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DEVELOPING A MODEL FOR MODULATING MIRROR FIXED ON ACTIVE SUPPORTS. DETERMINISTIC PROBLEM[1]

Abstract. We consider a problem of a modulating a mirror fixed on active supports. It is assumed that the mirror may have several defects. The problem is to find optimal locations of supports as well as control forces providing the best approximation of a given shape and phase of the oscillations for a homogeneous mirror as well as a plate with defects that have definite geometric and mechanical characteristics. The model of the Kirchhoff plate is chosen to describe the mirror. Defects are modeled by small inhomogeneities with changed elastic characteristics. An iterative technique for modeling finite-size defects in the Kirchhoff plate by point quadrupoles is developed. Isolated active supports are modeled by point forces. The optimization parameters are: the location of the supports and the amplitudes and phases of forces that generate vibrations. As an optimality criterion, the minimum of the root-mean-square deviation of the waveform of the plate from the given pattern is used.

Keywords: modulating mirror, defected plate, optimal excitation.

INTRODUCTION

In this work we generalize results obtained before in [1, 2]. We optimize parameters of mechanical devices for excitation and formation of wave motion. These devices can be used for generation, transformation, and transmission of information (and in more general sense transmission of wave energy). Particularly, we consider a problem of modulating a mirror fixed on active supports [2], and developed a model for optimizing of structure of these supports [1]. The problem is to find control forces and their characteristics (application points, amplitude and phase of oscillation), which provide the best approximation of a given shape and phase of the mirror oscillation taking into consideration structural inhomogeneities (defects) with unknown geometric and mechanical characteristics [3]. Mechanical properties of the defects are described by the following parameters: material density, Young's modulus of material, and cylindrical stiffness. We considered that any defect has elliptical form with stochastic parameters [4, 5]. To speed up calculations, we simplified the elliptical defect model and showed that in the first approximation equivalent body load does not depend on defect orientation [4]. We estimated error for the first approximation [5, 6]. We investigated how defect modeling accuracy depend on number of circular harmonics [6].

In this part of the study, we solve a deterministic problem in which it is assumed that the characteristics of defects are known. Thus, the problem of determining the shape of the plate vibration is reduced to a boundary problem with distributed point quadrupoles, which, in the first approximation, model inhomogeneities with known characteristics. Modeling of active supports by point forces makes it possible to use the Green's function method.

We also formulate an optimization problem in order to determine the best characteristics of the exciting forces, namely, their location, amplitudes and phases, which

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provide the smallest deviation of the mirror oscillation form from the given oscillation form. We use the minimum of the root-mean-square deviation as an optimality criterion.

DEFECTS MODELING

The section describes the approach to modeling of inhomogeneities in the plate using the apparatus of generalized functions [5, 6].

Let a defect with a cylindrical stiffness D_1 be located in the region $P(x, y) > 0$. A cylindrical stiffness of region $P(x, y) < 0$ without defects is D_0. The plate is loaded by external forces of intensity q, which are applied in points without defects. Suppose that $\vec{\nabla} P \big|_{P=0} \neq 0$, where P is simple connected and convex. Then the equations of harmonic oscillations of the plate can be written in the form [7]:

$$\begin{cases} \mathrm{L}(-D_0 \mathrm{L}_1)\mathrm{L}_2 w + \rho_0 \omega^2 w + q = 0, & P(x, y) < 0, \\ \mathrm{L}(-D_1 \mathrm{L}_1)\mathrm{L}_2 w + \rho_1 \omega^2 w = 0, & P(x, y) > 0, \end{cases} \tag{1}$$

where the following notations were introduced:

$$\mathrm{L}_0 = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}, \quad \mathrm{L}_1 = \begin{bmatrix} \frac{\partial}{\partial x} & \nu \frac{\partial}{\partial y} \\ \nu \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \frac{1-\nu}{2}\frac{\partial}{\partial y} & \frac{1-\nu}{2}\frac{\partial}{\partial x} \end{bmatrix}, \quad \mathrm{L}_2 = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix},$$

$$\mathrm{L} = \mathrm{L}_2^{\mathrm{T}} \mathrm{L}_0 = \begin{bmatrix} \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial y^2} & \frac{\partial^2}{\partial x \partial y} \end{bmatrix}. \tag{2}$$

Taking into consideration that $D_1 = D_0 - \Delta D$, $\Delta D / D_0 = \kappa_D$, $\rho_0 \omega^2 / D_0 = k^4$, (1) can be written as:

$$\begin{cases} \mathrm{L}(-\mathrm{L}_1)\mathrm{L}_2 w + k^4 w = -q, & P(x, y) < 0, \\ \mathrm{L}(-\mathrm{L}_1)\mathrm{L}_2 w + \frac{\rho_1}{\rho_0 (1-\kappa_D)} k^4 w = 0, & P(x, y) > 0. \end{cases} \tag{3}$$

We consider a piecewise homogeneous model of inhomogeneity, namely:

$$\begin{cases} D = D_0 (1 - \kappa_D \theta(P)), \\ \rho = \rho_0 (1 - \kappa_\rho \theta(P)), \end{cases} \tag{4}$$

where $\theta(P) = \begin{cases} 1, & P > 0, \\ 0, & P < 0, \end{cases}$ is a function of region, $\kappa_\rho = \frac{\rho_0 - \rho_1}{\rho_0}$.

Then (3) can be rewritten in the form of a single equation, valid in the whole area:

$$\mathrm{L}(-(1 - \kappa_D \theta(P))\mathrm{L}_1)\mathrm{L}_2 w + k^4 (1 - \kappa_\rho \theta) w = -q, \tag{5}$$

or

$$\mathrm{L}\mathrm{L}_1\mathrm{L}_2 w - k^4 w = \frac{q}{D_0} + \kappa_D \mathrm{L}\theta(P)\mathrm{L}_1\mathrm{L}_2 w - k^4 \kappa_\rho \theta w, \quad (x, y) \in \Omega. \tag{6}$$

For convenience, we introduce another notation:

$$\vec{v} = \mathrm{L}_1 \mathrm{L}_2 w = \begin{bmatrix} v_x \\ v_y \\ v_{xy} \end{bmatrix}.$$

Vector $\vec{v}$ has no physical meaning, however:

$$\vec{v} = \begin{cases} -\vec{M} / D_0, & P(x, y) < 0, \\ -\vec{M} / D_1 & P(x, y) > 0. \end{cases} \tag{7}$$

Let us consider the part of the right-hand side of (6) that describes the inhomogeneity of cylindrical stiffness in region $\theta(P)>0$.

It is easy to see that $\langle \mathrm{L}_2^{\mathrm{T}}\mathrm{L}_0\vec{v}\theta, \Phi\rangle = \langle \vec{v}, \mathrm{L}_2^{\mathrm{T}}\mathrm{L}_0\Phi\rangle$. Really, $\langle \mathrm{L}_2^{\mathrm{T}}\vec{u}, \Phi\rangle = -\langle \vec{u}^{\,\mathrm{T}}, \vec{\nabla}\Phi\rangle$, where $\vec{u} = \mathrm{L}_0\vec{v}$, and $-\langle \vec{u}\theta, \vec{\nabla}\Phi\rangle = -\langle (\mathrm{L}_0\vec{v})\theta, \vec{\nabla}\Phi\rangle = \langle \vec{v}\theta, \mathrm{L}_2^{\mathrm{T}}\mathrm{L}_0\Phi\rangle$.

According to the mean value theorem:

$$\langle \vec{v}, \mathrm{L}\Phi\rangle = S_P\vec{v}(X)\left(\frac{\partial^2\Phi}{\partial x^2}, 2\frac{\partial^2\Phi}{\partial x\partial y}, \frac{\partial^2\Phi}{\partial y^2}\right)\Bigg|_X + \overline{\overline{O}}\,(S_P)\,, \tag{8}$$

where $S_P = pow(P)$.

Remark 1. It can be shown that $\overline{\overline{O}}\,(S_P)$ can be replaced by $\underline{\underline{O}}(S_P^2)$. Thus,

$$\mathrm{L}\theta(P)\vec{v} =$$

$$= S_P^2\vec{v}\left(\frac{\partial^2\delta(x-X)}{\partial x^2}\delta(y-Y), 2\frac{\partial\delta(x-X)}{\partial x}\frac{\partial\delta(y-Y)}{\partial y}, \frac{\partial^2\delta(y-Y)}{\partial y^2}\delta(x-X)\right) + \underline{\underline{O}}(S_P^2)\,. \tag{9}$$

Volumetric loads equivalent to heterogeneity are interpreted as a superposition of double balanced dipoles (Fig. 1).

Thus, taking into account (7), (9), equation (6) takes the form:

$$\mathrm{L}\mathrm{L}_1\mathrm{L}_2 w - k^4 w = \frac{q}{D_0} + \kappa_D S_P + \left\{ v_x(X)\frac{\partial^2\delta(x-X)}{\partial x^2}\delta(y-Y) + \right.$$

$$\left. + 2v_{xy}(X)\frac{\partial\delta(x-X)}{\partial x}\frac{\partial\delta(y-Y)}{\partial y} + v_y(X)\frac{\partial^2\delta(y-Y)}{\partial y^2}\delta(x-X) \right\} -$$

$$- k^2\kappa_\rho S_\rho w(X)\delta(x-X)\delta(y-Y) + \underline{\underline{O}}(S_P^2)\,, \tag{10}$$

where $X = (X, Y)$ is a defect center with characteristics $\kappa_D, \kappa_\rho, S_\rho$.

We are looking for a solution in the form: $w = w_0 + \kappa w_1$, confining ourselves to the first two terms of the expansion in a small parameter $\kappa << 1$ ($\kappa << 1$ ($\kappa = \underline{\underline{O}}(\kappa_\rho, \kappa_D)$)). Neglecting change in ρ ($\kappa_\rho \equiv 0$). Then we have

$$\begin{cases} \mathrm{L}\mathrm{L}_1\mathrm{L}_2 w_0 - k^4 w_0 = \dfrac{q}{D_0}\,, \\ \mathrm{L}\mathrm{L}_1\mathrm{L}_2 w_1 - k^4 w_1 = v_{0x}(X)\dfrac{\partial^2\delta(x-X)}{\partial x^2}\delta(y-Y) + \\ +2v_{0xy}(X)\dfrac{\partial\delta(x-X)}{\partial x}\dfrac{\partial\delta(y-Y)}{\partial y} + v_{0y}(X)\dfrac{\partial^2\delta(y-Y)}{\partial y^2}\delta(x-X). \end{cases} \tag{11}$$

Equations (11) were obtained with accuracy $\underline{\underline{O}}(S_\rho^2)$. Obviously, the first equation (11) describes the deflection of the plate provided that there is no inhomogeneity.

Let the Green's function $w^*(x, X)$ be constructed as [2]:

$$\mathrm{L}\mathrm{L}_1\mathrm{L}_{2x} w^*(x, X) - k^4 w^*(x, X) = \delta(x - X)\,,\ x \in \Omega,\ X \in \Omega,\ X \neq x, \tag{12}$$

where $x = x(x, y)$ is the observation point, $X = X(X, Y)$ is the source point.

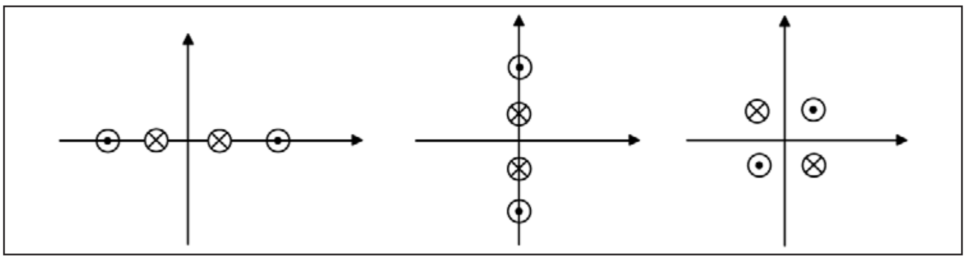

Fig. 1. Representation of inhomogeneity in the form of balanced dipoles

Let w^* satisfy homogeneous boundary conditions:

$$\begin{cases} w^*(x,X)=0, \\ M^*(x,X)=0, \end{cases} \quad x\in\Gamma. \tag{13}$$

Then, $w_1(x,X)$ satisfying (11) can be constructed as:

$$w_1(x,X)=\upsilon_{0x}(X)\frac{\partial^2 w^*(x,X)}{\partial X^2}+2\upsilon_{0xy}(X)\frac{\partial^2 w^*(x,X)}{\partial X\partial Y}+\upsilon_{0y}(X)\frac{\partial^2 w^*(x,X)}{\partial Y^2}. \tag{14}$$

In addition, it is easy to see that w_1 satisfies the boundary conditions (13). Indeed, for example:

$$\frac{\partial w^*(x,X)}{\partial X}=\lim_{\Delta x\to 0}\frac{w^*(x,X+\Delta X)-w^*(x,X)}{\Delta X},$$

but $w^*(x,X)$ satisfies (13) for $\forall\in\Omega$, hence, $\frac{\partial w^*(x,X)}{\partial X}$ as well as the higher derivatives (within the limits of smoothness) satisfy (13).

Let us move on to the polar coordinates. Let $(X,Y)\equiv(R,\Phi)$, $((x,y)\equiv(r,\varphi))$. Simple geometric transformations:

$$\begin{bmatrix}\upsilon_r\\ \upsilon_\varphi\\ \upsilon_{r\varphi}\end{bmatrix}=\begin{bmatrix}\cos^2\varphi & \sin^2\varphi & \sin 2\varphi\\ \sin^2\varphi & \cos^2\varphi & -\sin 2\varphi\\ -\frac{1}{2}\sin 2\varphi & \frac{1}{2}\sin 2\varphi & \cos 2\varphi\end{bmatrix}\begin{bmatrix}\upsilon_x\\ \upsilon_y\\ \upsilon_z\end{bmatrix} \tag{15}$$

as well as definition (7) allows one to obtain the polar components of vector $\vec{\upsilon}$:

$$\begin{cases} \upsilon_r=\dfrac{1}{r^2}\left(r^2\dfrac{\partial^2 w}{\partial r^2}+\nu\left(r\dfrac{\partial w}{\partial r}+\dfrac{\partial^2 w}{\partial\varphi^2}\right)\right), \\ \upsilon_\varphi=\dfrac{1}{r^2}\left(\nu r^2\dfrac{\partial^2 w}{\partial r^2}+\left(r\dfrac{\partial w}{\partial r}+\dfrac{\partial^2 w}{\partial\varphi^2}\right)\right), \\ \upsilon_{r\varphi}=\dfrac{(1-\nu)}{r^2}\left(r\dfrac{\partial^2 w}{\partial r\partial\varphi}-\dfrac{\partial w}{\partial\varphi}\right). \end{cases} \tag{16}$$

With this in mind, the first approximation of the boundary value problem (11), (13) has the form:

$$w_1(x,X)=\frac{1}{R^2}\left\{R^2\upsilon_{0r}(X)(R,\Phi)\frac{\partial^2 w^*(x,X)}{\partial R^2}+2\upsilon_{0r\varphi}(X)\times\right.$$

$$\left.\times\left(R\frac{\partial^2 w^*(x,X)}{\partial R\partial\Phi}-\frac{\partial w^*(x,X)}{\partial\Phi}\right)+\upsilon_{0\varphi}(X)(\left(R\frac{\partial w^*(x,X)}{\partial R}+\frac{\partial^2 w^*(x,X)}{\partial\Phi^2}\right)\right\}, \tag{17}$$

where υ_{0r}, $\upsilon_{0\varphi}$, $\upsilon_{0r\varphi}$ are determined through the zeroth order approximation.

Thus, to solve the direct problem you need:

1. Find w_0 (by solving the first equation (11) and satisfying the boundary conditions of the form (13)).
2. Construct the Green's function w^* from (12), (13).
3. Find υ_{0r}, $\upsilon_{0\varphi}$, $\upsilon_{0r\varphi}$ from (16).
4. Implement the formula (17).
5. As a result, we get the solution for a plate with defect with parameters R, Φ, S_P, κ_D, as a first approximation: $w=w_0+S_P\frac{\kappa_D}{1-\kappa_D}w_1$.

OPTIMIZATION PROBLEM. DETERMINISTIC SETTING

Let $I+1$ forces with intensities $F_k = u_k + iv_k$ $(k=\overline{0,I})$ be applied to the plate at points $\xi_k(r_{\xi_k}, \varphi_{\xi_k})$, $k=\overline{1,I}$, $\xi_0(0)$. In addition, the plate contains J defects $(r_{d_j}, \varphi_{d_j}, \kappa_j, s_j)$, $j=\overline{1,J}$. For a given profile function $W(r,\varphi)$ it is required to determine ξ_k, F_k from the optimization problem:

$$I(R) = \frac{1}{2\pi}\int_0^1 r dr \int_{-\pi}^{\pi} |w(r,\varphi) - W(r,\varphi)|^2 d\varphi \to \min . \quad (18)$$

Let

$$w(r,\varphi) = w_0(r,\varphi) + w_1(r,\varphi), \quad (19)$$

where $w_0(r,\varphi)$ is the optimal solution to the deterministic problem for a homogeneous plate with appropriate controls ξ_k^0, F_k^0, which minimize

$$I_0(R) = \frac{1}{2\pi}\int_0^1 r dr \int_{-\pi}^{\pi} |w_0(r,\varphi) - W(r,\varphi)|^2 d\varphi. \quad (20)$$

We are looking for a solution in the form $w_0(r,\varphi) = F_0^0 w_{00}^* + \sum_{k=1}^{I} F_k^0 w^*(r, r_{\xi_k}^0, \varphi - \varphi_{\xi_k}^0)$, where w_{00}^*, w^* are Green's functions, defined in the problem (12), (13). These functions, as well as solution, and implementation of this stage are very similar to methodology presented in [2]. In what follows, w_0 is considered as known. Moreover, $w_1(r,\varphi)$ is defined in (17) and looks like:

$$w_1(r,\varphi) = \sum_{j=1}^{J} \varepsilon_j \left\{ v_{0x}(X_j) \frac{\partial^2 w^*(x;X_j)}{\partial X^2} + 2v_{0xy}(X_j)\frac{\partial^2 w^*(x;X_j)}{\partial X \partial Y} + \right.$$

$$\left. + v_{0y}(X_j)\frac{\partial^2 w^*(x;X_j)}{\partial Y^2} \right\} \equiv \sum_{j=1}^{J} \frac{\varepsilon_j}{R_j^2} \left\{ R_j^2 v_{0r}(X_j) \frac{\partial^2 w^*(x;X_j)}{\partial R_j^2} + 2v_{0r\varphi}(X_j) \times \right.$$

$$\left. \times \left(R_j \frac{\partial^2 w^*(x;X_j)}{\partial R \partial \Phi} - \frac{\partial w^*(x;X_j)}{\partial \Phi} \right) + v_{0\varphi}(X_j) \left(R_j \frac{\partial w^*(x;X_j)}{\partial R} + \frac{\partial^2 w^*(x;X_j)}{\partial \Phi^2} \right) \right\}, \quad (21)$$

where v_0 is determined for w_0 by (16), and $\varepsilon_j = \frac{\kappa_j}{1-\kappa_j} S_j$, where κ_j is inhomogeneity measure, and S_j is the area of the jth inhomogeneity. We assume that $\sum_j^J \varepsilon_j << 1$, i.e., $\|w_1\| << \|w_0\|$.

REMARKS ON THE IMPLEMENTATION OF THE OPTIMIZATION PROBLEM

In a preliminary study of problem (20), we can take the known part of the control variables of the problem, namely, the points of application of forces: r_{ξ_k}, φ_{ξ_k}, $k=\overline{1,I}$. In this case, the optimization problem becomes quadratic and has an exact solution.

If we take $r_{\xi_k} = r_{\xi_k}^0, \varphi_{\xi_k} = \varphi_{\xi_k}^0$ (that is the optimal positions of the forces for a homogeneous plate), then we can obtain the first approximation of the problem of the complete problem (20). So, $I(R) = (w-W, w-W)$, and $w = w_0 + w_1$, where $\|w_1\| << \|w_0\|$. The solution to the problem can be represented as: $w_0 = \sum_{i=0}^{I} g_i F_i$,

$w_1 = \sum_{i=0}^{I} s_i F_i$, where g_i, s_i are some functions defined from (18), (20). Problem (20) is the classical optimal approximation problem in $L^2(\Omega)$ with basis $\{g_i, s_i\}_{i=0}^{I}$. The Gram system for this task has the form [8]:

$$\sum_{i=0}^{I} ((g_i + s_i)F_i, (g_s + s_s)) = (W, g_s + s_s),\ s = \overline{0, I}. \tag{22}$$

Wherein, in the absence of inhomogeneities, (20) has the form of a system of linear equations:

$$\mathrm{K}^0 \vec{F}^0 = \vec{b}^0,\ \mathrm{K}^0 = \{K_{is}\}_{i,s=0}^{I},\ \vec{b}^{0^{\mathrm{T}}} = \{b_s^0\}_{s=0}^{I},\ K_{is}^0 = (g_i, g_s),\ b_s^0 = (W, g_s). \tag{23}$$

The solution to this problem (see [2]) is denoted by $\vec{F}^{0\,\mathrm{opt}}$.

With this in mind, (22) is a system of linear equations of the form: $(\mathrm{K}^0 + \mathrm{K}^1)\vec{F} = \vec{b}^0 + \vec{b}^1$, where

$$\begin{cases} K_{is}^1 = (g_i, s_s) + (s_i, g_s) + (s_i, s_s), \\ b_s^1 = (W, s_s). \end{cases} \tag{24}$$

Let $\vec{F} = \vec{F}^0 + \vec{F}^1$, where $\vec{F}^0$ is defined in (23) and $\|\vec{F}^1\| << \|\vec{F}^0\|$. Then $(\mathrm{K}^0 + \mathrm{K}^1)(\vec{F}^0 + \vec{F}^1) = \vec{b}^0 + \vec{b}^1$. Neglecting a small member $\mathrm{K}^1\vec{F}^1$, with known $\vec{F}^0 = \vec{F}^{0\,\mathrm{opt}}$, solving (22) reduces to a system of linear equations to find $\vec{F}^1$:

$$\mathrm{K}^0 \vec{F}^1 = \vec{b}^1 - \mathrm{K}^1 \vec{F}^0 \tag{25}$$

with the solution $\vec{F}^1 = (\mathrm{K}^0)^{-1}(\vec{b}^1 - \mathrm{K}^1\vec{F}^0)$.

EXAMPLES OF CALCULATIONS

As calculation examples, the optimal excitation parameters of the plate with defects were found to achieve the shape of vibration defined by the expression:

$$W(k, r, \phi) = (-I_0(k) / J_0(k) J_0(kr) + I_0(kr)) + \\ + (-I_1(k) / J_1(k) J_1(kr) + I_1(kr)) e^{i\phi}. \tag{26}$$

The pattern (26) is presented in Fig. 2. As was shown in [2], this form provides a wave traveling in the circular direction.

In dimensionless form, the first resonant frequencies are: $k_{01} = 2.232$, $k_{02} = 5.455$, $k_{11} = 0$, $k_{12} = 3.734$.

The calculations were performed for the low frequency region ($k = 1.2324$ — quasi-static excitation), for the midrange frequency region ($k = 3.2324$ — between the first and the second resonant frequencies), and for the high frequencies ($k = 5.91$ — higher than the second resonant frequency) of excitation. The number of harmonics taken into account was determined automatically.

In each group, at a fixed excitation frequency, in addition to the central force, examples with different numbers of forces were considered. Forces were grouped by 2 forces located centrally symmetrically. Group radii, orientation angles (angles between axes of force groups and a fixed direction), amplitudes and phases of forces are taken as control parameters. As the location of the active supports, the optimal locations for a defect-free plate are taken. The optimal values of the excitation forces for a defect-free and defective plate are determined independently. Thus, the location of the supports, which is a design parameter, is determined based on the assumption that the plate is ideal. The presence of defects in the plate is proposed to be compensated by the selection of exciting forces.

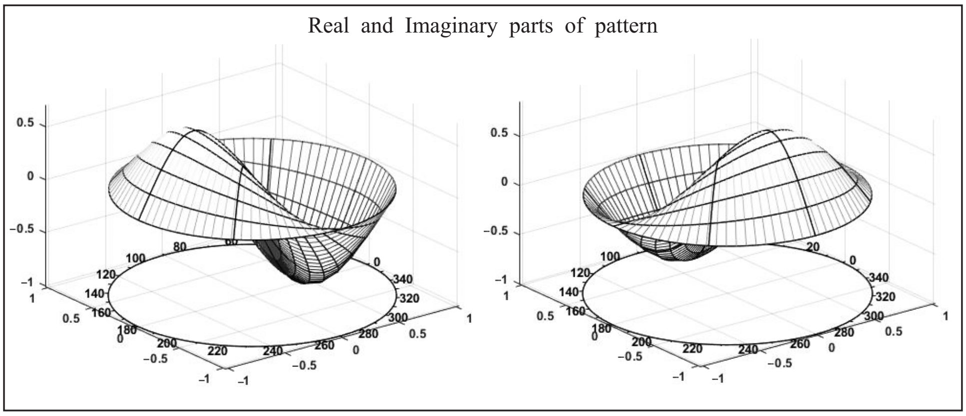

Fig. 2. Target surface shape

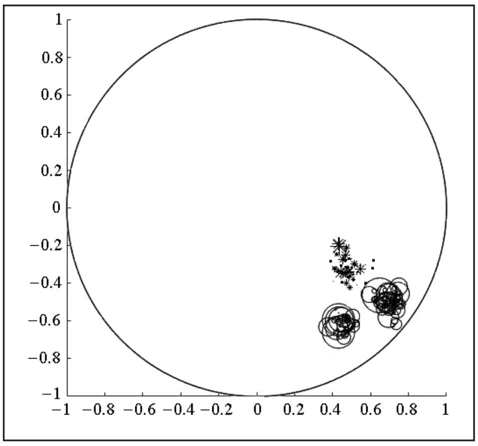

Fig. 3. The characteristic distribution of defects on the plate

We placed 3 groups of inhomogeneities on the plate. The centers of the groups were set randomly in accordance with the uniform distribution. Inhomogeneities in groups were placed randomly relative to the centers of groups in accordance with the normal distribution law with $\sigma = 0.03,\ 0.04,\ 0.05$. The total number of inhomogeneities in the groups was randomly selected (evenly distributed) within $N_{\min} = 150$, $N_{\max} = 250$. Measures of defect heterogeneity κ_D were chosen in accordance with the law of uniform distribution within $\kappa_{D\min} = -0.5$, $\kappa_{D\max} = 0.5$. Positive values of k_D correspond to weakening of the material in the defect area (circles in Fig. 3), negative values to toughening (asterisks in Fig. 3). Effective defect radii are uniformly distributed random variables with limits $r_{D\min} = 0.01$, $r_{D\max} = 0.08$. Some typical examples of defect distributions are shown in Fig. 3.

T a b l e 1. Results of calculations in low-frequency, medium-frequency and high-frequency ranges

k	key_def	$I(R)$	NF	R_g	SF	MF
1.2324	0	1.82e–04	5	0.443, 0.443	1.74	1.37
1.2324	1	3.2e–03	5	0.443, 0.443	2.05	7.3
1.2324	0	5.77e–05	7	0.545, 0.545, 0.545	1	1.18
1.2324	1	3.1e–03	7	0.545, 0.545, 0.545	2.17	5.94
3.2324	0	4.08e–04	5	0.447, 0.447	0.094	0.136
3.2324	1	6.85e–03	5	0.447, 0.447	0.128	0.152
3.2324	0	1.18e–05	7	0.546, 0.546, 0.546	0.072	0.089
3.2324	1	1.46e–03	7	0.546, 0.546, 0.546	0.086	0.133
5.91	0	1.32e–02	5	0.351, 0.351	0.437	0.628
5.91	1	1.07e–01	5	0.351, 0.351	0.518	0.595
5.91	0	2.04e–03	7	0.551, 0.551, 0.551	0.791	0.987
5.91	1	1.94e–02	7	0.551, 0.551, 0.551	0.886	1.05

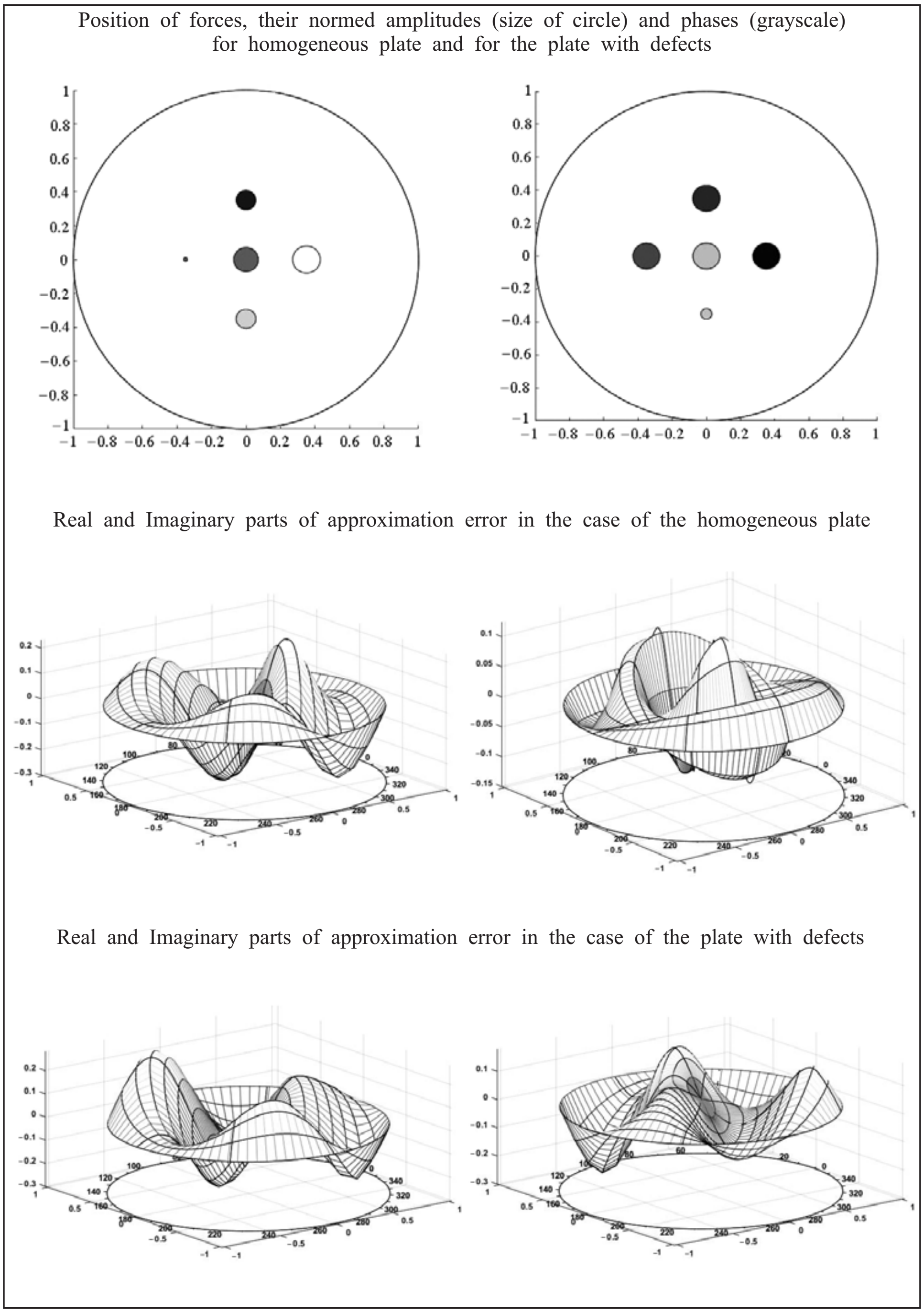

Fig. 4. The oscillation of the plate with 5 forces in the high frequency range ($k = 5.91$)

The computational procedure for each example is:

- finding the optimal location of the active supports and the values of the optimal excitation forces for a defect-free plate;
- introduction of a random set of defects into the plate;
- finding the optimal values of the optimal excitation forces for a defective plate with the optimal arrangement of active forces for a defect-free plate;
- finding waveforms for a defect-free and defective plate.

Calculation results are summarized in Table 1 and are illustrated in Figs. 4 and 5.

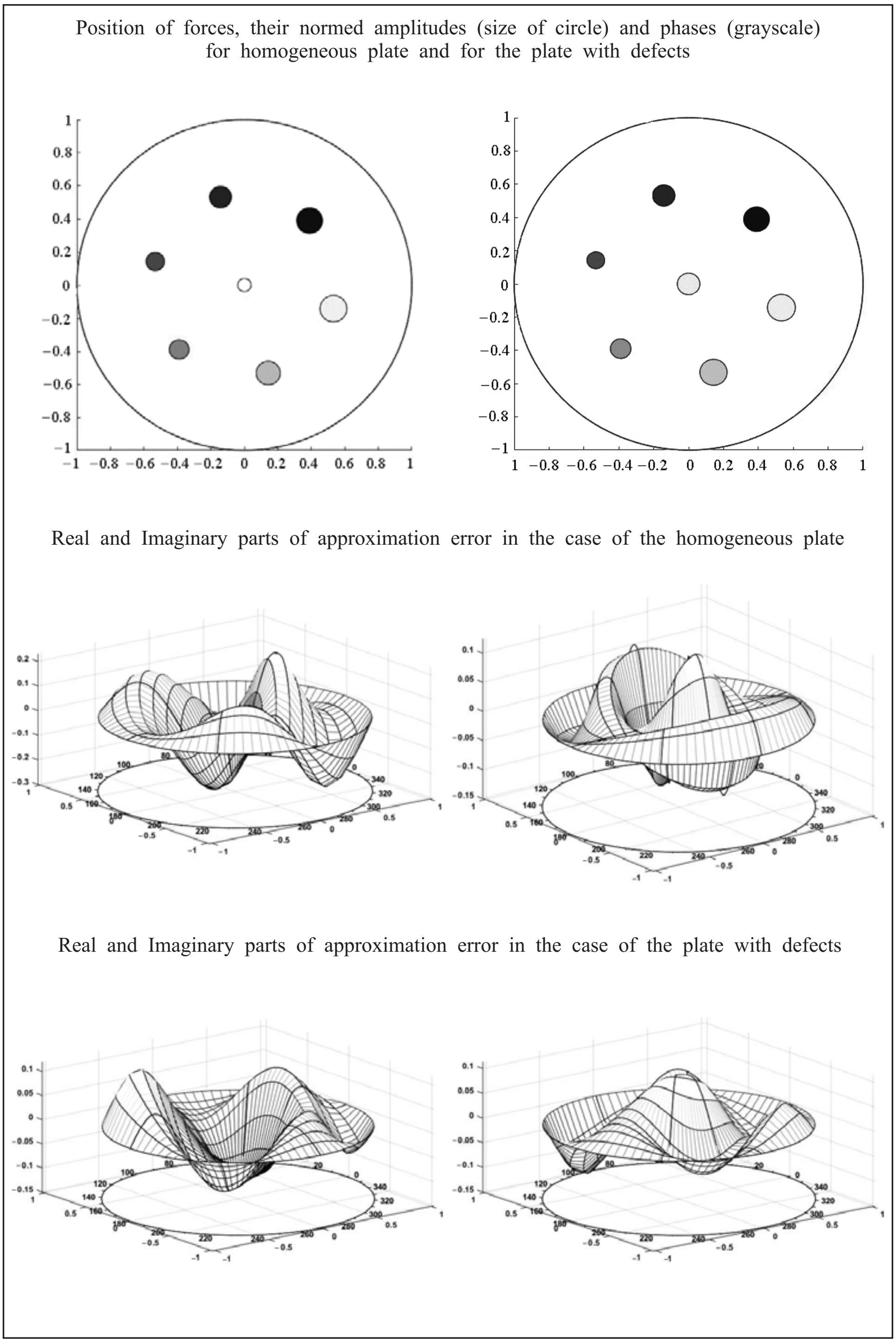

Fig. 5. The oscillation of the plate with 7 forces in the high frequency range ($k = 5.91$)

In Table 1 key_def means the presence (1) or absence of defects, $I(R)$ is the value of the root-mean-square deviation (formula (18), (20)), *NF* is the number of control forces, R_g is the optimal radii of force groups, *SF* is the root-mean-square value of forces amplitudes, *MF* is the maximum amplitude of forces, *k* is vibration excitation frequency.

The calculations were carried out for two and three groups of forces, each of which includes two forces located symmetrically. The frequency of excitation of oscillations was chosen in three frequency ranges.

• In low-frequency (quasistatic) range $k = 1.2324$ — below the value of the first resonant frequency $k_{01} = 2.232$, with the reference waveform corresponding to the first asymmetric waveform;

• In mid-frequency range $k = 3.2324$ — above the value of the first resonant frequency, but below the second resonance $k_{12} = 3.734$ with the reference waveform corresponding to the first asymmetric form of natural vibrations;

• In high-frequency range $k = 5.91$ — above the value of the frequency of the second symmetric resonance $k_{02} = 5.455$;

Visual interpretation of the results for k =5.91 is given in Figs. 4 and 5.

Results obtained are presented in Table 1.

CONCLUSIONS

The article presents the basic formulas which are necessary for calculating the control actions in the form of point active supports to obtain the optimal waveform of monoharmonic forced oscillations of a round plate. The optimization procedure is based on minimizing the root-mean-square deviation of the wave profile of the plate from the given one. The points of location of active supports, amplitudes and phases of forces are taken as control parameters. Since the optimization problem is not convex, additional studies have been carried out to substantiate the result. The article analyzes both an ideally homogeneous plate and a plate with randomly located inhomogeneities simulated by a set of small defects. The constructed theory has an applied character. An approximate solution of formal boundary value problems was sought in the form of segments of Fourier series along the circumferential coordinate. The inhomogeneities were modeled by point sources with a recursively increased singularity order.

The presented theory is the basis of the numerical implementation. Test calculations were carried out, and demonstrate the correctness of the problem statement and its solution. The optimization package PSG 3.2.0 [9] was used in the calculations. The calculation results are shown in Fig. 4, Fig. 5 and in Table 1. The figures show the optimal positions of the supports, the normalized values of the optimal amplitudes and phases of the control forces, as well as the deviations of the plate vibration modes from the given one in various frequency ranges, various numbers of supports, in the absence and presence of defects. Table 1 shows the dependences of the values of the root-mean-square deviation, the optimal radii of force groups, the root-mean-square values of the amplitudes of the optimal forces and the maximum values of the amplitudes of the optimal forces on the oscillation frequency, the presence or absence of defects, and the number of active supports.

As can be seen from the presented calculations, both for a homogeneous plate and for a plate with defects, it is possible to choose the optimal location and control of active supports, which provide a sufficiently high quality approximation of a given form of plate vibrations. In this case, the control forces and the quality of the approximation essentially depend on the presence of inhomogeneities and the oscillation frequency. At the same time, the optimal location of the supports weakly depends on the presence of inhomogeneities and the oscillation frequency. The location of the supports should be chosen based on the given form of vibrations. So, in the considered numerical example, in order to achieve the best result, the supports (not taking into account the central fixed support) should be placed symmetrically on one concentric circle, the radius of which depends significantly on the number of supports. The natural result is an improvement in the quality of the approximation with an increase in the number of forces. It should be noted that such conclusions are partly the result of the symmetry of the given form of vibrations, chosen in the form of a waveform traveling in a circumferential direction (Fig. 2).

The results of the article can be used in the design of active controllable reflective elements in multichannel information or energy transmission systems.

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РОЗРОБЛЕННЯ МАТЕМАТИЧНОЇ МОДЕЛІ МОДУЛЮВАЛЬНОГО ДЗЕРКАЛА, ЗАКРІПЛЕНОГО НА АКТИВНИХ ОПОРАХ. ДЕТЕРМІНОВАНА ЗАДАЧА

Анотація. Запропоновано математичну детерміновану модель модулювального дзеркала, закріпленого на активних опорах, за припущення, що дзеркало може містити дефекти. Задача полягає у знаходженні оптимального розташування опор, а також сил керування, які би забезпечили найкраще наближення заданої форми та розподілу фаз коливань як однорідного дзеркала, так і дзеркала з дефектами, що мають задані геометричні та механічні характеристики. Для опису дзеркала обрано модель пластини Кірхгофа. Моделювання дефектів виконано з використанням неоднорідностей малих розмірів зі зміненими пружними характеристиками. Розроблено ітераційний метод моделювання дефектів обмеженого розміру на пластині Кірхгофа з використанням точкових квадруполів. Моделювання ізольованих активних опор виконано точковими силами. Параметрами оптимізації є розташування опор, амплітуди та фази сил, що продукують коливання. Як критерій оптимальності використано мінімум середньоквадратичного відхилення хвильової форми пластини від заданої.

Ключові слова: модулювальне дзеркало, пластина з дефектами, оптимальне збудження.

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