\sin(3\theta);\ \psi_8=\overset{*}{\psi}_{31}\cos(3\theta);\ \psi_9=\overset{*}{\psi}_{12}\sin\theta;\ \psi_{10}=\overset{*}{\psi}_{12}\cos\theta,$$

where $\overset{*}{\psi}_{mk}$ is the solution of the refined problem (with satisfied boundary conditions on crests of waves on reservoir walls) of determination of normal modes of oscillations of a free surface with the angular number *m*, which is associated with the *k*-th eigenvalue (arrangement of coordinate functions was accepted in ascending order of eigenvalues).

For investigation of nonlinear dynamics of combined motion of the system «hyperboloid reservoir – liquid», a number of numerical experiments were performed.

2. Results of numerical experiments

Let us consider hyperboloid reservoir $r=\dfrac{a}{c}\sqrt{(x+H+c)^2-c^2}$ with vertical longitudinal axis *Oz*, which performs translational motion in the plane *xOy*. A step of numerical

Table 1

Frequency ratios

Modes of oscillations ψ_i	ω / ω_1
ψ_1, ψ_2	1,000000
ψ_3	1,637210
ψ_4, ψ_5	1,329786
ψ_6	1,581338
ψ_7, ψ_8	1,801157
ψ_9, ψ_{10}	2,154455

integration was specified as $\Delta t = 0{,}01$ s. For solving the problem of determination of coordinate functions, we used decomposition of solution with respect to $N = 22$ harmonic polynomials. The ratio of masses of reservoir and liquid is $R = M_{res}/M_{liq} = 0{,}1$, the liquid depth parameter is $H/R_0 = 1$. The natural frequencies for selected values of parameters of the mechanical system are given in Table 1. Process of oscillation is investigated over the time interval of 100 periods of oscillation of the main normal mode; graphs are shown for 30 periods. Motion is initiated by horizontal force, applied to reservoir walls, which is changed according to the harmonic function $F_x = A\cos(pt)$, the initial excitation of a free surface of liquid is absent. For all numerical examples, the value of amplitude of external horizontal force applied to the reservoir was accepted in such a way, that oscillations of a free surface of liquid hit in nonlinear range of variation of wave amplitudes (i.e. elevation of a free surface of liquid was $(0{,}2 \div 0{,}25)R_0$).

We consider nonlinear oscillations of free surfaced liquid when frequency of external force is under resonance given below (Fig. 1, 2), i.e., $p = 0{,}7\omega_1$, where $\omega_1 = 3{,}503581$ is partial frequency of the first antisymmetric normal mode. Normal (resonant) frequency of the mechanical system for the specified mass ratio $M_{res}/M_{liq} = 0{,}1$ is equal to $\omega^* = 1{,}58\omega_1$.

Time dependence of the amplitude of perturbation of the free surface is exposed to discrete Fourier transform; the resulting frequency spectrum $A = |A(\omega/\omega_1)|$ was analyzed for the presence of harmonics equal or multiples to normal, matching, or forced frequencies.

Graph of amplitude perturbation of the free surface on tank walls is shown in Fig. 1*a*, *b* illustrates frequency spectrum amplitude perturbation of the liquid free-surface motion. Fig. 1*a* illustrates that free surface oscillations occur with noticeable amplitude modulation with the presence of the varied in time mean value.

The presence of amplitude modulation is explained by the presence of two amplitude harmonics on external $(p = 0{,}7\omega_1)$ and normal frequencies of the system $(p = 1{,}58\omega_1)$.

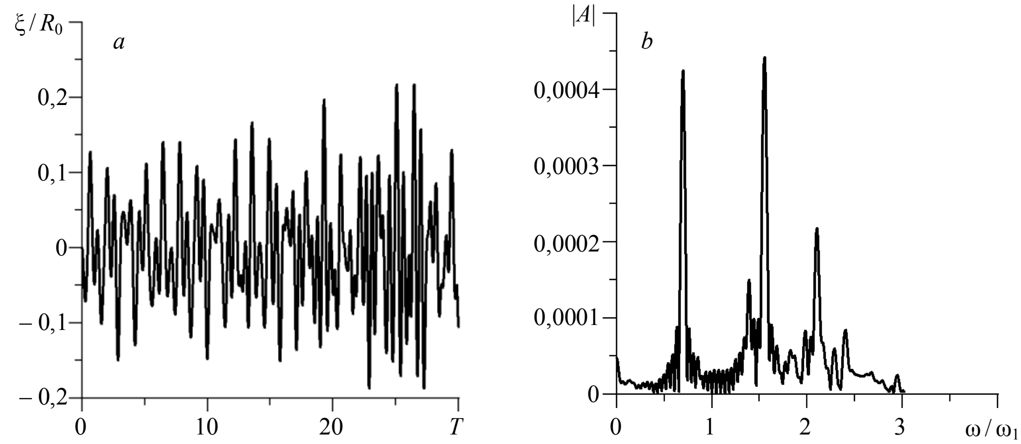

Fig. 1. Forced oscillations of liquid $F_x = A\cos(pt)$, $p = 0{,}7\omega_1$
a) amplitude of oscillations of liquid on tank wall, *b*) frequency spectrum

Variation of mean value in time can be explained by the presence of time harmonics with extremely low (close to zero) frequencies, caused by difference of two values of frequencies (Fig. 1*b)*. Moreover, the frequency spectrum (Fig. 1*b*) contains high spectral harmonics, both in their own and in combination frequencies. Frequency spectrum includes four peaks: frequency of external force $\left(p = 0,7\omega_1\right)$, natural frequency of the mechanical system ($p = 1,58\omega_1$) and combination frequencies $\left(p = 1,4\omega_1\right.$, $\left.p = 2,1\omega_1\right)$. The presence of amplitude modulation, time variable mean value and high harmonics with frequencies, which are not multiple to the frequency of external disturbance, made it possible to draw conclusion that excitation of motion in below resonance range steady oscillations in the system «hyperboloid reservoir – liquid» are absent. Fig. 2*a* shows a graphical time-dependent vector main forces fluid pressure on the shell wall. Graph of variation in time of the liquid horizontal response on tank walls has amplitude modulation and time variable mean value. Spectrum (Fig. 2*b*) slightly differs from spectrum of amplitude perturbation of free surface (Fig. 1*b*). Frequency spectrum of liquid response contains only two pronounced peaks corresponding to external force $\left(p = 0,7\omega_1\right)$ and natural frequency of the mechanical system $\left(p = 1,58\omega_1\right)$, since only antisymmetric oscillations corresponding number $m = 1$ can create horizontal hydrodynamic response tank in the case of combined motion of the reservoir and liquid.

Let us consider nonlinear oscillations of a free surface of liquid, when frequency of the external force is in a close vicinity of resonant frequency (Fig. 3, 4), namely, $p = 1,58\omega_1$ for ratio of masses $M_{res}/M_{liq} = 0,1$. As it is seen from Fig. 3*a*, oscillations of a free surface have pronounced amplitude modulation and time variable mean value. This property is also confirmed by structure of frequency spectrum, namely, all dominating harmonics are focused near frequencies of disturbance (Fig. 3*b*). Graphs of variation of the main vector of pressure on reservoir walls (liquid response) have

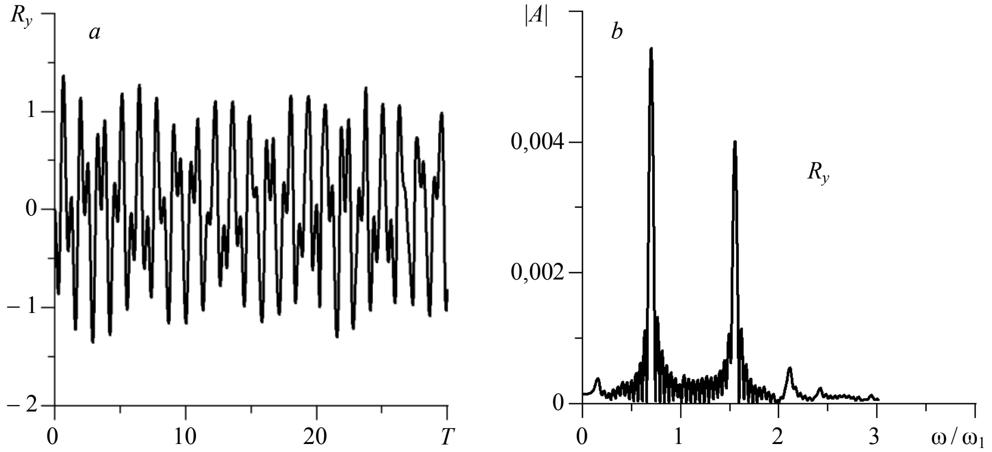

Fig. 2. Forced oscillations of liquid $F_x = A\cos(pt)$, $p = 0{,}7\omega_1$
a) total liquid response on the wall of a tank, *b*) frequency spectrum

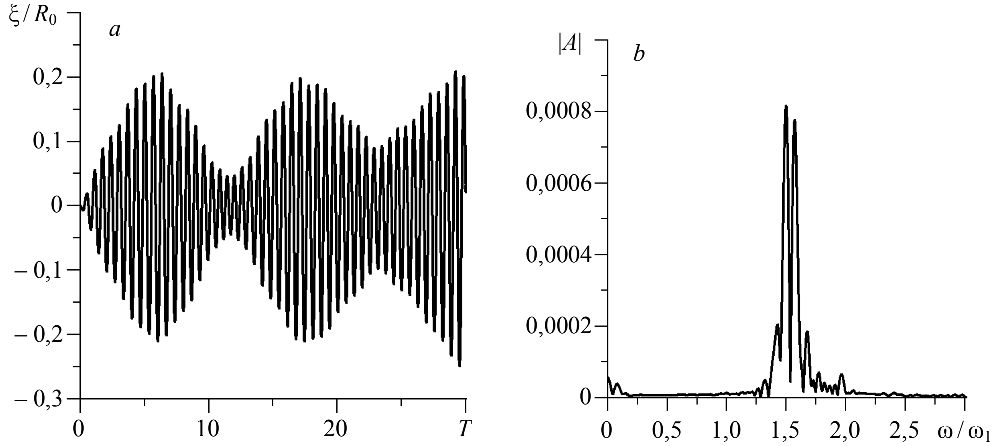

Fig. 3. Forced oscillations of liquid $F_x = A\cos(pt)$, $p = 1{,}58\omega_1$
a) amplitude of oscillations of liquid on tank wall, *b*) frequency spectrum

also periodic character (Fig. 4*a*). Noticeable amplitude modulation and the absence of manifestation of high frequency modes are peculiar also. Frequency spectrum of vector of liquid horizontal response on tank is similar (Fig. 4*b*). Therefore, in the case of system disturbance in a small vicinity of resonant frequency transition of the system to steady mode of oscillations in its classical sense does not occur.

Fig. 5*a* shows variation in time of amplitude of elevation of a free surface of liquid, when system motion is disturbed by force, which varies with the above resonance frequency $\left(p = 1{,}8\omega_1\right)$. As it is seen from Fig. 5*a*, graph is characterized by amplitude modulation and variable in time mean value of amplitudes. Frequency spectrum

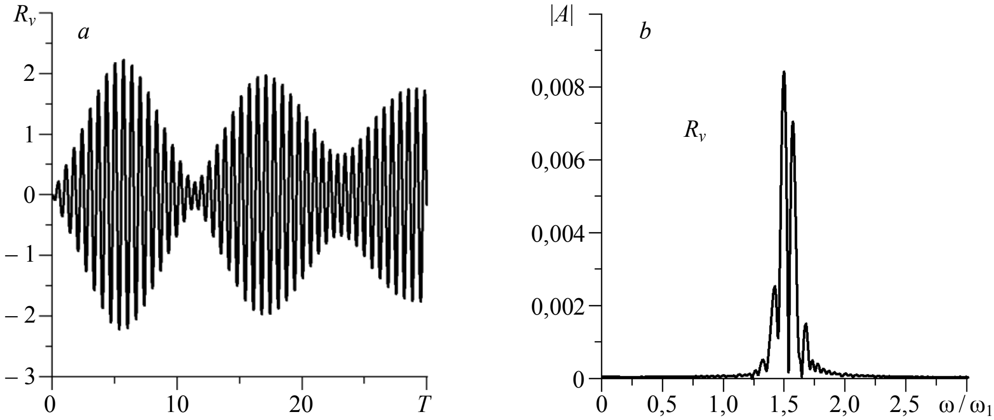

Fig. 4. Forced oscillations of liquid $F_x = A\cos(pt)$, $p = 1{,}58\omega_1$
a) liquid response on tank walls, *b*) frequency spectrum

(Fig. 5*b*) has four peaks: frequency of external force ($p = 0{,}7\omega_1$), natural frequency of the mechanical system ($p = 1{,}58\omega_1$) and combination frequencies ($p = 2{,}3\omega_1$, $p = 2{,}5\omega_1$). The presence of harmonics at low frequencies explains variation in time of mean value. Graph of variation of horizontal response of liquid on tank walls has amplitude modulation (Fig. 6*a*). The spectrum of this liquid response is similar to the case of below resonant disturbance (two dominant peaks, corresponding to external frequency and natural frequency of the system). Behaviour of «hyperbolic tank – liquid» in above resonance range of motion disturbance does not lead to steady mode of oscillations. We consider the case, when ratio of mass of the reservoir to mass of liquid is $M_{res}/M_{liq} = 0{,}1$ (this ratio is often met in practice), this differs from research results

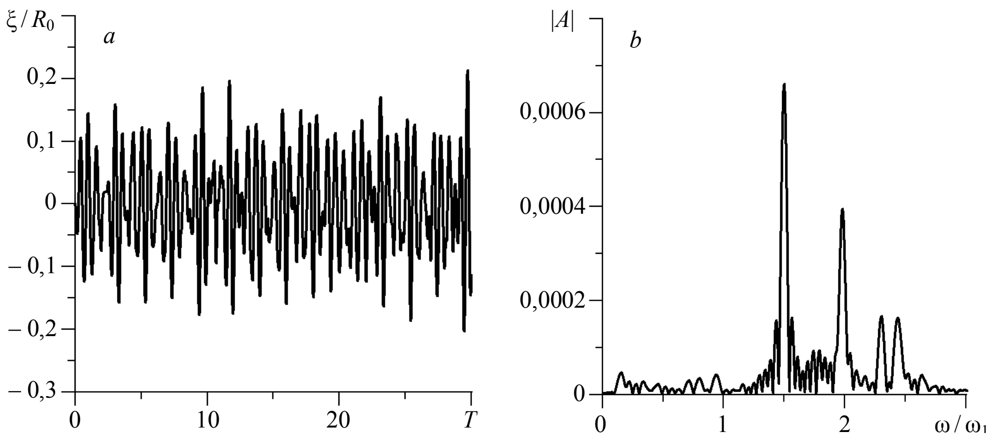

Fig. 5. Forced oscillations of liquid $F_x = A\cos(pt)$, $p = 2{,}3\omega_1$
a) amplitude of oscillations of liquid on tank wall, *b*) frequency spectrum

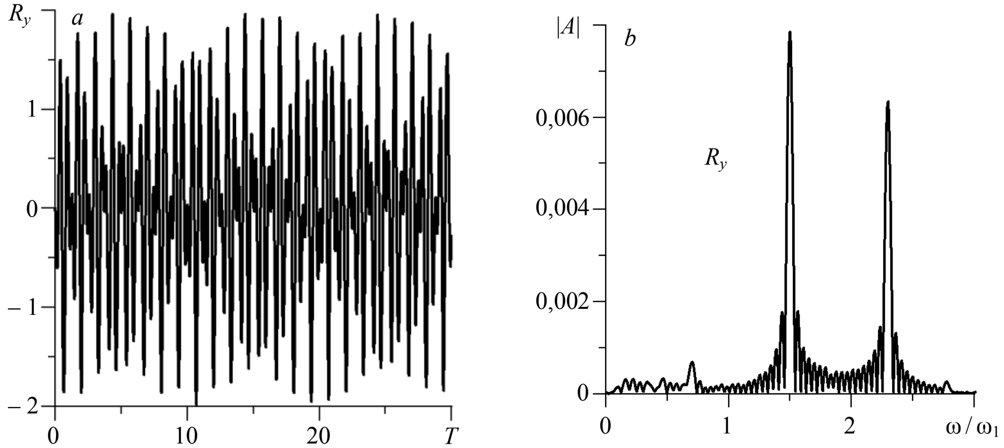

Fig. 6. Forced oscillations of liquid $F_x = A\cos(pt)$, $p = 2{,}3\omega_1$
a) total force of water acting on the wall of a tank, *b*) frequency spectrum

[1, 2, 4], when law of reservoir motion is given, i.e., this corresponds to reservoir with infinitely great mass, and free surface oscillations of liquid do not affect the reservoir motion. In this case, nonlinear interaction of liquid sloshing with translational motion of reservoir leads to quasi-periodic motion characterized by the effects of variation in time of mean value and amplitude modulation.

Conclusion. We consider a problem of modelling of nonlinear forced motion of liquid with a free surface in movable hyperboloid reservoir. Behaviour of the system is considered under horizontal periodic disturbance, when frequency of external force is in a vicinity of resonance, namely, below, near and above resonance values. Frequency spectrum of elevation near reservoir walls and spectrum of liquid response on tank walls were compared. It was shown that transition to steady mode of motion in the considered nonlinear multifrequency systems of «hyperboloid reservoir – liquid» type is not manifested at all.

References

[1] *Микишев Г. Н., Рабинович Б. И.* Динамика твердого тела с полостями, частично заполненными жидкостью — Москва: Машиностроение, 1968. — 532 с.

[2] *Нариманов Г. С., Докучаев Л. В., Луковский И. А.* Нелинейная динамика летательного аппарата с жидкостью. — Москва: Машиностроение, 1977. — 208 с.

[3] *Ibrahim R. A.* Liquid sloshing dynamics: theory and applications. — Cambridge: Cambridge University Press, 2005. — 950 p.

[4] *Limarchenko O. S.* Peculiarities of application of perturbation techniques in problems of nonlinear oscillations of liquid with a free surface in cavities of non - cylindrical shape // Укр. мат. журн. — 2007. — Т. 59, № 1. — С. 44-70.

[5] *Limarchenko O., Semenova I.* Forced oscillations of liquid in a reservoir of paraboloidal shape // Фіз.-мат. моделювання та інформ. технології. — Львів: 2013. — Вип. 18. — С. 106-112.

Вимушені нелінійні коливання рідини в гіперболоїді обертання

Ірина Семенова

У роботі досліджено особливості поведінки механічної системи «резервуар – рідина з вільною поверхнею» за збурення руху резервуара горизонтальною гармонічною силою. Задача про вимушені коливання вивчається на основі нелінійної багатомодової моделі, що описує сумісний рух резервуара з рідиною під дією активних зовнішніх сил. Показано, що вихід системи на усталений режим коливань не відбувається.

Вынужденные нелинейные колебания жидкости в гиперболоиде вращения

Ирина Семенова

В работе исследованы особенности поведения механической системы «резервуар – жидкость со свободной поверхностью» при возбуждении движения резервуара горизонтальной гармонической силой. Задача о вынужденных колебаниях жидкости изучается на основе многомодовой нелинейной динамики совместного движения ограниченного объема жидкости со свободной поверхностью и резервуара гиперболической формы. Показано, что выход системы на установившийся режим не происходит.

Представлено професором Є. Чаплею

Отримано 24.11.14