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A LINEAR SYSTEM OUTPUT TRANSFORMATION FOR SPARSE APPROXIMATION[1]

Abstract. We propose an approach that provides a stable transformation of the output of a linear system into the output of a system with a desired basis. The matrix of basis functions of a linear system has a large condition number, and the series of its singular numbers gradually decreases to zero. Two types of methods for stable output transformation are developed using approximation of matrices based on the truncated Singular Value Decomposition and on the Random Projection with different types of random matrices. It is shown that the use of the output transformation as a preprocessing makes it possible to increase the accuracy of solving sparse approximation problems. An example of using the method in the problem of determining the activity of weak radiation sources is considered.

Keywords: sparse approximation, discrete ill-posed problem, random projection, singular value decomposition.

INTRODUCTION

In practical applications related to the recovery of signals from the results of indirect measurements, the following problem is often encountered. The signal emitted by the object of measurement is fed to the input of a linear measuring system. The result of measurements is a measurement vector (output). The matrix of a linear input "/" output transformation (a set of basis functions, which are non-orthogonal in the general case) is known. It describes the interaction of the measured signal with the environment, as well as the properties of the measuring means. The problem is to obtain the input vector given the output vector.

Here we consider a downstream problem using the recovered input vector. In particular, the set of basic functions of some measuring system may not meet user requirements or may be incompatible with downstream processing methods. However, if one knows a set of basis functions which would give the output of a measuring system with the required properties (resolution, accuracy), there appears a problem of transforming the output of a real system into the output of a system with

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such a desired basis. This could be achieved by passing the recovered input through the matrix of desired basis functions.

The matrices of real basis functions of many measuring means have a large condition number and an indefinite rank. This leads to an unstable and therefore inaccurate solution of the input recovery problem when using the pseudoinverse of the basis functions' matrix. In its turn, the solution for the output transformation problem also becomes unstable and inaccurate, and, therefore, does not provide the expected advantages of using the obtained output that approximate the output of the measuring system with the desired basis.

This paper develops stable methods for solving the problem of transforming the output of a linear measuring system for matrices of basis functions with a high condition numbers and indefinite rank. Two types of stable output transformation methods are being developed. These methods use matrix approximations based on the truncated Singular Value Decomposition and based on the Random Projection with different types of random matrices. It is shown how the transformation of the linear system output into the output of a system with a desired basis allows increasing the accuracy of solving sparse approximation problems. An example is considered that uses the proposed method in the problem of determining the activity of weak radiation sources.

1. THE PROBLEM OF OUTPUT TRANSFORMATION

Let the signal $\mathbf{b}$ be obtained from the output of an existing linear system that performs the transformation $\mathbf{Ax} + \boldsymbol{\varepsilon} = \mathbf{b}$, where $\mathbf{A} \in \Re^{m \times n}$, $\mathbf{x} \in \Re^{n}$, $\mathbf{b} \in \Re^{m}$, $\boldsymbol{\varepsilon} \in \Re^{m}$ (the vector of noise with the components that are realizations of a Gaussian independent random variable with zero mean and variance σ^2), and $\mathbf{Ax} = \mathbf{b}_0$. Let us denote $\mathbf{d}_0$ the output of the linear system C that performs the transformation $\mathbf{Cx} = \mathbf{d}_0$. In order to obtain a solution, i.e., an estimate of the output of the system C by using $\mathbf{b}$, we first obtain an estimate $\mathbf{x}^*$ of the input $\mathbf{x}$ by solving the inverse problem: $\mathbf{x}^* = \mathbf{P}\,\mathbf{b}$, where $\mathbf{P}$ is the operator (matrix) that transforms the output $\mathbf{b}$ to $\mathbf{x}^*$. Then we get an estimate $\mathbf{d}^*$ of the system C output: $\mathbf{d}^* = \mathbf{C}\mathbf{x}^* = \mathbf{CP}\,\mathbf{b} = \mathbf{T}\,\mathbf{b}$. Thus, the operator $\mathbf{CP}$ transforms $\mathbf{b}$ to $\mathbf{d}^*$. The transformation matrix $\mathbf{T} = \mathbf{CP}$ is called the reduction matrix in [1, 2].

The specific form of $\mathbf{P}$ depends on the properties of the matrix $\mathbf{A}$. If the series of singular values of $\mathbf{A}$ decreases smoothly and the condition number is large, the problem is classified as a discrete ill-posed problem. Approximate solutions of discrete ill-posed problems as least squares problems using numerical methods of linear algebra, such as the LU, Cholesky, QR decompositions, are unstable. This means that small perturbations in the input data lead to large perturbations in the solution. Thus, it is important to develop stable methods of output transformation.

We develop an approach for a stable solution of the output transformation problem using matrix approximations by the truncated Singular Value Decomposition and the Random Projection.

1.1 Method of transformation of the output vector based on Singular Value Decomposition. Let us consider the approach to a stable solution of the output transformation problem based on the truncated Singular Value Decomposition (SVD). For a stable regularized solution, we obtain the operator $\mathbf{CP}$ using the matrix $\mathbf{P}$ obtained as $\mathbf{P}_k = \mathbf{A}_k^{+} = \mathbf{V}_k \mathbf{S}_k^{-1} \mathbf{U}_k^{\mathrm{T}}$. Here $\mathbf{A}_k = \mathbf{U}_k \mathbf{S}_k \mathbf{V}_k^{\mathrm{T}}$ is the approximation of the matrix $\mathbf{A} \in \Re^{m \times n}$ obtained by the k ($k < n$) components of SVD, $\mathbf{U}_k = (\mathbf{u}_1, \ldots, \mathbf{u}_k)$ is

the matrix of the left singular vectors, $\mathbf{V}_k = (\mathbf{v}_1, \ldots, \mathbf{v}_k)$ is the matrix of the right singular vectors, $\mathbf{S}_k = \mathrm{diag}\,(s_1, \ldots, s_k)$ the matrix of singular values. The estimated output of the system C, obtained using the k components of SVD of $\mathbf{A}$, is $\mathbf{d}_k^* = \mathbf{C}\mathbf{A}_k^+ \mathbf{b} = \mathbf{T}_k \mathbf{b}$, $\mathbf{T}_k = \mathbf{C}\mathbf{A}_k^+ = \mathbf{C}\mathbf{V}_k \mathrm{diag}\,(s_k^{-1})\mathbf{U}_k^{\mathrm{T}}$. The number of the SVD components will be considered optimal if it provides the minimum of the mean squared error of the output transformation represented as $e_{\mathrm{SVD\,OT}}(k) = \mathrm{E}_\varepsilon \| \mathbf{d}_k^* - \mathbf{d}_0 \|^2 = \mathrm{E}_\varepsilon \| \mathbf{T}_k \mathbf{b} - \mathbf{d}_0 \|^2$, where E_ε is the averaging over the noise realizations. In the expression for the mean squared error of the output transformation, we distinguish the deterministic $\| \mathbf{T}_k \mathbf{b}_0 - \mathbf{d}_0 \|^2$ and the stochastic $\mathrm{E}_\varepsilon \| \mathbf{T}_k \varepsilon \|^2 = \sigma^2 \mathrm{trace}\,(\mathbf{T}_k^{\mathrm{T}} \mathbf{T}_k)$ parts:

$$e_{\mathrm{SVD\,OT}}(k) = \mathrm{E}_\varepsilon \| \mathbf{T}_k \mathbf{b} - \mathbf{d}_0 \|^2 = \| \mathbf{T}_k \mathbf{b}_0 - \mathbf{d}_0 \|^2 + \mathrm{E}_\varepsilon \| \mathbf{T}_k \boldsymbol{\varepsilon} \|^2.$$

The dependence of the value of the stochastic error part on the number k of the truncated SVD components of $\mathbf{A}$ was analytically studied in [3].

For the stochastic error part represented as $\sigma^2 \mathrm{trace}\,(\mathbf{T}_k^{\mathrm{T}} \mathbf{T}_k) = \sigma^2 \mathrm{trace}\,(\mathbf{H}\mathbf{M}_k)$, where $\mathbf{H} = \mathbf{C}^{\mathrm{T}}\mathbf{C}$, $\mathbf{M}_k = \mathbf{A}_k^+ \mathbf{A}_k^{+\mathrm{T}}$, the recursive expression for $\mathbf{M}_k$ can be written as follows [3]: $\mathbf{M}_k = \mathbf{M}_{k-1} + \mathbf{v}_k \mathbf{s}_k^{-2} \mathbf{v}_k^{\mathrm{T}}$.

The recursive expression for $\mathrm{trace}\,(\mathbf{H}\mathbf{M}_k)$ has the form [3]:

$$\mathrm{trace}\,(\mathbf{H}\mathbf{M}_k) = \mathrm{trace}\,(\mathbf{H}\mathbf{M}_{k-1}) + s_k^{-2} \mathrm{trace}\,(\mathbf{v}_k^{\mathrm{T}} \mathbf{H} \mathbf{v}_k) =$$

$$= \mathrm{trace}\,(\mathbf{H}\mathbf{M}_{k-1}) + s_k^{-2} \mathrm{trace}\,(\mathbf{v}_k^{\mathrm{T}} \mathbf{C}^{\mathrm{T}} \mathbf{C} \mathbf{v}_k).$$

The value of $\mathbf{v}_k^{\mathrm{T}} \mathbf{C}^{\mathrm{T}} \mathbf{C}\, \mathbf{v}_k$ is positive, and so $s_k^{-2} \mathrm{trace}\,(\mathbf{v}_k^{\mathrm{T}} \mathbf{C}^{\mathrm{T}} \mathbf{C}\, \mathbf{v}_k) > 0$. From the recursive expression for $\mathrm{trace}\,(\mathbf{H}\mathbf{M}_k)$ and the positivity of $s_k^{-2} \mathrm{trace}\,(\mathbf{v}_k^{\mathrm{T}} \mathbf{C}^{\mathrm{T}} \mathbf{C}\, \mathbf{v}_k)$, it follows that the stochastic error part increases with k.

Let us consider the case when the vector $\mathbf{b}_0$ is represented by the realization of a random process ξ with the Gaussian distribution of zero mean and variance ν^2. In this case, averaging can be carried out over the realizations of ξ. An analytical study of the dependence of the deterministic part of the output transformation error on the number k of the truncated SVD components was carried out in [3].

For the deterministic component of the error, we have $\| \mathbf{T}_k \mathbf{b}_0 - \mathbf{d}_0 \|^2 = \| \mathbf{C} \Delta \mathbf{A}_k^+ \mathbf{b}_0 \|^2$, $\Delta \mathbf{A}_k^+ = \mathbf{A}^+ - \mathbf{A}_k^+$. To study the dependence of $\mathrm{E}_\varepsilon \| \mathbf{C} \Delta \mathbf{A}_k^+ \boldsymbol{\xi} \|^2$ on k, the expression for the deterministic part of the error $\mathrm{E}_\varepsilon \| \mathbf{C} \Delta \mathbf{A}_k^+ \boldsymbol{\xi} \|^2 = \nu^2 \mathrm{trace}\,(\Delta \mathbf{A}_k^{+\mathrm{T}} \mathbf{H} \Delta \mathbf{A}_k^+)$ can be represented in a recursive form:

$$\mathrm{E}_\varepsilon \| \mathbf{C} \Delta \mathbf{A}_k^+ \boldsymbol{\xi} \|^2 = \nu^2 \mathrm{trace}\,(\mathbf{N}_{k-1}) - \nu^2 s_k^{-2} \mathrm{trace}\,(\mathbf{v}_k^{\mathrm{T}} \mathbf{H} \mathbf{v}_k) =$$

$$= \nu^2 \mathrm{trace}\,(\mathbf{N}_{k-1}) - \nu^2 s_k^{-2} \mathrm{trace}\,(\mathbf{v}_k^{\mathrm{T}} \mathbf{C}^{\mathrm{T}} \mathbf{C} \mathbf{v}_k), \text{ where } \mathbf{N}_{k-1} = \mathbf{H} \Delta \mathbf{A}_{k-1}^+ \Delta \mathbf{A}_{k-1}^{+\mathrm{T}}.$$

From this recursive representation and the positivity of $\nu^2 s_k^{-2} \mathrm{trace}\,(\mathbf{v}_k^{\mathrm{T}} \mathbf{C}^{\mathrm{T}} \mathbf{C} \mathbf{v}_k)$, it follows that in this case the deterministic part of the error decreases with k.

It has been experimentally shown [3] that the dependence $e_{\mathrm{SVD\,OT}}(k)$ has a minimum at $k < n$. The decomposition of the error into the two parts and the analytical study of their behavior depending on k explains the nature of the presence of a minimum.

1.2. Output transformation method using Random Projection. The stability of the output transformation is determined by the stability of the estimation $\mathbf{x}^*$ of the input signal. A stable estimate $\mathbf{x}^*$ of the input signal can be obtained (in addition to the

considered truncated SVD-based approach) using the approach for solving a discrete ill-posed problem based on random projection (RP), proposed by us in [4 –10]. See also preliminary work in [11].

For this case, we left-multiply both parts of the approximate equation $\mathbf{Ax} \approx \mathbf{b}$ by the matrix with the elements that are realizations of a random variable with the Gaussian distribution of zero mean and unit variance. Thus, we do the transformation (random projection) into a new space, the coordinate axes of which are random vectors. We obtain the expression $\mathbf{R}_k\mathbf{Ax} = \mathbf{R}_k\mathbf{b}$, where $\mathbf{R}_k\mathbf{A} \in \Re^{k\times n}$, $\mathbf{R}_k\mathbf{b} \in \Re^k$. The number of columns n is determined by the size of $\mathbf{A}$, the number of rows is a priori unknown. The solution of the least squares problem is obtained using the pseudo-inverse matrix $(\mathbf{R}_k\mathbf{A})^+$ as $\mathbf{x}_k^* = (\mathbf{R}_k\mathbf{A})^+\mathbf{R}_k\mathbf{b}$.

Therefore, the estimate $\mathbf{d}_k^*$ of the system C output is

$$\mathbf{d}_k^* = \mathbf{C}\mathbf{x}_k^* = \mathbf{C}(\mathbf{R}_k\mathbf{A})^+\mathbf{R}_k\mathbf{b} = \mathbf{T}_k\mathbf{b},$$

where $\mathbf{T}_k = \mathbf{C}(\mathbf{R}_k\mathbf{A})^+\mathbf{R}_k$. The expression for the mean squared error of the output vector transformation has the form:

$$e_{\text{RP OT}}(\mathbf{R}_k) = ||\mathbf{C}(\mathbf{R}_k\mathbf{A})^+\mathbf{R}_k\mathbf{b}_0 - \mathbf{Cx}||^2 + \sigma^2\,\text{trace}\,(\mathbf{R}_k^{\text{T}}(\mathbf{R}_k\mathbf{A})^{+\text{T}}\mathbf{C}^{\text{T}}\mathbf{C}(\mathbf{R}_k\mathbf{A})^+\mathbf{R}_k).$$

If we make random projection by the matrix $\mathbf{\Omega}_k \in \Re^{k\times n}$ obtained by the SVD of $\mathbf{R}$ (i.e., $\mathbf{R} = \mathbf{\Omega\Sigma\Psi}^{\text{T}}$), the error of the output vector transformation has the following form:

$$e_{\text{RP OT}}(\mathbf{\Omega}_k) = ||\mathbf{C}(\mathbf{\Omega}_k\mathbf{A})^+\mathbf{\Omega}_k\mathbf{b}_0 - \mathbf{Cx}||^2 + \sigma^2\,\text{trace}\,(\mathbf{\Omega}_k^{\text{T}}(\mathbf{\Omega}_k\mathbf{A})^{+\text{T}}\mathbf{C}^{\text{T}}\mathbf{C}(\mathbf{\Omega}_k\mathbf{A})^+\mathbf{\Omega}_k),$$

and, given the orthonormality of the $\mathbf{\Omega}_k$ columns, we obtain:

$$e_{\text{RP OT}}(\mathbf{\Omega}_k) = ||\mathbf{C}(\mathbf{\Omega}_k\mathbf{A})^+\mathbf{\Omega}_k\mathbf{b}_0 - \mathbf{Cx}||^2 + \sigma^2\,\text{trace}\,(\mathbf{C}^{\text{T}}\mathbf{C}(\mathbf{\Omega}_k\mathbf{A})^+(\mathbf{\Omega}_k\mathbf{A})^{+\text{T}}).$$

Let us consider the error of the output vector transformation based on the approach of Deterministic Random Projection (DRP) [7], i.e., using analytical averaging over random matrices. The estimate of the input vector by DRP is $\mathbf{x}_k^* = \mathbf{A}^{\text{T}}\mathbf{U}\mathbf{D}_k\mathbf{U}^{\text{T}}\mathbf{b}$, where $\mathbf{D}_k$ is the diagonal matrix that regularizes the estimate equivalent to the random projection by $\mathbf{R}_k$ with averaging [7].

The estimate $\mathbf{d}_k^*$ of the system C output is $\mathbf{d}_k^* = \mathbf{C}\mathbf{A}^{\text{T}}\mathbf{U}\mathbf{D}_k\mathbf{U}^{\text{T}}\mathbf{b} = \mathbf{T}_k\mathbf{b}$, where $\mathbf{T}_k = \mathbf{C}\mathbf{A}^{\text{T}}\mathbf{U}\mathbf{D}_k\mathbf{U}^{\text{T}}$. The squared norm of the output transformation error vector, averaged over noise realizations, has the form:

$$e_{\text{DRP OT}}(\mathbf{D}_k) = ||\mathbf{C}(\mathbf{V}\mathbf{S}^2\mathbf{D}_k\mathbf{V}^{\text{T}} - \mathbf{I})\mathbf{x}||^2 + \sigma^2\,\text{trace}\,(\mathbf{C}^{\text{T}}\mathbf{C}\mathbf{V}\mathbf{S}^2\mathbf{D}_k^2\mathbf{V}^{\text{T}}).$$

2. SELECTION OF THE OPTIMAL NUMBER OF MODEL COMPONENTS FOR THE OUTPUT TRANSFORMATION PROBLEM

One of the possible approaches to choosing the optimal number of model components is to use the existing model selection criteria (see in [6, 12]). A model selection criterion is designed so that the number of model components providing its minimum value is close to the number of model components that provide the error minimum for the model that approximates the output of a linear system.

For the system A, the output that approximates $\mathbf{b}_0$ is given by $\mathbf{A}\mathbf{x}_k^*$, where $\mathbf{x}_k^*$ is obtained by some model. So, the error of the output approximation is given by

$$e_{\text{A}}(k) = \text{E}_\varepsilon||\mathbf{A}\mathbf{x}_k^* - \mathbf{b}_0||^2 = \text{E}_\varepsilon||\mathbf{APb} - \mathbf{b}_0||^2.$$

Note that such an error for the system C, i.e., $e_{\mathrm{C}}(k) = \mathrm{E}_{\varepsilon} \, ||\mathbf{C}\mathbf{x}_k^* - \mathbf{d}_0||^2 = \mathrm{E}_{\varepsilon}||\mathbf{C}\mathbf{P}\mathbf{d} - \mathbf{d}_0||^2$, could not be calculated since $\mathbf{d}$ (the real output of C) is unknown. The error of the system A output transformation into the output of the system C is given by

$$e_{\mathrm{OT}}(k) = \mathrm{E}_{\varepsilon} \, ||\, \mathbf{d}_k^* - \mathbf{d}_0 \, ||^2 = \mathrm{E}_{\varepsilon} \, ||\, \mathbf{T}_k \mathbf{b} - \mathbf{d}_0 \, ||^2.$$

In order to be able to use a model selection criterion that gives us the value of k close to the optimal one for A (that is, $k_{\mathrm{opt\,A}}$ that minimizes $e_{\mathrm{A}}(k)$), it is important that $k_{\mathrm{opt\,A}}$ should be close to $k_{\mathrm{opt\,OT}}$ which minimizes the output transformation error $e_{\mathrm{OT}}(k)$.

To compare $k_{\mathrm{opt\,A}}$ and $k_{\mathrm{opt\,OT}}$, we use the function $J(k)$: $J(k) = \dfrac{\Delta e_{\mathrm{d}}(k)}{\sigma^{-2}\Delta e_{\mathrm{s}}(k)} = \dfrac{e_{\mathrm{d}}(k-1) - e_{\mathrm{d}}(k)}{\sigma^{-2}(e_{\mathrm{s}}(k) - e_{\mathrm{s}}(k-1))}$, where $e_{\mathrm{d}}(k)$ is the deterministic error part, $e_{\mathrm{s}}(k)$ is the stochastic error part.

For $e_{\mathrm{A}}(k)$, $e_{\mathrm{A}}(k) = ||(\mathbf{A}\mathbf{P}_k - \mathbf{I})\mathbf{b}_0||^2 + \sigma^2 \mathrm{trace}(\mathbf{P}_k^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{P}_k)$, $e_{\mathrm{d}}(k) = ||(\mathbf{A}\mathbf{P}_k - \mathbf{I})\mathbf{b}_0||^2$, $e_{\mathrm{s}}(k) = \sigma^2 \mathrm{trace}(\mathbf{P}_k^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{P}_k)$. For $e_{\mathrm{OT}}(k)$, $e_{\mathrm{d}}(k) = ||\mathbf{T}_k\mathbf{b}_0 - \mathbf{d}_0||^2$, $e_{\mathrm{s}}(k) = \sigma^2 \mathrm{trace}(\mathbf{T}_k^{\mathrm{T}}\mathbf{T}_k)$.

Note that for a monotonically decreasing $J(k)$ there is one (global) error minimum [6].

Figure 1 shows the experimentally obtained functions $J(k)$ for the output approximation error and for the output transformation error for the truncated SVD method, denoted by J_{SVD} and $J_{\mathrm{SVD\,OT}}$ respectively. The dependencies $J_{\mathrm{SVD}}(k)$ and $J_{\mathrm{SVD\,OT}}(k)$ are close to each other, so the positions of the $e_{\mathrm{SVD\,A}}(k)$ and $e_{\mathrm{SVD\,OT}}(k)$ minima (i.e., $k_{\mathrm{opt\,A}}$ and $k_{\mathrm{opt\,OT}}$) are also close. Closeness of $k_{\mathrm{opt\,A}}$ and $k_{\mathrm{opt\,OT}}$ makes it possible to use model selection criteria for the system A output to determine the optimal number of model components that minimizes the output transformation error.

Note, that the non-monotonic character of dependencies $J_{\mathrm{SVD}}(k)$ and $J_{\mathrm{SVD\,OT}}(k)$ indicates that the error dependencies $e_{\mathrm{A}}(k)$ and $e_{\mathrm{OT}}(k)$ will have local minima.

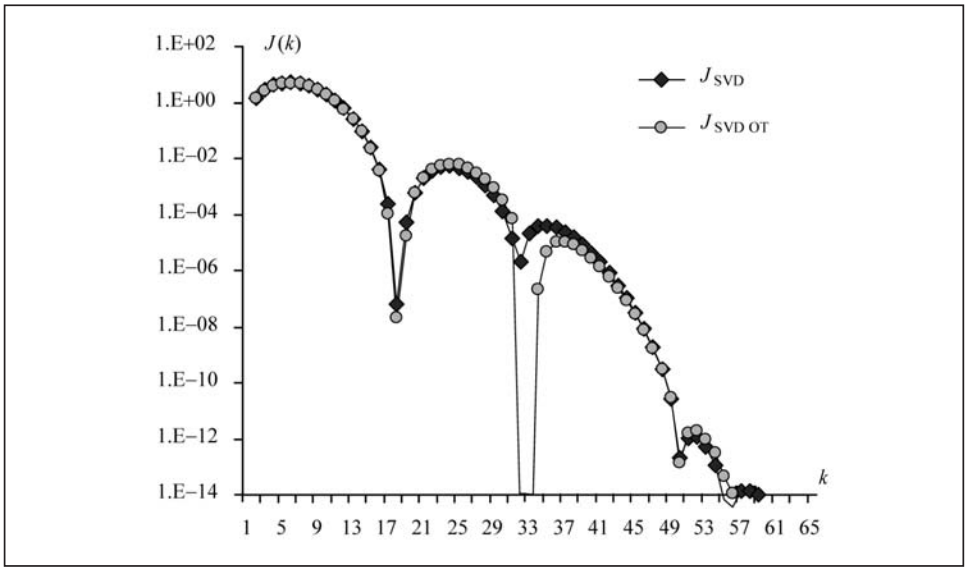

Fig. 1. The function $J_{\mathrm{SVD}}(k)$ for the output approximation error (system A) and $J_{\mathrm{SVD\,OT}}(k)$ for the output transformation error (from A to C) obtained by the truncated SVD method

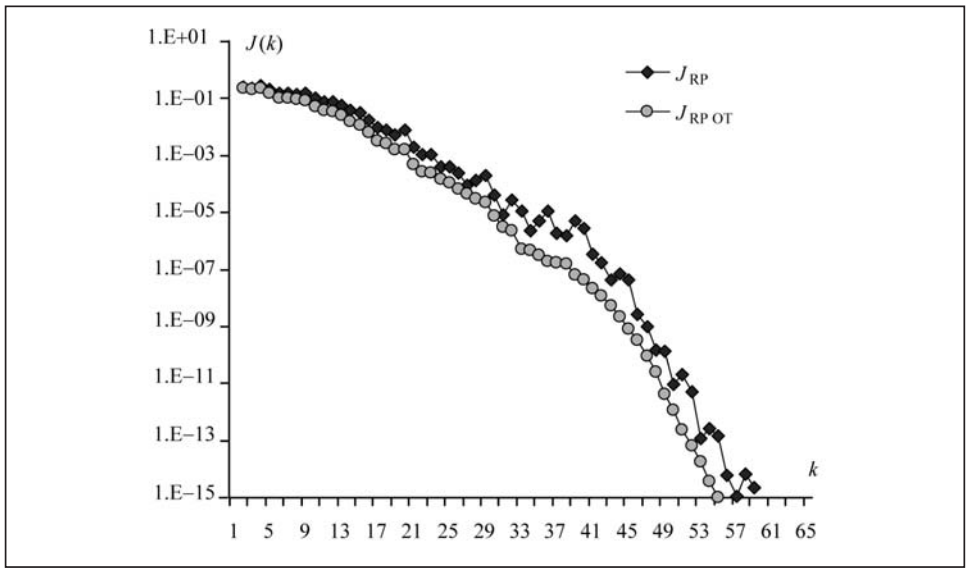

Fig. 2. The function J_{RP} for the output approximation error (system A) and $J_{\mathrm{RP\,OT}}$ for the output transformation error (from A to C) obtained by the Random Projection method

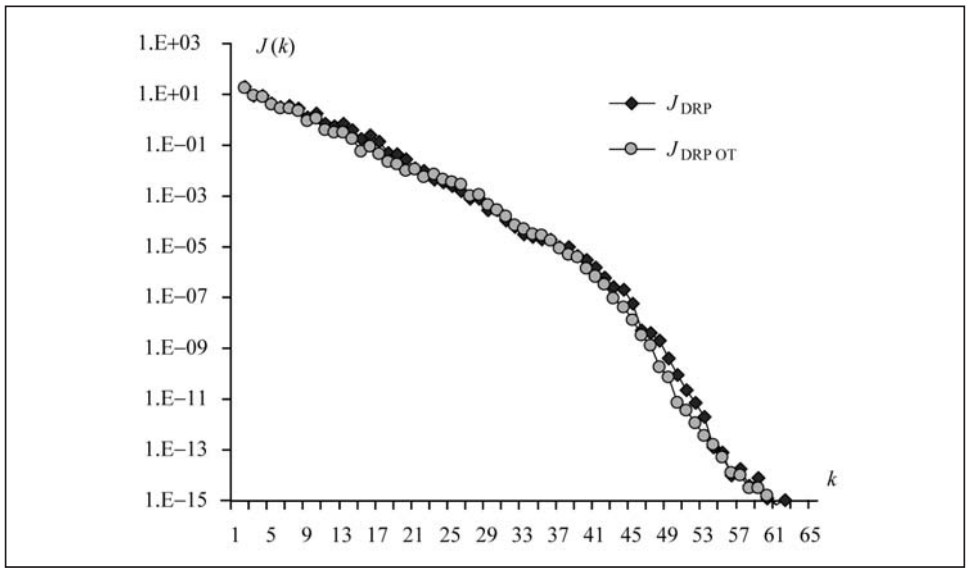

Fig 3. The function J_{DRP} for the output approximation error (system A) and $J_{\mathrm{DRP\,OT}}$ for the output transformation error (from A to C) obtained by the Deterministic Random Projection method

Figures 2 and 3 show the functions $J(k)$ for the output approximation error of the linear system A obtained with the RP and DRP methods (J_{RP} and J_{DRP}), as well as for the output transformation by the same methods ($J_{\mathrm{RP\,OT}}$ and $J_{\mathrm{DRP\,OT}}$). The dependency for RP was obtained experimentally, whereas for DRP it was obtained analytically by averaging over random matrices. That is, the number of matrices used for the averaging was finite for the RP method, whereas it was infinite for the DRP method.

The dependence $J_{\mathrm{RP}}(k)$ is somewhat non-monotonic. However, as the number of random matrices over which averaging is performed increases, the dependence $J_{\mathrm{RP}}(k)$ becomes smoother. The dependence is quite smooth for averaging over 100 random matrices as shown in Fig. 2. The smoothness of $J_{\mathrm{RP}}(k)$ and $J_{\mathrm{RP\,OT}}(k)$ also indicates that for the RP method the dependences will have fewer local minima than

the dependences for the truncated SVD-based method. The increased smoothness of $J_{\text{DRP}}(k)$ and $J_{\text{DRP OT}}(k)$ also indicates that the corresponding error dependencies almost do not have local minima.

Thus, for the output approximation error of the existing linear system and the error of transforming the output into the output of a system with a desired basis, we have experimentally studied the relative position of the minima, the minimal value of the error, and the number of local minima. For this purpose, we used the recovery of the input signal by the output signal of the existing linear system (the system A) using the models obtained by the truncated SVD, RP, and DRP methods.

For the truncated SVD, as well as for the DRP method, we observed very close positions (i.e., the values of k) of the error minima for the case of the existing system output approximation and for the case of the output transformation. For the RP method, the distance between the minima positions were slightly larger than for the truncated SVD method.

For the truncated SVD and DRP methods, the closeness of the error minimum positions for the output approximation and for the output transformation allows the use of the optimal number of model components determined for output approximation problem in the output transformation problem. For the truncated SVD method, the error dependence on the number of model components has numerous local minima, which makes the use of SVD to transform the output less attractive compared to DRP. The dependence of the error on the number of model components for the DRP method very rarely has local minima and provides accuracy (error value) at the level of the truncated SVD, making it desirable to use DRP in the output transformation problem.

3. IMPROVING THE ACCURACY OF ESTIMATING THE VECTOR OF PARAMETERS BY A LINEAR SYSTEM OUTPUT TRANSFORMATION

When solving various tasks related to the processing of information received from various sensors, there appears a problem of effective analysis of noisy signal mixtures. In a number of such problems, the measured data are the result of summing the effects generated by the physical process and weighted by some coefficients (parameters), and are therefore described by the models linear in parameters.

If a possible set of basic functions is known, but it is unknown which of them formed the observed output, the approximation problem solution can be obtained by sparse approximation methods [13]. For the output vector $\mathbf{y}_0$ that is not distorted by noise, the sparse approximation problem is set as the problem of minimizing the number of nonzero components in the parameter vector, provided that $\mathbf{y}_0 = \mathbf{\Phi\beta}$, where $\mathbf{\Phi} \in \Re^{L\times N}$ is the matrix of basis functions, and $\boldsymbol{\beta} \in \Re^N$ is the parameter vector. If the output vector $\mathbf{y}$ is distorted by noise, the problem of sparse approximation is set as the problem of minimizing the number of non-zero components in the parameter vector, provided that $||\mathbf{y} - \mathbf{\Phi\beta}|| \le \delta$, where δ is a (small) value proportional to the norm of the noise vector $\boldsymbol{\varepsilon}$.

In context of sparse approximation of the noisy output vector, the concept of the "l_0-optimal solution" was introduced, i.e., a solution that provides both the minimum approximation error and the maximum possible sparseness. The test proposed in [14] allows making an l_0-optimality test of solution $\boldsymbol{\beta}^*$ obtained by some algorithm. The disadvantage of the l_0-optimality test is that it cannot be applied to any (arbitrary) system of basis functions. Below, we propose to use the "output transformation" as a preprocessing for sparse approximation problems that use basis functions which do not satisfy the conditions of the l_0-optimality test.

4. MATCHING PURSUIT USING THE TRANSFORMATION OF THE LINEAR SYSTEM OUTPUT

To solve the sparse approximation problem in the case of a noisy output vector, we propose to modify the original method of Matching Pursuit [15].

The Modified Matching Pursuit (MMP) method works as follows. As in the original Matching Pursuit, starting with $k=0$ and $\mathbf{f}_0=0$, on the $k+1$th pass of the algorithm, it selects the vector $\boldsymbol{\varphi}_{k+1}\in\Re^L$ (some basis function, i.e., some column of $\boldsymbol{\Phi}$) and calculates the parameter β^*_{k+1} that minimize the square of the residual norm: $(\beta^*_{k+1},\boldsymbol{\varphi}_{k+1})=\arg\min_{\beta\boldsymbol{\varphi}}||\mathbf{r}_k-\beta\boldsymbol{\varphi}||^2$, where $\mathbf{r}_k=\mathbf{y}-\mathbf{f}_k$, $\mathbf{f}_k$ is the output approximation at the pass k. The solution for β_{k+1} in the closed form is presented below. Then the next output approximation is calculated as $\mathbf{f}_{k+1}=\mathbf{f}_k+\beta^*_{k+1}\boldsymbol{\varphi}_{k+1}$.

The modification of the original Matching Pursuit consists in l_0-optimality testing the vector of parameters $\boldsymbol{\beta}^*_k$ obtained at the kth pass. If the conditions of l_0-optimality are satisfied, the method stops.

The l_0-optimality test uses the cumulative coherence function defined for normalized vectors $\boldsymbol{\varphi}$ [14] as $\mu(s)=\max_{\mathrm{card}(I)\le s}\max_{j\notin I}\sum_{i\in I}|\langle\boldsymbol{\varphi}_j,\boldsymbol{\varphi}_i\rangle|$, where s is the number of nonzero parameters; I is the set of indices of functions that form a subspace; i indexes the elements of the subspace, for all possible $\mathrm{card}(I)$-member decompositions of $\mathbf{y}$, $\mathrm{card}(I)\le s$ means that the cardinality of the set of indices (subspace dimension) varies from 1 to s.

Our modified test for the l_0-optimality is as follows: the solution vector $\boldsymbol{\beta}^*_k$ (with k components) is the solution with the maximum possible sparseness and with the smallest approximation error, if $|\mathbf{r}|_1+|\mathbf{r}|_{2K}<0.5(1-\mu(2k-1))\max_i|\beta_i|$ and $\mu(2k-1)<1$, where $|\mathbf{r}|_K:=\left(\sum_{j\in\{K\}}|\langle\mathbf{r},\boldsymbol{\varphi}_j\rangle|^2\right)^{1/2}$. Here the set of indices $\{K\}$ corresponds to the basis functions with largest dot product values with the residual $\mathbf{r}$.

This test is different from that of [14] in our using max instead of min in [14].

In the case when the "basis coherence" condition $\mu(2k-1)<1$ is not satisfied for the particular system of basis functions, we proposed [12] to stop the MMP method according to some other model selection criterion, different from the l_0-optimality. However, our comparative experimental studies showed that the accuracy of estimating the parameter vector by MMP using other model selection criteria is worse than for the l_0-optimality test. So, we propose to transform the available output vector to the output of a linear system with the basis functions that satisfy the condition of basis coherence.

The MMP algorithm using the transformation of the system A output vector consists of the following steps:

Step 1.1. Form a matrix of basis functions $\mathbf{A}\in\Re^{L\times N}$, $L<<N$, where N is the number of basis functions.

Step 1.2. For $\mathbf{A}$, calculate the cumulative coherence function $\mu(s)$.

Step 1.3. Check the basis coherence condition: $\mu(2s-1)<1$ for $s=1,\ldots,0.5N$.

If the basis coherence condition is not met, go to step 1.5.

Step 1.4. Initialize the matrix of basis functions $\boldsymbol{\Phi}=\mathbf{A}$ and go to step 2.1.

Step 1.5. Form a matrix of basis functions $\mathbf{B} \in \Re^{L \times N}$, $L << N$, satisfying the basis coherence condition.

Step 1.6. Perform the SVD of $\mathbf{A}$. Check the condition number and the behavior of the singular value series.

- If the condition number is small, go to step 1.6.1.
- If the condition number is large and the singular value series gradually decreases to zero, go to step 1.6.2.

Step 1.6.1. Calculate the transformation matrix $\mathbf{T}_k = \mathbf{B}\mathbf{A}^+$ and go to step 1.7.

Step 1.6.2. Calculate the transformation matrix $\mathbf{T}_k$ for the method used, for example, $\mathbf{T}_k = \mathbf{B}\mathbf{A}_k^+ = \mathbf{B}\mathbf{V}_k\mathbf{S}_k^{-1}\mathbf{U}_k^{\mathrm{T}}$ for SVD; $\mathbf{T}_k = \mathbf{B}(\mathbf{R}_k\mathbf{A})^+\mathbf{R}_k$ for RP; $\mathbf{T}_k = \mathbf{B}\mathbf{A}^{\mathrm{T}}\mathbf{U}\mathbf{D}_k\mathbf{U}^{\mathrm{T}}$ for DRP.

Step 1.7. Perform the transformation of the output vector to the system of basic functions B as $\mathbf{d}_k^* = \mathbf{T}_k\mathbf{b}$.

Step 1.8. Initialize the matrix of basis functions $\mathbf{\Phi} = \mathbf{B}$.

Step 2.1. Initialize $\mathbf{f}_0 = 0$, $\mathbf{r}_k = \mathbf{y}$. Normalize the columns of $\mathbf{\Phi}$.

Step 2.2. In $\mathbf{\Phi}$, find the index of the basis function (the column) for which $\gamma_k = \arg\max\limits_{i=1,\ldots,N} |\langle \mathbf{\Phi}(\cdot, i), \mathbf{r}_k \rangle|$.

Step 2.3. Form $\mathbf{\Phi}_k = \{\mathbf{\Phi}_{k-1}, \boldsymbol{\varphi}_{\gamma_k}\}$. Check the condition number of $\mathbf{\Phi}_k$ and the behavior of the singular value series.

- If the condition number is small, go to step 2.4.
- If the condition number is large and the singular value series gradually decreases to zero, go to step 2.5.

Step 2.4. Calculate the values of the parameter vector $\boldsymbol{\beta}_k = (\mathbf{\Phi}_k^{\mathrm{T}}\mathbf{\Phi}_k)^{-1}\mathbf{\Phi}_k^{\mathrm{T}}\mathbf{r}_k$, go to step 2.6.

Step 2.5. Calculate the values of the parameter vector $\boldsymbol{\beta}_k$, using the chosen regularization method.

Step 2.6. Calculate the new residual vector $\mathbf{r}_{k+1} = \mathbf{r}_k - \beta_{\gamma_k}\mathbf{\Phi}(:, \gamma_k)$.

Step 2.7. Calculate $|\mathbf{r}|_1 + |\mathbf{r}|_{2K} = |\langle \mathbf{r}_{k+1}, \boldsymbol{\varphi}_i \rangle| + \left(\sum\limits_{i \in \{2K\}} |\langle \mathbf{r}_{k+1}, \boldsymbol{\varphi}_i \rangle|^2 \right)^{1/2}$, where $\{2K\}$ is the set of indices with the $2k$ largest dot products $|\langle \mathbf{r}_{k+1}, \boldsymbol{\varphi}_i \rangle|$.

Step 2.8. Test if $|\mathbf{r}|_1 + |\mathbf{r}|_{2K} < 0.5(1 - m(2k-1)) \max\limits_i |\beta_i|$.

If the test is satisfied, the resulting k-term linear model gives the solution with the maximum possible sparseness and the smallest approximation error based on the optimal sparseness test. Otherwise, continue the formation of the model by going to the next pass, i.e., step 2.2.

5. THE EXPERIMENTAL STUDY

Let us consider an example of applying the transformation of the linear measuring system output to the output of a system with a desired basis. It is applied in one of the topical tasks of radiation monitoring, i.e., to the problem of identifying and determining the activity of weak radioactive sources. Let the detector of the linear measuring gamma-spectrometric system A have a lower resolution than the detector of the system C. Let us investigate the accuracy of determining the radionuclide activity using the method of transforming the output of the measuring system.

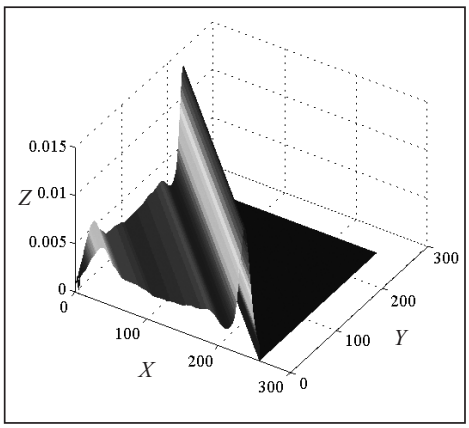

Fig. 4. An example of basis functions

In the test task, the spectra were formed by the radionuclides cesium-137 (Cs^{137}), cesium-134 (Cs^{134}) and cobalt-60 (Co^{60}). The vector of parameters (proportional to the activities of radionuclides) was as follows: $x_{Cs^{137}} = 1.5$, $x_{Cs^{134}} = 0.5$, $x_{Co^{60}} = 0.26$ (spectrum line 1), $x_{Co^{60}} = 0.25$ (spectrum line 2).

We studied the error of the parameter vector estimate for the real measuring system output (with the "wide" detector response function) and for the transformed measuring system output (with the "narrow" detector response function). An example of basis functions is shown in Fig. 4, and an example of the system output is shown in Fig. 5. The X axis in both figures is the spectrometer channel number. The Y axis in Fig. 4 is the basis function number. The Z (vertical) axis is the number of registered gamma-quants (in Fig. 4, per unit area).

At the two levels (0.01 and 0.02) of the intrinsic noise, the real spectra were measured, the output was transformed, and the accuracy (the mean squared error e) of the parameter vector estimation was calculated. The results are shown in Table 1, with the following notations. The upper index denotes the system for which the error was calculated. The lower index (Cp, MDL, L_0) denotes the model selection criterion used to estimate the parameter vector (Mallows, Minimum Description Length, and the test for l_0-optimality, correspondingly). True denotes the error obtained using the parametric least squares regression with the true model.

The measurements of spectra at the noise levels 0.01 and 0.02 were performed in the laboratory. When measured in the field conditions, due to the ambient temperature changes, the noise level usually increases. Since the measurement of the intrinsic noise of the measuring system in the field is difficult, we simulated the increase of the intrinsic noise by adding noise (the Gaussian noise with zero mean and variance equal to the noise level) to the measured spectra, in the range 0.03–0.09.

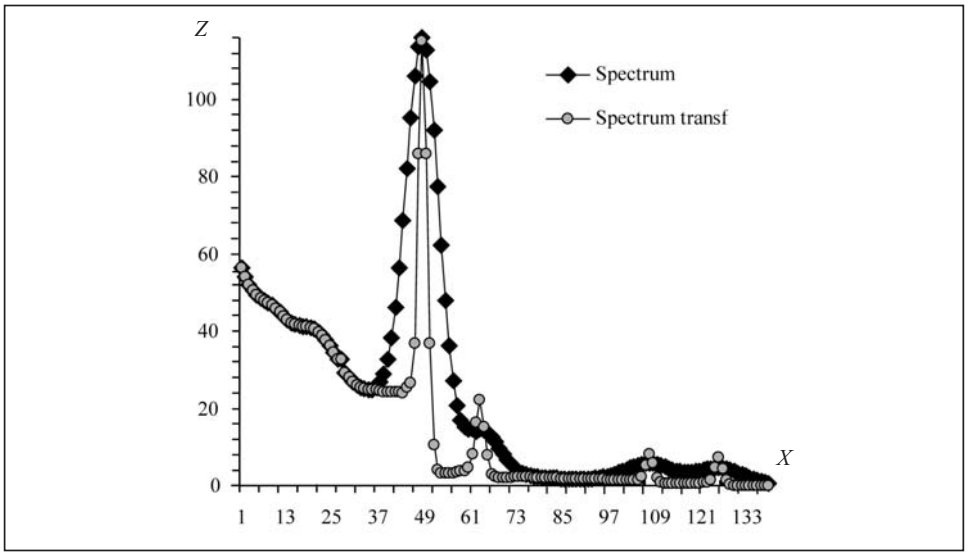

Fig. 5. The output of the system (Spectrum) A and the transformed output (Spectrum transf)

Table 1. Accuracy (the mean squared error e) of parameter vector estimation

Noise level	e^{A}_{Cp}	e^{C}_{Cp}	e^{A}_{MDL}	e^{C}_{MDL}	$e^{C}_{L_0}$	e^{A}_{True}	e^{C}_{True}
0.01	0.011	0.021	0.011	0.013	0.011	0.011	0.013
0.02	0.257	0.254	0.025	0.024	0.024	0.025	0.024
0.03	0.283	0.281	0.043	0.036	0.036	0.044	0.036
0.04	0.308	0.297	0.119	0.097	0.051	0.062	0.051
0.05	0.332	0.325	0.203	0.192	0.064	0.082	0.064
0.06	0.348	0.351	0.255	0.252	0.075	0.110	0.076
0.07	0.363	0.357	0.280	0.289	0.086	0.132	0.087
0.08	0.375	0.368	0.298	0.291	0.095	0.162	0.098
0.09	0.391	0.383	0.301	0.294	0.106	0.182	0.106

The results of the study are shown in Table 1. They were obtained for the DRP regularization. As the intrinsic noise level increases, the parameter vector estimation error increases for all the methods of parameter estimation. However, we see that the accuracy with the l_0-optimality test is higher than for other model selection criteria.

CONCLUSIONS

We consider a linear system in which the output vector is formed by a linear transformation of the input vector and adding noise. The matrix of basis functions of this linear system has a high condition number and its singular values gradually decrease to zero. For this case, the proposed methods allowed a stable transformation of the observed output into the output of a linear system with a known set of basic functions. This was achieved by employing various regularization methods with discrete regularization parameters (model complexity).

For the mean squared error of the output transformation problem solution, the decomposition into deterministic and stochastic parts has been performed. For the method based on the truncated Singular Value Decomposition, we have analytically shown the increase of the stochastic error value vs the number of the SVD components. An analytical study of the behavior of the deterministic error for the case when the input vector is the realization of a random process showed that the deterministic part decreases vs the number of the SVD components.

We have conducted an experimental study of the minimum error position for the methods based on the truncated Singular Value Decomposition, Random Projection, and Deterministic Random Projection. The study showed that for the truncated Singular Value Decomposition and Deterministic Random Projection, the positions of the output approximation error minimum of the existing system and the output transformation error are very close. Using the output transformation as a pre-processing can increase the accuracy of solving sparse approximation problems. For these tasks there is a test for l_0-optimality which gives a very accurate estimate of the optimal complexity of the model. However, this test requires certain properties of basic functions, which we obtained by the output transformation method. The improved accuracy was confirmed in the problem of determining the activity of weak radiation sources. We consider future applications in the areas of Unmanned Aerial Vehicles [16–18] and Electrocardiogram-related signal processing [19, 20].

For regularization, we employed the Random Projection methods that are a family of randomized algorithms. Randomization of calculations is widely used to increase the efficiency of calculations with a slight decrease in accuracy. Elsewhere, random projections are widely used to reduce the dimensionality of vector representations while maintaining the vector similarity values for some similarity measures, see, e.g., [21] and references therein. Random projections are also used as a stage in algorithms that transform the input vectors to the binary format, as exemplified in [22, 23]. Even earlier, randomized methods have been used to perform a nonlinear transform of the original vector data into binary high-dimensional vectors, making possible to apply linear classifiers for solution on non-linear problems, such as in [24, 25]. Also, those and other randomized transforms are widely used in the field known as Hyperdimensional Computing or Vector Symbolic Architectures, see [26–28]. We believe that randomized transformations and computations are promising for Artificial Intelligence systems.

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ПЕРЕТВОРЕННЯ ВИХОДУ ЛІНІЙНОЇ СИСТЕМИ ДЛЯ РОЗРІДЖЕНОЇ АПРОКСИМАЦІЇ

Анотація. Розглянуто підхід, що забезпечує стійке перетворення виходу лінійної системи у вихід системи із заданим базисом. Матриця базисних функцій лінійної системи має високе число обумовленості, і ряд її сингулярних чисел плавно спадає до нуля. Розроблено два типи стійких методів перетворення виходу з використанням апроксимації матриць на основі усіченого сингулярного розкладання та на основі випадкової проєкції з різними типами випадкових матриць. Показано, що за рахунок перетворення виходу як попереднього оброблення можна збільшити точність розв'язання задач розрідженої апроксимації. Розглянуто приклад використання методу в задачі визначення активності слабких джерел радіоактивного випромінювання.

Ключові слова: розріджена апроксимація, дискретна некоректна задача, випадкова проєкція, сингулярне розкладання.

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