1M

UDK 514.116

L.P. Mironenko
Donetsk National Technical University
Ukraine, 83000, Donetsk, Artema st., 58, mironenko.leon@yandex.ru

A New Approach to the Fundamental Limits in the Theory of Limits

Л.П. Мироненко
Донецкий национальный технический университет, Украина
Украина, 83000, г. Донецк, ул. Артема, 58, *mironenko.leon@yandex.ru*

Новый подход для фундаментальных пределов в теории пределов

Л.П. Мироненко
Донецький національний технічний університет, Україна
Україна, 83000, м. Донецьк, вул. Артема, 58

Новий підхід до фундаментальних границь у теорії границь

The purpose of the paper is an alternative way of obtaining of the fundamental limits in the elementary theory of limits. The approach uses two double inequalities. The method is applied to the both limits simultaneously, that makes the theory universal. Proof of the limits is quite new and original. Apart from that our theory gives many new representations of the limits (total 37).
Keywords: methodic, theory of limits, fundamental limits, function, sine, hyperbolic sine, method, inequalities, standard limits.

В работе предложен единый подход к фундаментальным пределам в теории пределов. Подход основан на использовании двойных неравенств и предельном переходе в них. Метод применим к обоим фундаментальным пределам одновременно, что значительно упрощает общепринятые подходы. Доказательства обоих пределов дано в одном стиле и носит оригинальный характер. Кроме того, получены новые следствия из фундаментальных пределов.
Ключевые слова: методика, теория пределов, фундаментальные пределы, неравенство, функция, синус, гиперболический синус, двойное неравенство.

Метою статті є підхід, який забезпечує єдиний спосіб отримання фундаментальних границь у теорії границь. Підхід заснований на використанні нерівностей і граничному переході в них. Підхід достатньо простий у використанні. Єдина система нерівностей забезпечує одночасне отримання формул першого та другого границь і, практично, всіх наслідків з них. Більш того, теорія призводить до великої кількості нових наслідків.
Ключові слова: методика, теорія границь, фундаментальні границі, функція, синус, гіперболічний синус, нерівність, граничний перехід.

Introduction

The fundamental limits in mathematical analysis are known as the first and second fundamental (”wonderful”) limits. They are used mainly for the determination of derivatives of the elementary functions such as $\sin x, \cos x, e^x, a^x$ and so on [1], [2]. The fundamental limits (we shall call simply limits) are used practically in all branches of mathematics, also for a calculation of the entire class of limits.

Each of the limits is proved by two different ways. The first limit is based on a limit procedure in the trigonometric circle. Newton's binomial is used for the second limit. Each of the approaches is effective and attractive. Nevertheless these approaches are not universal [1], [2].

However there are universal methods for the limits. For example, the first and second limits can be connected one with another by Euler's formula [3].

The next universal proof method is based on the so-called undetermined coefficient method which allows finding limits by the standard expansion of the functions $\sin x$ and e^x [4].

At last there are approaches that look alike as the method in this paper. They are also based on inequalities and application limits to them. Such methods have some disadvantages and there is some difficulty in the proof of the inequalities. This is the main serious disadvantage of such approaches [6].

The inequalities must be proved by elementary mathematical methods. Other words, the proof must be performed without using of a derivative concept.

We found a simple method to prove such limits. For this we used a deep analogy between the trigonometric and hyperbolic functions from one side and the classical proof method of the first limit from another side. Such way has occurred effective and helpful.

1 An auxiliary theorem of the method

The theorem that we shall use in our theory is well-known theorem about a function $f(x)$ which is placed between two given functions $f_1(x)$ and $f_2(x)$ [4].

Theorem. If the function $f(x)$ satisfies in a vicinity of the point x_o to the inequalities

$$f_1(x) \le f(x) \le f_2(x) \tag{1}$$

and $\lim\limits_{x\to x_o} f_1(x) = \lim\limits_{x\to x_o} f_2(x) = a$ then the limit $\lim\limits_{x\to x_o} f(x)$ exists and it is equal to a.

It is clear also, if the function $f(x)$ is continuous at $x = x_o$ then $a = f(x_o)$.

The inequalities (1) have a very important property of symmetry if all functions of the inequalities are odd. If to replace in the inequalities x by $-x$

$$-f_1(x) \le -f(x) \le -f_2(x) \Rightarrow f_1(x) \ge f(x) \ge f_2(x).$$

The sings of the inequalities (1) have changed.

If the functions are even

$$f_1(-x) \le f(-x) \le f_2(-x) \Rightarrow f_1(x) \le f(x) \le f_2(x),$$

The sings of the inequalities (1) does not change.

We shall use the theorem for a proof of the limits in the next item.

2 Inequalities for the fundamental limits

It is well-known the inequalities for the first fundamental limit follow from the trigonometric circle (Fig. 1)

$$\sin x \le x \le \tan x. \tag{2}$$

The inequalities are obvious for big values of $x \ge 0$ (Fig.2). It follows from a behavior of the graphs of the functions (2) at big values of x. This fact will be used later for a proof the inequalities for any values of x, including small values of x.

The proof will be done by the controversy method.

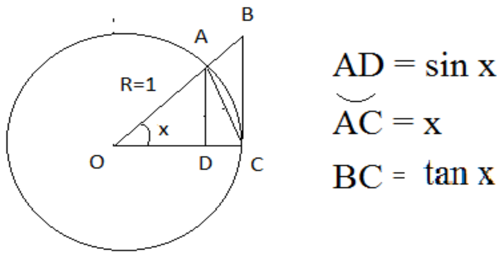

Figure 1 – The geometrical proof of the inequalities $\sin x \le x \le \tan x$.

The first inequality $\sin x \le x$ will be proved if the equation $\sin x = x$ has the unique solution $x_o = 0$. This follows directly from the figure (Fig. 2). Assume that there is a root $x_o \ne 0$ of the equation $\sin x = x$. Execute some elementary transformations to confirm our assumption.

$$\sin x = x \Rightarrow 2\sin\frac{x}{2}\cos\frac{x}{2} = x \Rightarrow \sin\frac{x}{2}\cos\frac{x}{2} = \frac{x}{2}.$$

We assumed that the equation $\sin\frac{x}{2}\cos\frac{x}{2} = \frac{x}{2}$ has a non-zero root x_o. Therefore we may to account the equality $\sin\frac{x_o}{2} = \frac{x_o}{2}$ in the last equation. In the result we get the equation $\cos\frac{x_o}{2} = 1$. From here $x_o = 0$ is an unique root of the equation $\sin x = x$.

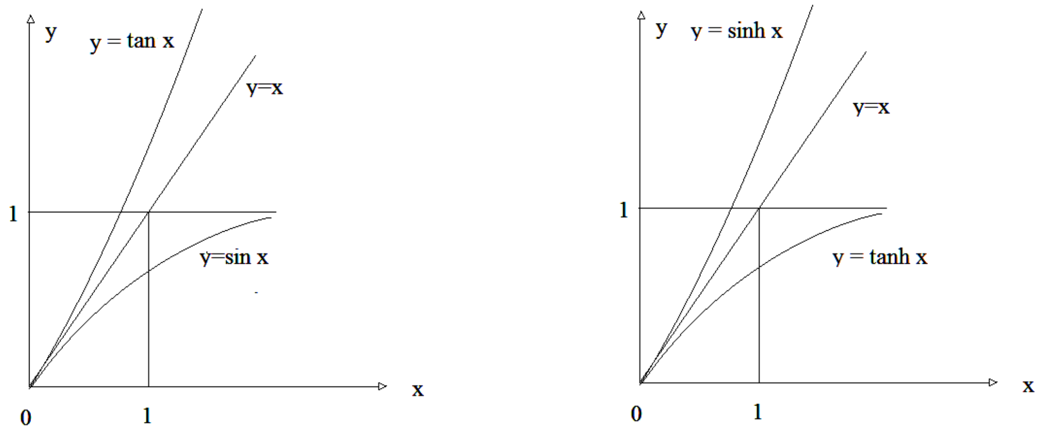

Figure 2 – The trigonometric and hyperbolic functions demonstrate the inequalities $\sin x \le x \le \tan x$ and $\tanh x \le x \le \sinh x$

The second inequality $x \le \tan x$ is proved as well as the previous inequality. In that case again we assume that there is a root $x_o \ne 0$ but now of the equation $x = \tan x$,

$$x = \tan x \Rightarrow x = \frac{\sin x}{cosx} \Rightarrow x = \frac{2\sin\frac{x}{2}cos\frac{x}{2}}{cos^2\frac{x}{2} - \sin^2\frac{x}{2}} \Rightarrow \frac{x}{2} = \frac{\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}} \Rightarrow 1 = \frac{1}{1-\tan^2\frac{x}{2}}.$$

The last equation is satisfied with only the root $x = 0$.

The inequality (2) has proved complitly.

The first limit follows from the inequalities (w2) and the theorem. For this the inequality (2) has to be divided by $\sin x > 0$ and limited at $x \to +0$.

$$\sin x \le x \le \frac{\sin x}{\cos x} \Rightarrow 1 \le \frac{x}{\sin x} \le \frac{1}{\cos x} \Rightarrow \cos x \le \frac{\sin x}{x} \le 1 \Rightarrow \lim_{x\to+0} \cos x \le \lim_{x\to+0} \frac{\sin x}{x} \le 1 \Rightarrow \lim_{x\to+0} \frac{\sin x}{x} = 1.$$

The method works for the hyperbolic functions. As a result we shall get one of a form of the second limit. For this consider inequalities like to the inequalities (2) (Fig. 2)

$$\tanh x \le x \le \sinh x. \quad (3)$$

Repeat the method as well as in the case of the first limit. Now use the formulas $\sinh x = 2\sinh\frac{x}{2} cosh\frac{x}{2}$, $\cosh x = cosh^2\frac{x}{2} + \sinh^2\frac{x}{2}$. For example,

$$\sinh x = x \Rightarrow 2\sinh\frac{x}{2}\cosh\frac{x}{2} = x \Rightarrow \sinh\frac{x}{2}\cosh\frac{x}{2} = \frac{x}{2} \Rightarrow \cosh\frac{x}{2} = 1 \Rightarrow x = 0.$$

The inequality (3) divide by $\sinh x > 0$, account also $\tanh x = \sinh x / \cosh x$. We have

$$\frac{1}{\cosh x} \le \frac{x}{\sinh x} \le 1 \Rightarrow 1 \le \frac{\sinh x}{x} \le \cosh x.$$

Taking into account also that $\lim\limits_{x\to 0} \cosh x = 1$, we have in the limit at $x \to 0$,

$$\lim_{x\to 0} \frac{\sinh x}{x} = 1. \quad (4)$$

This is one of the representations of the second fundamental limit.

The formulas (2) and (4) were received at $x \to +0$, but they are written for arbitrary $x \to 0$. It is cleared they do not change at $x \to -0$. In that case all functions in the inequalities (2) and (3) are odd, therefore for negative x we have

$$\tanh(-x) \le -x \le \sinh(-x) \Rightarrow -\tanh x \le -x \le -\sinh x \Rightarrow \sinh x \le x \le \tanh x.$$

Now show how to come from the form (4) to the standard form of the second limit. Let us rewrite the limit (r4) according to the definition of the function $\sinh x = \frac{e^x - e^{-x}}{2}$

$$\lim_{x\to 0} \frac{e^x - e^{-x}}{2x} = 1 \Rightarrow \lim_{x\to 0} \frac{e^x}{e^x}\frac{e^x - e^{-x}}{2x} = 1 \Rightarrow \lim_{x\to 0} \frac{1}{e^x} \lim_{x\to 0} \frac{e^{2x} - 1}{2x} = 1 \Rightarrow \lim_{x\to 0} \frac{e^x - 1}{x} = 1.$$

The last limit is known well [1-3]. It represents one of the corollaries from the second limit. Other forms of the second limit follow from this and they are described in many handbooks.

The corollaries of the fist fundamental limit are well-known and they are described in literature. We restrict ourselves only new corollaries (Tab. 1).

.

Table 1 – The corollaries from the first and second fundamental limits.

First standard limit $\lim_{x\to 0}\frac{\sin x}{x}=1$		Second standard limit $\lim_{x\to 0}\frac{\sinh x}{x}=1$	
Replacement or changing of the variable	Result	Replacement or changing of the variable	Result
$x=\arcsin y$	$\lim_{x\to 0}\frac{\arcsin x}{x}=1$	$x=a\sinh y$	$\lim_{x\to 0}\frac{a\sinh x}{x}=1$
$\tan x=\frac{\sin x}{\cos x}, \lim_{x\to 0}\cos x=1$	$\lim_{x\to 0}\frac{\tan x}{x}=1$	$\tanh x=\frac{\sinh x}{\cosh x}, \lim_{x\to 0}\cosh x=1$	$\lim_{x\to 0}\frac{\tanh x}{x}=1$
$x=\arctan y$	$\lim_{x\to 0}\frac{\arctan x}{x}=1$	$x=a\tanh y$	$\lim_{x\to 0}\frac{a\tanh x}{x}=1$

Results

1. It is developed a new effective method of the proof of the first and second fundamental limits in the classical limit theory.
2. The approach generalizes usual theory of the fundamental limits and from our point of view the theory is more universal.
3. The approach is generalized that leads to many equivalent forms of the limits (from the first limit we have got 3 corollaries, from the second limits 34). The most of them are new.
4. The proposed method essentially improves the classical limit theory.

Список литературы

1. Кудрявцев Л.Д. Математический анализ / Кудрявцев Л.Д. – Наука, 1970. – Том I. – 571 с.
2. Ильин В.А. Основы математического анализа / В.А. Ильин, Э.Г. Поздняк. – Москва : Изд. ФМЛ, 1956. – Том 1. – 472 с.
3. Мироненко Л.П. Эквивалентность стандартных пределов в теории пределов / Л.П. Мироненко // Искусственный интеллект. – 2012. – № 2.
4. Мироненко Л.П. Стандартные пределы и метод неопределенных коэффициентов / Л.П. Мироненко, И.В. Петренко // Искусственный интеллект. – 2012. – № 2.
5. Фихтенгольц Г.М. Курс дифференциального и интегрального исчисления / Фихтенгольц Г.М. – Наука, «ФМЛ», 1972 – Том 1. – 795 с.
6. Mironenko L.P. A compact system of inequalities for the standard limits in the theory of limits / L.P. Mironenko, A.Y. Vlasenko // Искусственный интеллект. – 2013 – № 2. – P. 61-70.

References

1. Kudriavtsew L.D. Matematichesky analiz. Том I., Nauka,1970 - 571 p.
2. Ilyin V.A., Pozdniak E.G. Oscnowy matematicheskogo analiza, том 1, Izd. FML, Moskwa, 1956. – 472 p.
3. Mironenko L.P. Ekvivalentnost standartnih predelov v teoryi predelov// Iskustvenyi intellekt, 2, 2012– p. 123-128.
4. Mironenko L.P., Petrenko I.V. Standartnie predeli i metod neopredelennih koefficientov// skustvenyi intellekt, 3, 2012– p. 284-291.
5. Fihtengolts G.M. Kurs differentsialnogo i integralnogo ischislenia, том 1, Nauka, «FML», 1972 - 795 p.
6. Mironenko L.P., Vlasenko A.Y. A compact system of inequalities for the standard limits in the theory of limits // Artificial intelligence -2, 2013– p. 61-70.

RESUME

L.P. Mironenko

A New Approach to the Fundamental Limitsin the Theory of Limits

Background: there is a deep analogy between the trigonometric and hyperbolic functions. The fist fundamental limit is connected with trigonometry. It should expect an universal method of a proof of the both fundamental limits. We found such way by using the method of the proof from a controversy.

Materials and methods: it is used so-called the method of a proof from a controversy. Besides in the paper we apply usual properties of limits such as taking a limit in inequalities, some methods of elementary mathematics.

Results: it is developed a new effective method of a proof of the first and second fundamental limits in the classical limit theory. The approach generalizes usual theory. From our point of view the theory is more simple and effective. The approach is more general then existing that we have got many equivalent forms of the limits (from the first limit 3 corollaries, from the second limits 34). The most of them are new. The proposed method essentially improves the classical fundamental limit theory.

Conclusion: it is found more effective method of a proof of two fundamental limits in the theory of limits. The method is universal and very simple. The result can be used by professors for students for a studying of the limit theory in colleges and universities.

The paper is received by the edititorial 05.04.2014.