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The system of powers of conformal mappings and biorthogonal to them systems of the functions

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In this article we review the methods of power summation of factors. The degree of factors which are arbitrary powers of summation indices are considered. We show that using the Poisson-Abel method only those series can be summarized the order of member increase of which is proportional to the exponent depending on the summation index. At the same time the Gauss-Weierstrass method and other power factors methods can be also applied to the series the terms of which increase in proportion to the exponential dependence of the indices summation.

Keywords: biorthogonal systems; conformal mapping; functional series; Helmholtz equation.

Introduction. The systems of the functions, biorthogonal on closed lines in mono-connected domains in the complex plane determine bases in spaces of analytic functions in these areas [1-3]. The presentation of the functions by series by systems of polynomials and other systems of functions were investigated using contour integrals and conformal mappings in works [3-5]. In the works [6-8] biorthogonal systems of the functions are used for constructing solutions of plane and spatial boundary value problems for the Helmholtz equation.

1. Biorthogonal functions of the system. Let:

$$w = \varphi(z) = \varphi_0(z - z_0) \tag{1}$$

be conformal mapping of mono-connected domain D of extended complex plane z in the circle $K : |w| < 1$ of complex plane w; $\varphi(z_0) = 0$; $\varphi'(z_0) = 1$; $z = h(w)$— inverse mapping; border $L = \partial D$ is displayed on the circle $C : |w| = 1$.

The system of the functions $\left\{w^n\right\}_{n=0}^{\infty}$ and conjugate (associated) with it system $\left\{1/w^{m+1}\right\}_{m=0}^{\infty}$ is biorthogonal on a closed conour:

$$\frac{1}{2\pi i}\int_{C_r}\frac{w^n}{w^{m+1}}dw=\delta_{nm}\,. \tag{2}$$

Here $C_r:|w|=r,\ 0<r<1$. The system $\left\{w^n\right\}$ is a basis in a broad sense [3, p. 612] in the space of functions which are analytic in a circle $K:|w|<1$. And the system $\left\{1/w^{m+1}\right\}$ is a basis in the space of functions, which are analytic outside the circle $\overline{K}$.

Substituting the expression of the mapping $w=\varphi(z)$ in the orthogonality condition (2), we obtain such two relations:

$$\frac{1}{2\pi i}\int_{L_r}\varphi^n(z)\frac{\varphi'(z)}{\varphi^{m+1}(z)}dz=\delta_{nm};\quad \frac{1}{2\pi i}\int_{L_r}\frac{d}{dz}\frac{\varphi^{n+1}(z)}{n+1}\frac{1}{\varphi^{m+1}(z)}dz=\delta_{nm}, \tag{3}$$

where $L_r\subset D$ is a prototype of a circle C_r by the mapping (1).

We introduce the systems of the functions

$$\left\{g_n(z)=\varphi^n(z)\right\}_{n=0}^{\infty}.\ \left\{g_n^*(z)=\frac{1}{n+1}\frac{d}{dz}\varphi^{n+1}(z)\right\}_{n=0}^{\infty},\ z\in D. \tag{4}$$

Functions $g_n(z)$ i $g_n^*(z)$ are analytical in the neighborhood of the point z_0. As a result of analyticity of the function $\varphi(z)$ in D and their expansion in Taylor's series are possible in the neighborhood of this point. Appropriate conjugates to (4) systems of the functions $\left\{\omega_m(z-z_0)\right\}_{m=0}^{\infty}$, $\left\{\omega_m^*(z-z_0)\right\}_{m=0}^{\infty}$ in accordance with (3), we can select by principal parts of Laurent series in the neighborhood of the point z_0 function $\varphi'(z)\varphi^{-(m+1)}(z)=\omega_m(z-z_0)+g_m(z-z_0)$. Function $\varphi^{-(m+1)}(z)=\omega_m^*(z-z_0)+g_m^*(z-z_0)$, where $g^*(z-z_0)$. $g_m^*(z-z_0)$ are the right of the expansis of the functions.

Theorem 1. Let $w=\varphi(z)$ be conformal mapping (1) i $|z-z_0|<l$ is the largest circle in which Taylor's series of this function convergent, and which is in the domain D. Then the systems of the functions (4) bases in the space $E_r, 0<r<l$, functions which are analyticin the circle $|z-z_0|<r$.

Proof. Consider the function

$$F(\xi)=\frac{1}{\varphi_0(l/\xi)},$$

where $\xi=z-z_0$. It is heand univalent in the domain $|\xi|>l$ and satisfies conditions $F(\infty)=\infty$, $\lim\limits_{z\to\infty} F(\xi)/\xi=1$. Let $\psi(\xi)$ is univalent function in the fdomain $|\xi|>l$ i $\psi(\infty)=1$. Then, by Theorem 10 [3. Art. 616] system of polynomials $\{p_n(\xi)\}_{n=0}^{\infty}$ is basis in $E_r, r>l$, where $p_n(\xi)$ are the main parts of the Laurent' series of functions $\frac{F^n(\xi)F'(\xi)}{\psi(\xi)}$ in the vicinity of the infinitely far point. Conjugates to these polynomials are functions $\widetilde{\omega}_m(\xi)=\frac{\psi(\xi)}{F^{m+1}(\xi)}, m=0,1,\ldots$. Also, the system functions $\{\widetilde{\omega}_m(z)\}_{m=0}^{\infty}$ is basis in $\widetilde{E}_\rho$ functions, which are analytic in the field of $|\xi|>\rho, \rho\geq l$ and equal to zero at infinitely remote point, and $\{p_n(\xi)\}_{n=0}^{\infty}$ is conjugate to it system of functions.
Biorthogonality conditions are:

$$\frac{1}{2\pi i}\int\limits_{|\xi|=r'\geq r} p_n(\xi)\widetilde{\omega}_m(\xi)d\xi=\delta_{nm}.$$

Transform these conditions using the mapping $\xi=l/(z-z_0)$:

$$\frac{1}{2\pi i}\int\limits_{\Gamma^+}\widetilde{\omega}_m\left(\frac{l}{z-z_0}\right)p_n\left(\frac{l}{z-z_0}\right)\frac{l\,dz}{(z-z_0)^2}=\delta_{nm}. \qquad (5)$$

Here $\Gamma^+ : |z-z_0|=l/r'$ is circle oriented counterclockwise.

The system of the functions $\left\{\frac{\widetilde{\omega}_m(l/z-z_0)}{z-z_0}\right\}$ is basis in the space of functions which are analytic in a circle $|z-z_0|<l$ and $\left\{\frac{lp_n(l/(z-z_0))}{z-z_0}\right\}$ is its conjugate system of polynomials in negative powers of the variable,

$$\frac{\widetilde{\omega}_m(l/(z-z_0))}{z-z_0}=\frac{\psi(l/(z-z_0))}{zF^{m+1}(l/(z-z_0))}=\varphi^m(z)\frac{\varphi(z)\psi(l/(z-z_0))}{z}; \tag{6}$$

where $F'(\xi)=F'(l/(z-z_0))=\frac{(z-z_0)^2\varphi'(z)}{l\varphi^2(z)}$; $\Gamma[g(z)]$ is the main part of the Laurent series $g(z)$ in the neighborhood of point z_0.

If you choose $\psi(l/(z-z_0))=\frac{z-z_0}{\varphi(z)}$ and $\psi(l/(z-z_0))=\frac{(z-z_0)\varphi'(z)}{\varphi(z)}$, then from (5), we obtain the conditions of biorthogonality respectively for the first and second functions (4) and coupled to them functions of the systems:

$$\omega_n(z-z_0)=\frac{lp_n(l/(z-z_0))}{z-z_0}=\Gamma\left[\frac{\varphi'(z)}{\varphi^{n+1}(z)}\frac{z-z_0}{\varphi(z)\psi(l/(z-z_0))}\right]=\Gamma\left[\frac{\varphi'(z)}{\varphi^{n+1}(z)}\right];$$

$$\omega_n^*(z-z_0)=\frac{lp_n(l/(z-z_0))}{z-z_0}=\Gamma\left[\frac{\varphi'(z)}{\varphi^{n+1}(z)}\frac{z-z_0}{\varphi(z)\psi(l/(z-z_0))}\right]=\Gamma\left[\frac{1}{\varphi^{n+1}(z)}\right]. \tag{7}$$

Thus, if the function $f(z)$ is analytic in the circle $|z-z_0|<l$, it decomposes inside the circle in a uniformly convergent series in the systems (4),

$$f(z)=\sum_{n=0}^{\infty}a_n g_n(z).\ f(z)=\sum_{n=0}^{\infty}a_n g_n(z),\ |z-z_0|\le r<l, \tag{8}$$

where $a_n=\frac{1}{2\pi i}\int\limits_{|\xi-z_0|=r'\le r} f(\xi)\omega_n(\xi-z_0)d\xi$; $b_n=\frac{1}{2\pi i}\int\limits_{|\xi-z_0|=r'\le r} f(\xi)\omega_n^*(\xi-z_0)d\xi$.

Theorem 2. The system of the functions (4) are the bases in the space of functions, which are analytical in the domain D.

Proof. We show that the series (8) converge uniformly in any field $\overline{D'}\subset D$, if the function $f(z)$ is analytic in D. Since (1) represents the domain D on the unit circle, then any point $z \in \overline{D}'$ is on line L_ρ, which is a prototype of a circle $|w| = \rho < 1$. Then, we have the estimates for functionsof systems(4):

$$|g_n(z)| = |\phi(z)|^n = \rho^n, |g_n^*(z)| = |\phi(z)|^n |\phi'(z)| \le A_\phi \rho^n, z \in L_\rho, \quad (9)$$

where $A_\varphi = \max\limits_{z \in L_\rho} |\varphi'(z)|$.

Let L_{ρ_0} be a prototype of a circle $|w| = \rho_0$, $\rho < \rho_0 < 1$, $F = \max\limits_{z \in L_{\rho_0}} |f(z)|$, $A_\psi = \max\limits_{|w| = \rho_0} |h'(w)|$. We transform the expression for the coefficients of the series (8) taking into considerating that the integral along the contour $L_{\rho_0} \subset D$ from the function, which is analytic in the domain D is equal zero:

$$a_n = \frac{1}{2\pi i} \int\limits_{|\xi| = L_{\rho_0}} f(\xi) \omega_n(\xi - z_0) d\xi = \frac{1}{2\pi i} \int\limits_{|\xi| = L_{\rho_0}} \frac{f(\xi)\varphi'(\xi) d\xi}{\varphi^{n+1}(\xi)} = \frac{1}{2\pi i} \int\limits_{|w| = \rho_0} \frac{f(h(w)) dw}{w^{n+1}}$$

$$b_n = \frac{1}{2\pi i} \int\limits_{|\xi| = L_{\rho_0}} f(\xi) \omega_n^*(\xi - z_0) d\xi = \frac{1}{2\pi i} \int\limits_{|\xi| = L_{\rho_0}} \frac{f(\xi) d\xi}{\varphi^{n+1}(\xi)} = \frac{1}{2\pi i} \int\limits_{|w| = \rho_0} \frac{f(h(w)) h'(w) dw}{w^{n+1}}$$

and find the assessment:

$$|a_n| \le \frac{1}{2\pi} \int\limits_{|w| = \rho_0} \frac{|f(h(w))||dw|}{|w|^{n+1}} \le \frac{F}{\rho_0^n};$$

$$|b_n| \le \frac{1}{2\pi} \int\limits_{|w| = \rho_0} \frac{|f(h(w))||h'(w)||dw|}{|w|^{n+1}} \le \frac{F A_\psi}{\rho_0^n}. \quad (10)$$

Estimating the sum of series (8) with the estimates (9), (10) and inequality $\rho < \rho_0$, we see that these series convergence in the domain $\overline{D}' \subset D$:

$$\sum_{n=0}^{\infty} |a_n||g_n(z)| \le F \sum_{n=0}^{\infty} \left(\frac{\rho}{\rho_0}\right)^n = \frac{F\rho_0}{\rho_0 - \rho}. \quad \sum_{n=0}^{\infty} |b_n||g_n^*(z)| \le F A_\psi \sum_{n=0}^{\infty} \left(\frac{\rho}{\rho_0}\right)^n = \frac{F A_\psi \rho_0}{\rho_0 - \rho}$$

and analytic sums of series in this domain. Since the circle $|z| < l$ also belongs to the domain D, sums of the series (8) continue analytically the function $f(z)$ in the domain.

Example 1. Let domain D be the half-plane $\operatorname{Im} z > 0$. Conformal mapping of the domain D on the unit circle $K: |w| < 1$ and inverse mapping functions are given by the functions [9]

$$w = i\frac{2z - i}{2z + i}. \qquad z = \frac{i}{2}\frac{i + w}{i - w}. \tag{11}$$

At the same time, points $z = \pm 1/2$, $z = 0$, $z = \infty$ match point $w = \pm 1$, $w = -i$, $w = i$ and the limit $L = \partial D$, which is the axis Oy is displayed in a circle $C: |w| = 1$.

We introduce in accordance with a first relation (3) the system of the functions

$$\left\{ g_n(z) = i^n \left(\frac{2z - i}{2z + i} \right)^n \right\}_{n=0}^{\infty}, \ z \in D. \tag{12}$$

Functions $g_n(z)$ are analytic in the neighborhood of the point $z_0 = i/2$. The functions $\varphi'(z)/\varphi^{m+1}(z) = = 4i^{m-1}(2z + i)^{m-1}/(2z - i)^{m+1}$ have the poles of the order $m + 1$ in this point.

First, we find the decomposition of the function $g_n(z)$ into Maclaurin series, which converges in the circle $|z - i/2| < 1/2$, which belongs to the domain D,

$$g_0(z) = 1, g_n(z) = i^n \left(\frac{z - i/2}{z + i/2} \right)^n = \sum_{k=0}^{\infty} i^k C_{n+k-1}^k \left(z - \frac{i}{2} \right)^{n+k}, \ n \geq 1. \tag{13}$$

Now, we construct the system of functions, which is conjugate to (12) and (13). First, we find the expansion of the function $\varphi'(z)/\varphi^{m+1}(z) = 4i^{-m+1}(2z + i)^{m-1}/(2z - i)^{m+1}$ into a Laurent series in the neighborhood of the point $z_0 = i/2$,

$$\frac{4i^{-m+1}(2z + i)^{m-1}}{(2z - i)^{m+1}} = \left(z - \frac{i}{2} \right)^{-(m+1)} \sum_{k=0}^{m-1} C_{m-1}^k i^{k-m+1} \left(z - \frac{i}{2} \right)^{m-1-k}.$$

$$= \sum_{k=0}^{m-1} C_{m-1}^k i^{k-m+1} \left(z - \frac{i}{2} \right)^{-k-2}$$

Hence, we find the main part of the Laurent series of this function

$$\omega_0(z)=\left(z-\frac{i}{2}\right)^{-1},$$

$$\omega_m(z)=\frac{i^{-m+1}(z+i/2)^{m-1}}{(z-i/2)^{m+1}}=\sum_{k=0}^{m-1}C_{m-1}^{k}i^{k-m+1}\left(z-\frac{i}{2}\right)^{-k-2},\ m\ge 1. \tag{14}$$

Note that only the number of the zero member of the original system has the correct part. The systems of the functions (13) and (14) are biorthogonal on any circle encompassing the point $z=i/2$ and lying in a domain D. According to Theorem 1, the system of the series sums (13) is a basis in the space of analytic functions in a circle $|z-i/2|<1/2$, and by Theorem 2, the system (12) is basis in the space of the functions, which areanalytic in the halfplane $\operatorname{Im} z>0$.

Note that the sum of the series

$$\sum_{m=0}^{\infty}g_m(z)\omega_m(t)=\frac{2}{2t-i}+4i\sum_{m=1}^{\infty}\left(\frac{2z-i}{2z+i}\right)^m\frac{(2t+i)^{m-1}}{(2t-i)^{m+1}}=$$

$$=\frac{2}{2t-i}+4i\left(\frac{2z-i}{2z+i}\right)\frac{1}{(2t-i)^2}\sum_{m=0}^{\infty}\left(\frac{2z-i}{2z+i}\right)^m\left(\frac{2t+i}{2t-i}\right)^m$$

$$=\frac{2}{2t-i}+\frac{1}{(2t-i)}\frac{2z-i}{t-z}=\frac{1}{t-z}$$

converges uniformly with respect to z in domain D_r with abroad L_r (prototype of circle $|w|=r,\ 0<r\le 1$) for any $t\in L_r$. Then, for the function, which is analytic in the halfplane $\operatorname{Im} z>0$ we obtain an expansion of the function

$$f(z)=\frac{1}{2\pi i}\int_{L_r}\frac{f(t)dt}{t-z}=\sum_{n=0}^{\infty}a_n g_n(z),\ \text{where}\ a_n=\frac{1}{2\pi i}\int_{L_r}f(t)\omega_n(t)dt.$$

Example 2. If we have the real part of the function on axis Ox ,which is analytic in the halfplane $\operatorname{Im} z>0$, then according to Schwartz formula [9, p. 219], we have

$f(z)=\frac{1}{\pi i}\int\limits_{-\infty}^{\infty}\frac{u(t)dt}{t-z}+Ci$, where C is real constant; $u(t)$ is piecewise continuous function, $u(t)=o\left(1/|t|^{\alpha}\right)$ at $|t|\to\infty$, $\alpha>0$.

Then, for the coefficients of the expansion functions $f(z)$ by the system $\{g_n(z)\}$, we obtain formula

$$a_0=\frac{1}{\pi i}\int\limits_{-\infty}^{\infty}u(t)\omega_0(t)dt\,,\; a_n=\frac{1}{\pi i}\int\limits_{-\infty}^{\infty}u(t)\omega_n(t)dt\,.$$

Example 3. We find the decomposition by the system (12) of the function $f(z)$, which is analytic in the half-plane $\operatorname{Im} z>0$, its real part takes values $u(t)=1/\left(t^2+h^2\right), 0<h<\infty$ on the real axis. By formulas (15) and (16) we find the expansion of the function and its coefficients

$$f(z)=\frac{1}{\pi i}\int\limits_{-\infty}^{\infty}\frac{u(t)dt}{t-z}=-2\sum_{\operatorname{Im} t_k<0}\underset{t=t_k}{res}\frac{u(t)}{t-z}=-2\underset{t=-hi}{res}\frac{1}{\left(t^2+h^2\right)(t-z)},$$

$$=-\frac{1}{hi(z+hi)}=\sum_{n=0}^{\infty}a_n g_n(z)$$

where $a_0=\frac{1}{\pi i}\int\limits_{-\infty}^{\infty}\frac{1}{\left(t^2+h^2\right)(z-i/2)}dz=-2\underset{z=-hi}{res}\frac{1}{\left(t^2+h^2\right)(z-i/2)}=\frac{1}{h(h+1/2)}$;

$$a_n=\frac{1}{\pi i}\int\limits_{-\infty}^{\infty}\frac{1}{\left(t^2+h^2\right)}\frac{i^{-n+1}(t+i/2)^{n-1}}{(t-i/2)^{n+1}}dt=-2\underset{z=-hi}{res}\frac{1}{\left(h^2+t^2\right)}\frac{i^{-n+1}(t+i/2)^{n-1}}{(t-i/2)^{n+1}}$$

$$=\frac{-i^{-n}(h-1/2)^{n-1}}{h(h+1/2)^{n+1}}$$

Thus, we have

$$f(z) = -\frac{1}{hi(z+hi)} = \frac{1}{h(h+1/2)} - \sum_{n=1}^{\infty} \frac{(h-1/2)^{n-1}}{h(h+1/2)^{n+1}} \left(\frac{2z-i}{2z+i}\right)^n .$$

If $h = 1/2$, then $f(z) = -\dfrac{2}{i(z+i/2)} = 2g_0(z) + 2ig_1(z)$.

2. The Dirichlet problem for the Helmholtz equation.

Let us find a solution to this problem in a mono-connected domain D. Let: $w = \varphi(z)$ be conformal mapping (1). We write the Helmholtz equation using variables $w, \overline{w}$:

$$4\frac{\partial^2 U}{\partial w \partial \overline{w}} + U = 0, \tag{15}$$

where $\kappa = const$; $U = U(u,v) = U(w,\overline{w})$ is a real function.

The solution of this equation in the circle $K : |w| < 1$ can be written as [6, 8]

$$U = \operatorname{Re} \sum_{m=0}^{\infty} c_m w^m J_m^*(w\overline{w}) , \tag{16}$$

where $J_m^*(w\overline{w}) = J_m^*(|w|^2) = \sum_{n=0}^{\infty} \dfrac{(-1)^n |w|^{2n}}{2^{2n+m}(n+m)!n!}$, c_m are arbitrary constants. Functions $J_m^*(w\overline{w})$, if $\kappa > 0$ are directly expressed in terms of Bessel functions of m th order, $J_m(|w|) = |w|^m J_m^*(|w|^2)$.

We now turn to the new variables $z = h(w)$. $\overline{z} = \overline{h}(w)$. As $\varphi'(z) \neq 0$, $z \in D$, we obtain the equation:

$$4\frac{\partial^2 U}{\partial z \partial \overline{z}} + \varphi'(z)\overline{\varphi}'(z)U = 0. \tag{17}$$

The solutions of equation (17) we obtain from (16) by replacing the variables $w = \varphi(z)$, $\overline{w} = \overline{\varphi}(z)$:

$$U(z,\bar{z}) = \operatorname{Re} \sum_{m=0}^{\infty} c_m \varphi^m(z) J_m^*(\varphi(z)\overline{\varphi}(z)). \tag{18}$$

Let us find the solution of equation (17) in the domain D on condition

$$U(z,\bar{z})\big|_L = u(t), t \in L, \tag{19}$$

where $u(t)$ is the real part of a function which splits into evenly convergent series of the first system (4),

$$f(z) = \sum_{n=0}^{\infty} a_n g_n(z), \; z \in \overline{D}. \tag{20}$$

From this condition it follows [9, p. 192] the analycity of function $f(z)$ in domain D and its continuity at the boundary of the area.

The solution of the problem we search as a sum of a series (18). Coefficients of this series we find from the conditions (9). Substituting formula (18) i (20) with the boundary condition depending $g_n(z) = \varphi^n(z)$, $\varphi(t)\overline{\varphi}(t) = \left|\varphi(t)^2\right| = 1$, $\varphi(t) = e^{i\psi}$, $0 \le \psi < 2\pi$, we obtain the equation :

$$\sum_{n=0}^{\infty} c_n e^{in\psi} J_n^*(1) = \sum_{n=0}^{\infty} a_n e^{in\psi}.$$

Hence, we find $c_n = a_n / J_n^*(1)$ and write down the solution of the problem :

$$U(z,\bar{z}) = \operatorname{Re} \sum_{n=1}^{\infty} \frac{a_n}{J_n^*(1)} g_n(z) J_n^*(\varphi(z)\overline{\varphi}(z)). \tag{21}$$

Uniform convergence of the series (21) in the domain D follows from the limitation of the functions $J_n^*(\varphi(z)\overline{\varphi}(z))$, as $0 \le |\varphi(z)| \le 1$, $z \in D$ and uniform convergence of the series (20).

Example 4. We write the solution of equation (17) in the domain D on condition

$$U(z,\bar{z})\big|_L = \operatorname{Re} g_n(t), t \in L. \tag{22}$$

From the expressions (19) and (21) we find:

$$U(z,\bar{z}) = \frac{1}{J_n^*(1)} \operatorname{Re} g_n(z) J_n^*(\varphi(z)\overline{\varphi}(z)).$$

If $f(t) = g_0(t) = 1$, then

$$U(z,\bar{z}) = \frac{1}{J_0^*(1)} J_0^*(\varphi(z)\overline{\varphi}(z)),$$

where $J_0^*(\varphi(z)\overline{\varphi}(z)) = J_0(|\varphi(z)|)$ is Bessel function of zero order.

3. NeumannproblemfortheHelmholtzequation.

Let us find the solution of equation (17) in the domain D on condition

$$\left.\frac{\partial U(z,\bar{z})}{\partial n}\right|_L = f(t),\ t \in L, \tag{23}$$

Where $\partial/\partial n$ is derivative in the direction of the normal to the curve L. We assume that the boundary function is represented as

$$f(t) = \operatorname{Re}\frac{\partial F(t)}{\partial n},$$

where $F(z)$ is analytic function in the domain $\overline{D}$ and its expansion in a uniformly convergent series of the system $\{\varphi^n(z)\}_{n=0}^{\infty}$ looks as:

$$F(z) = \sum_{n=0}^{\infty} d_n \varphi^n(z) \,. \tag{24}$$

First, we find the derivative of the function $F(z)$ in a direction of the normal to L_r: $z = h(w)$, $r = const$ (the prototype of the circumference $|w| = r,\ 0 < r \le 1$). Considering the formula $\partial_n \varphi(z) = -i\partial_\tau \varphi(z)$, $\rho = |\varphi(z)|$, $h'(w)\varphi'(z) = 1$, $w = re^{i\psi}$, where $\partial_\tau = \partial/\partial\tau$ is derivative in a direction of tangent to L_r, we find:

$$\frac{\partial\phi(z)}{\partial n}=-i\phi'(z)\frac{\partial z}{\partial\tau}=-\frac{i\phi'(z)\partial_\psi z}{|\partial_\psi z|}=\phi'(z)\frac{h'(w)e^{i\psi}}{|h'(w)|}=\frac{e^{i\psi}}{|h'(w)|}.$$

$$=\frac{1}{|h'(w)|}\frac{w}{r}=|\phi'(z)|\frac{\phi(z)}{|\phi(z)|}$$

$$\frac{\partial(\varphi\overline{\varphi})}{\partial n}=\frac{2}{r}(\varphi\overline{\varphi})|\varphi'(z)|=2\frac{|\varphi'(z)|}{|\varphi(z)|}(\varphi\overline{\varphi}). \qquad (25)$$

Considering first the formula (24), we obtain the derivative of $F(z)$:

$$\frac{\partial F(z)}{\partial n}=F'(z)\frac{\partial z}{\partial n}=F'(z)\frac{h'(w)e^{i\psi}}{|h'(w)|}=\frac{F'(z)}{\varphi'(z)}\frac{|\varphi'(z)|\varphi(z)}{|\varphi(z)|}=\frac{\partial F(z)}{\partial\varphi}\frac{|\varphi'(z)|\varphi(z)}{|\varphi(z)|},$$

Which takes the following form on the contour L due to conditions $|\varphi(z)|\big|_L=1$:

$$\left.\frac{\partial F(z)}{\partial n}\right|_L=|\varphi'(z)|\varphi(z)\partial_\varphi F(z)\big|_L. \qquad (26)$$

The boundary condition (22) in view of formula (25) can be written as:

$$\left.\frac{\partial U(z,\bar z)}{\partial n}\right|_L=|\varphi'(t)|\mathrm{Re}\left[\varphi(t)\partial_\varphi F(t)\right],t\in L. \qquad (27)$$

Now, we build the solution (17), (26). Function, whose real partis the solution of equation (17) , $U(z,\bar z)=\mathrm{Re}\,\Phi(z,\bar z)$, we look as the sum of a series:

$$\Phi(z,\bar z)=\sum_{m=0}^{\infty}c_m\varphi^m(z)J_m^*\left(\varphi(z)\overline{\varphi(z)}\right).$$

The derivative of this function in the direction of the normals to L_r, we obtain using formulas (24) as:

$$\partial_n\Phi(z,\bar z)=\frac{|\varphi'(z)|}{|\varphi(z)|}\sum_{m=0}^{\infty}c_m\varphi^m(z)\left[mJ_m^*(\varphi\overline{\varphi})-(\varphi\overline{\varphi})J_{m+1}^*(\varphi\overline{\varphi})\right].$$

Substituting this expression into the condition (26) in view of the expansion (23), we find:

$$|\varphi'(t)| \operatorname{Re} \sum_{m=0}^{\infty} c_m \varphi^m(t)\left[m J_m^*(\varphi\overline{\varphi}) - (\varphi\overline{\varphi}) J_{m+1}^*(\varphi\overline{\varphi})\right] = |\varphi'(t)| \operatorname{Re} \sum_{m=1}^{\infty} m d_m \varphi^m(t),$$

$t \in L$

or

$$\operatorname{Re} \sum_{m=0}^{\infty} c_m e^{im\psi}\left[m J_m^*(1) - J_{m+1}^*(1)\right] = \operatorname{Re} \sum_{m=1}^{\infty} m d_m e^{im\psi} .$$

Hence, we find $c_m = m\, d_m \Big/ \left[m J_m^*(1) - J_{m+1}^*(1)\right]$ and write down the solution of the problem

$$U(z,\bar{z}) = \operatorname{Re} \sum_{m=1}^{\infty} \frac{m d_m}{m J_m^*(1) - J_{m+1}^*(1)} \varphi^m(z) J_m^*\left(\varphi(z)\overline{\varphi(z)}\right). \tag{28}$$

The uniform convergence of the series (27) in domain $\overline{D}$ should be with limited functions $J_m^*\left(|\varphi|^2\right)$ and the uniform convergence of (23) in this domain.

Example 5. Let us find the solution of the problem (17), (26) for the case of function $F(z) = \varphi(z)$ and correspondingly, $f(t) = |\varphi'(t)|\varphi(t)$. Then, from (27), we obtain:

$$U(z,\bar{z}) = \frac{1}{J_1^*(1) - J_2^*(1)} J_1^*\left(|\varphi(z)|^2\right) \operatorname{Re} \varphi(z) .$$

If $\varphi(z) = 1/z = r^{-1} e^{-i\psi}$ and correspondingly, $f(t) = \cos\psi$, we have

$$U(z,\bar{z}) = \frac{1}{J_1^*(1) - J_2^*(1)} \frac{\cos\psi}{r} J_1^*\left(\frac{1}{\rho^2}\right) = \frac{\cos\psi}{J_1(1) - J_2(1)} J_1\left(\frac{1}{\rho}\right).$$

The resulting representation is a solution of the corresponding problem (17), (26) in a plane with a circular hole.

Conclusion. Essential in the construction of bases in spaces of analytic functions is domain which is simply connected, which defines the functions and representations in explicit form of the conformal mappings of domains in the circle. For the construction of solutions of the boundary value problems are used rather strict condition for the uniform convergence of the corresponding series on the boundary of the domain. These conditions can besimplified by consideration the generalized solutions of the problems.

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Системи степенів конформних відображень і біортогональні до них системи функцій

Галина Івасик

Використовуючи конформні відображення однозв'язних областей на круг, побудовано базиси в просторах функцій, аналітичних в цих областях. Многочлени Фабера біортогональні з базисними функціями. Грунтуючись на розкладах аналітичних функцій в ряди, побудовано розв'язки граничних задач для рівняння Гельмгольца, граничні значення яких збігаються з граничними значеннями цих функцій.

Системы степеней конформных отображений и биортогональные с ними системы функций

Галина Ивасык

Используя конформные отображения односвязных областей на круг, построены базисы в пространствах функций, аналитических в этих областях. Многочлены Фабера биортогональны с базисными функциями. Основываясь на разложениях аналитических функций в ряды, построены решения граничных задач для уравнения Гельмгольца, граничные значения которых совпадают с граничными значениями этих функций.

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