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FEATURES OF MODELING AND IDENTIFICATION OF COGNITIVE MAPS UNDER UNCERTAINTY

Abstract. A process of complex systems identification is examined in this paper. It was established that it is impossible to create a universal identification method. Only for a well-identifiable system with a high signal-to-noise ratio for each individual system mode, a high-quality model can be reconstructed. In other cases, if modes with sufficiently small signal-to-noise ratio exist, only a surrogate model can be obtained. For cognitive maps, theoretical foundations are developed, which may be used in approaches to find a surrogate model and then to improve the result using different tuning and learning algorithms. Numerical simulation was used to analyze the identification process.

Keywords: cognitive map, system identification, subspace method, complex system, ill-conditioning, regularization.

PROBLEM STATEMENT

Linear time-invariant (LTI) models are widely used for simulation of impulse processes in cognitive maps (CM). Many different ecological, social, economical, educational, financial and other systems can be modeled and analyzed based on a CM. Mathematically CM is an oriented graph with nodes representing complex systems coordinates (concepts) and edges describing cause–effect relations between the nodes. We consider weighted CM where edges are weighted depending on significance of corresponding relation. See, for example, Fig. 1 (CM of IT company) [1].

During complex system operation under different disturbances CM coordinates change in time. Each CM node R_i is set to values $z_i(t)$ in discrete times $t = 0, 1, 2, \ldots$. The next value $z_i(t+1)$ is determined by current value $z_i(t)$ and coordinates increments of other nodes R_j connected to R_i at time t. Change of nodes R_j coordinates $P_j(t) = \Delta z_j(t) = z_j(t) - z_j(t-1)$, $t > 1$, is called an impulse. Propagation of impulses over CM nodes is called impulse process and according to [2] is described by the equation

$$z_i(t+1) = z_i(t) + \sum_{j=1}^{n} a_{ij} P_j(t), \quad i = 1, \ldots, n, \tag{1}$$

where a_{ij} is a weight of edge from R_j to R_i.

Another way, CM nodes coordinates' evolution rule (1) may be formulated as first-order difference equation in increments:

$$\Delta z_i(t+1) = \sum_{j=1}^{n} a_{ij} \Delta z_j(t), \quad i = 1, \ldots, n. \tag{2}$$

Equation (2) may be written in a vector form:

$$\Delta z(t+1) = A \Delta z(t), \tag{3}$$

where A is a transposed CM adjacency matrix, Δz is a vector of coordinates

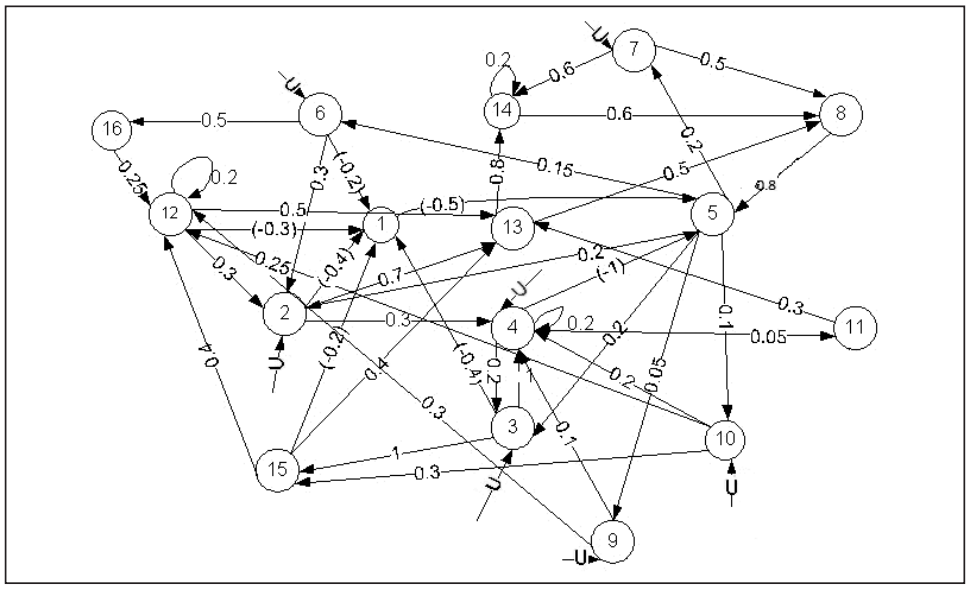

Fig. 1. CM of IT company. Nodes: 1 — project duration; 2 — innovation expenses; 3 — salaries, bonuses; 4 — project's budget; 5 — profit; 6 — manager group expenses; 7 — marketing expenses; 8 — revenue of projects' sells; 9 — staff re-assessment expenses; 10 — staff training expenses; 11 — technical control; 12 — intelligent assets; 13 — quality of a final product; 14 — competitive ability; 15 — job satisfaction; 16 — experience exchange

increments. Models (2), (3) describe multivariate dynamic discrete system in free motion of CM nodes.

To describe a forced motion equation under impulse process the following equation was proposed in [3]:

$$\Delta z_i(t+1) = \sum_{j=1}^{n} a_{ij} \Delta z_j(t) + b_i u_i(t), \;\; i = 1, \ldots, n, \tag{4}$$

where $u_i(t)$ is external input increment which is implemented by means of varying resources of R_i node, b_i is usually equal to 1 or 0 depending on whether it is possible to directly affect the ith node. Then equation (3) for the forced motion case can be written as

$$\Delta z(t+1) = A \Delta z(t) + Bu(t), \tag{5}$$

where $u(t)$ is a vector of external inputs; in the rows of the matrix B, corresponding to the numbers of nodes to which the external action is applied, units are set (in the corresponding columns), and the remaining elements are assumed to be equal to zero.

Also, it is quite natural that some nodes of CM are not measured. Then equation (5) can be augmented by the measurement equation

$$y(t) = C \Delta z(t), \tag{6}$$

where $y(t)$ is a vector of increments of measurable coordinates; knowing the nodes with measured coordinates, one can put units in the matrix C in their respective places, and the remaining elements are equal to zero.

The cognitive maps impulse process models (5), (6) are associated with multi-input multi-output (MIMO) LTI systems and their dimension in some cases may be large. So subspace-based state-space system identification (4SID) method is the numerically reliable way for model reconstruction directly from measured data. No nonlinear search is performed nor is a canonical parametrization used. Computational

complexity is modest compared to other existing identification methods. Theoretical base for 4SID is a state-space realization theory. Its origin is linked with classical contribution [4] where a scheme for recovering the system matrices from impulse response measurements is outlined. Up to now there are many publications devoted to different realizations of this method and its applications. Here we mention only two contributions where subspace method is outlined in details [5, 6].

In this paper we use a subspace concept and its developments including software for research of identification process features in CM. There are many versions of subspace method but it is not clear how to choose the most appropriate one in a specific case. From the general insight it is supposed that each existing method has its own field of applications which is defined by dynamic system properties, informativeness of the input and noses in available data. According to consistence theory the persistently exiting input under presence of white type of noises provides nonbiased model parameters estimation [5, 7]. But in recent years a shift in paradigm of stochastic identification (see [8]) happened, caused by ill-conditioning of the matrices to be inverted in subspace and others methods of identification. With increasing of a model dimension, it is well known that conditioning of these matrices become worse and the regularization procedure has to be used.

It is worth mentioning that regularization parameters used for this depend not only on model dimension but also on noise level and its type (distribution, variance and so on). Very often, as a regularizing procedure, the choice of an appropriate model dimension is used, which may turn out to be smaller than the real system dimension, if one exists in a given class of models.

The most difficult problem is the problem of separating the signal subspace and the noise subspace. In measurements, only the signal-to-noise ratio (SNR) can be estimated for each output variable. A contribution of each individual mode to the outputs remains unknown. First of all, the SNR of individual modes is affected by the dynamic features of the data generating system, namely, the eigenvalues and their location on the complex plane with respect to the unit circle.

In this paper, all the features of the identification process based on the Subspace approach and its dependence on the structural features of the system and a number of other factors are investigated in detail.

FEATURES OF COGNITIVE MAP MODELING

Impulse processes in cognitive maps are considered as LTI and are described in the state space by the following system of difference equations, taking into account (5), (6):

$$x(t+1) = Ax(t) + Bu(t), \;\; y(t) = Cx(t), \tag{7}$$

where t is the dimensionless discrete time; $x(t) = \Delta z(t)$ is the system state vector at the time t of dimension n, which consists of CM nodes coordinates increments Δz_i; $y(t)$ is the vector of measurable variables (increments of measured nodes coordinates) at the moment t of dimension M; $u(t)$ is the vector of input impulses at the moment t of dimension R; A, B, and C are constant matrices of corresponding dimensions representing system parameters.

From description (7) based on the Cauchy formula for discrete systems, one can go to the equivalent matrix representation [9, 10]

$$Y = \Gamma_L X + \Phi_L U, \tag{8}$$

where trajectory data are used on the observation interval $[l, l+T-1]$, i.e., duration T with starting point l. A cascade vector is formed from the vectors $y(t)$ on the observation interval

$$y(t, L) = [y_t' \;\; y_{t+1}' \ldots \; y_{t+L-1}']' \tag{9}$$

of dimension ML ("′" is the transposition operation). By analogy with (9), a column vector $u(t,L)$ is written, composed of a sequence of vectors $u(t)$:

$$u(t,L)=[u_t'\ u_{t+1}'\dots\ u_{t+L-1}']'. \tag{10}$$

Cascade vectors (9), (10) actually represent a piece of the system (7) trajectory from time t to $t+L-1$, i.e., $\{y(t),u(t)\}_t^{t+L-1}$. The matrices Γ_L and Φ_L in (8) are represented as

$$\Gamma_L=\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{L-1}\end{bmatrix},\ \Phi_L=\begin{bmatrix} 0 & \dots & 0 & 0 \\ CB & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ CA^{L-2}B & \dots & CB & 0\end{bmatrix}. \tag{11}$$

Equation (8) also includes Hankel trajectory matrices

$$Y=[y(l,L),\ y(l+1,L),\dots,\ y(l+T-L,L)],$$

$$U=[u(l,L),u(l+1,L),\dots,u(l+T-L,L)],$$

as well as the Hankel matrix of initial conditions of the form

$$X=[x(l),x(l+1),\dots,x(l+T-L)].$$

Matrix equation (8) represents the shift set of system trajectories pieces on the observation interval. It connects a set of trajectories with system parameters in the form of observability and impulse response matrices, and each of these trajectories is determined by its initial state. Matrix equation (8) is the initial direct 4SID method for identifying MIMO systems from trajectory data.

If the matrix A is reduced to a Jordan form, then using the same Cauchy formula, one can transform (7) into the input-output relations.

Let $A=(A_Q\ A_P)$, where A_Q is the block-matrix of real eigenvalues α_q $(q=\overline{1,Q})$ located on its diagonal; A_P is a matrix with Jordan blocks $A_P=\begin{pmatrix}\alpha_p & -\beta_p\\ \beta_p & \alpha_p\end{pmatrix}$ $(p=\overline{1,P})$ on the diagonal corresponding to complex conjugate eigenvalues $\lambda_p=\alpha_p\pm i\beta_p$. The matrices B and C are in the form

$$B=(b_1\dots b_Q\ b_1^c\ b_1^s\dots b_P^c\ b_P^s)',$$

$$C=(c_1\dots c_Q\ c_1^c\ c_1^s\dots c_P^c\ c_P^s),$$

where b_q $(q=\overline{1,Q})$, b_p^c, b_p^s $(p=\overline{1,P})$ are column vectors of dimension R, and c_q $(q=\overline{1,Q})$, c_p^c, c_p^s $(p=\overline{1,P})$ are column vectors of dimension M.

Then the Cauchy formula for the discrete case will be written as the following expression [11]:

$$y_m(t)=\sum_{q=1}^{Q}g_{mq}^0\alpha_q^{(t-l)}+\sum_{p=1}^{P}h_{mp}^0(t-l)+\sum_{j=l}^{t-1}\sum_{q=1}^{Q}\sum_{r=1}^{R}g_{mqr}\alpha_q^{(t-j)}\cdot u_r(j)+$$

$$+\sum_{j=l}^{t-1}\sum_{p=1}^{P}\sum_{r=1}^{R}h_{mpr}(t-j)u_r(j),\ m=\overline{1,M}, \tag{12}$$

where

$$h_{mp}^0(t-l)=\rho_p^{(t-l)}[f_{mp}^{0c}\cos\omega_p(t-l)+f_{mp}^{0s}\sin\omega_p(t-l)],$$

$$h_{mpr}(t-j)=\rho_p^{(t-l)}[f_{mp}^{rc}\cos\omega_p(t-l)+f_{mpr}^{rs}\sin\omega_p(t-l)],$$

$$\rho_p=|\lambda_p|=\sqrt{\alpha_p^2+\beta_p^2},\ \omega_p=\arg(\lambda_p),$$

$$g_{mq}^0=c_{mq}x_q(l),\ f_{mp}^{0c}=c_{mp}^c x_p^c(l)+c_{mp}^s x_p^s(l),$$

$$f_{mp}^{0s}=c_{mp}^c x_p^s(l)-c_{mp}^s x_p^c(l),\ f_{mp}^{rc}=c_{mp}^c b_{rp}^c+c_{mp}^s b_{rp}^s,\ f_{mp}^{rs}=c_{mp}^c b_{rp}^s-c_{mp}^s b_{rp}^c,$$

$x_q(l)$, $x_p^c(l)$, $x_p^s(l)$ are components of the initial state vector, namely

$$x(l)=(x_1(l)\dots x_Q(l)\,x_1^c(l)\,x_1^s(l)\dots x_P^c(l)\,x_P^s(l))'.$$

Formula (12) is written for the case where there are no multiple roots. In their presence (12) becomes more cumbersome. Like the eigenvalues, the parameters g_{mq}^0, f_{mp}^{0c}, f_{mp}^{0s}, f_{mp}^{rc}, and f_{mp}^{rs} in (12) are invariants.

If the trajectory data $\{y(t),u(t)\}_l$ are known, then from (12) it is possible to create a system of equations appropriate for the study of identification processes.

As follows from the above, for the implementation of the CM corresponding to the directed graph (see Fig. 1), the matrices B and C in (7) become known, and both of these matrices consist of zeros and ones. In accordance with the realization theory [12], the product of matrices CB is invariant under a nonsingular transformation. Therefore, for any other realization obtained from the original one with the help of a non-singular transformation T, we can write the relation

$$C'B'=CB, \tag{13}$$

where $C'=CT^{-1}$, $B'=TB$.

Since for the implementation of the matrix represented by the directed graph C and B are known, then from (13) we obtain a matrix equation with a known right-hand side, which, along with (7), (8), (12), can be used to identify the CM.

There is also another invariant

$$CAB=C'A'B'. \tag{14}$$

If for some implementation we know matrices C', A' and B', then (14) gives additional equations for finding the initial matrix A corresponding to the CM graph (see Fig. 1).

ACTIVE EXPERIMENTS WITH INFORMATIVE INPUTS

Usually data for identification are taken from active or passive experiments. In active experiments, it is planned and organized in order to obtain the best result of a model reconstruction that is adequate to the data obtained in experiments. In passive experiments, the input action is implemented without participation of the experimenter, and the data are obtained from measurements of the object, including the parameters that determine the input. It is obvious that the result essentially depends on what process was realized in the experiment. In this case, the quality of the identified model may turn out to be unsatisfactory. In this research, the main attention is paid to the study of the identification process itself and its features depending on various factors, including the dynamic properties of the object generating data. Research is carried out by numerical simulation methods with imitation of different conditions implemented in experiments. It is best to do this in active experiments that allow us to identify all the features and nuances of identification in all its diversity. Therefore, we begin with a description of the

experiments in which the data and conditions necessary for this are formed. Particular attention is paid to the formation of input actions favorable for identification [10]. The existing 4SID identification methods usually use a persistently exciting input. It is widely used not only in 4SID methods, but also in many others. There are different interpretations of what is meant by a constantly exciting effect of the order equal to or greater than the expected dimension of the desired model.

The informativeness of the input signal during identification is evaluated by the signal-to-noise ratio (SNR). This is usually done for measured quantities. However, they are an estimate of the total contribution of individual modes, and among them there may be modes with large and weak signals, and the identification result is determined by the SNR of each of these modes. Therefore, it is proposed to excite the system in such a way as to be able to enhance the response of one mode or another. To do this, we will use different methods, including the resonant properties of modes with an oscillatory process corresponding to complex conjugate eigenvalues. For real roots, a signal in the form of a rectangular pulse of the maximum allowable amplitude is suitable. This assumes that the noise signal does not increase proportionally to the excitation signal. Then, by varying the pulse length, one can control the response of each of these modes. At short durations, the fastest modes have the greatest signal, to which, as the excitation interval increases, slower modes are added.

Based on the foregoing, the following plan for active experiments is proposed to obtain data for identifying the system under study. Depending on the possibilities available, it can be implemented in two ways. In the first method, many separate experiments are carried out, in each of which two intervals are formed. At the first stage the system is excited, and at the second stage the system relaxes at zero input (it is assumed that the system is stable), i.e., free movement from some initial state. Due to the variety of excitation signals (rectangular pulses, harmonics of different frequencies), it is possible to obtain different initial conditions in which the system is located at the moment the relaxation process begins. The larger and more diverse the set of initial conditions, the better the identification results will be. The second method of obtaining informative data for identification is implemented in one long-term experiment. It alternates successively intervals of excitation with intervals of relaxation. At each excitation interval, different signals are used, both rectangular pulses and harmonics with a variable frequency. The smaller the step of varying the lengths of rectangular pulses and harmonic frequencies, the more likely it is to obtain an informative set of initial states for relaxation intervals. And this makes it possible to more effectively solve the problem of identification.

The experiment with the first method of excitation is most effective when the system is at rest at each moment of the excitation signal, i.e. with zero initial data. The second method is more general and allows an acceptable solution of the problem with non-zero initial data. In any case, to solve the problem based on the subspace approach, it is necessary that the length of the relaxation intervals be the same and preferably not less than the duration of the transition process of the slowest mode in order to have the most informative signals. Let u_0 be the admissible maximum value of the input signal. Then we take the amplitude of rectangular pulses equal to u_0, and it is desirable that their number be not less than the number of aperiodic modes with a variable length in the range from the minimum to the maximum length of the transient. When such information is not available, one should take a sufficiently large interval with a uniform distribution of the length of rectangular pulses on it, which is guaranteed to contain transients of all real modes. With harmonic excitation, one

harmonic is used in each interval, varying in the interval $\left(0, \frac{\pi}{2}\right]$. Since (7) is a dimensionless description, the frequencies which exceed $\frac{\pi}{2}$ lose their oscillation, and the long-wavelength harmonics appear to be frequencies close to zero. At the same time, in order to ensure their information content, the duration of the experiment should be increased accordingly with decreasing frequency.

REALIZATION-BASED SUBSPACE IDENTIFICATION

We apply Realization-based subspace identification for generalized observability matrix identification using data on relaxation intervals.

Let each ith relaxation interval start from the point t_i. Since the length of all of them is the same equal to l, exceeding or equal to the time of the transient process, for identification we will use the data on the set of points $[t_i \;\; t_{i+1} \ldots \; t_{i+l-1}]$, where t_i is the initial and t_{i+l-1} is the last point of the relaxation interval. The total number of relaxation intervals is K, i.e., $i = \overline{1, K}$. There will be the same number of excitation intervals.

The value K must exceed the expected dimension of the model. From the measured output in the relaxation intervals, we form a cascade vector of the form

$$y(t_i, l) = (y'_{t_i} \;\; y'_{t_{i+1}} \ldots \; y'_{t_{i+l-1}})'.$$

Then, according to the Cauchy formula, for the discrete case of the LTI system, the following equation is held for each relaxation interval:

$$y(t_i, l) = \Gamma_l x_{t_i}, \tag{15}$$

where Γ_l is the generalized observability matrix and x_t is the initial state of the system for the ith relaxation interval. From the vectors $y(t_i, l)$ we form a matrix Y_{relax} of dimensions $l \times K$ in the following form

$$Y_{\text{relax}} = (y(t_1, l) \;\; y(t_2, l) \ldots \; y(t_K, l)). \tag{16}$$

As a result, according to (15) and (16), we arrive at the matrix equation

$$Y_{\text{relax}} = \Gamma_l X_0, \tag{17}$$

where X_0 is a matrix of dimension $n \times K$ (n is the estimated dimension of the model), which has the form

$$X_0 = (x_{t_1} \;\; x_{t_2} \ldots \; x_{t_K}), \tag{18}$$

i.e., matrix of initial states on relaxation intervals. Since instead of accurate measurements we have noisy measurements containing random errors, instead of (17) we have

$$\widetilde{Y}_{\text{relax}} = \Gamma_l X_0 + N, \tag{19}$$

where N is the corresponding Y_{relax} error matrix of the measured components of the cascade vectors $y(t_i, l)$. Based on these approximate data, it is necessary to find an estimate for the matrices A, C and X_0. In subspace identification methods, a realization-based approach is used for this, which begins with the singular value decomposition (SVD) of the matrix $\widetilde{Y}_{\text{relax}}$, i.e.,

$$\widetilde{Y}_{\text{relax}} = Q \Sigma V', \tag{20}$$

where Q and V are square orthogonal matrices, and Σ is a rectangular matrix of singular numbers located on the diagonal in non-increasing order [13]. In the

presence of noise, i.e., in case (19), (20) will have the representation

$$\widetilde{Y}_{\text{relax}} = Q_s \Sigma_s V_s' + Q_e \Sigma_e V_e', \tag{21}$$

where the first term allocates the signal subspace corresponding to the output data generated by the system, and the second term allocates errors in the data. Separation (21) is done by the values of singular numbers and by partitioning into blocks of matrices Q, Σ, V. At low noise, when the SNR is large, the noise part corresponds to small singular values located at the end of the diagonal of the matrix Σ. However, it should be noted here that individual modes with an SNR less than unity may appear in the second term. This means that these modes have little effect on the dynamics of the system and it is difficult to identify them at the background of noise. As a result of identifying systems with such singularities, approximate models of reduced order can be found.

After selecting the signal subspace according to (21), it is possible to derive the equations for identification of observability matrix Γ_l and matrix X_0 using the realization-based subspace approach. As far as all matrices Q_s, Σ_s, and V_s have full rank there is such a realization that

$$Q_s \Sigma_s = \Gamma_l,\ V_s' = X_0. \tag{22}$$

According to the realization theory, it is possible to obtain equations similar to (22) for a different realization, if represent $Q_s \Sigma_s V_s'$ as a product of two full-rank matrices differently.

The first equation in (22) is used to find the matrices C and A for some realization corresponding to (22), while the second one gives directly the initial states on all relaxation intervals. The first M rows $Q_s \Sigma_s$ represent the matrix C of the corresponding (22) implementation. The matrix A can be found from a system of equations derived from the shift invariance of the LTI system. It is written as

$$(Q_s \Sigma_s)_1 A = (Q_s \Sigma_s)_2, \tag{23}$$

where the matrix $(Q_s \Sigma_s)_1$ is created based on the matrix $Q_s \Sigma_s$ by deleting the last M rows, and $(Q_s \Sigma_s)_2$ consists of the first M rows of the same matrix. It is proposed to find a solution (23) using the SVD decomposition of the matrix $(Q_s \Sigma_s)_1$. We write it in the form

$$(Q_s \Sigma_s)_1 = Q_1 \Sigma_1 V_1', \tag{24}$$

where Q_1 and V_1 are square matrices of dimensions $l \times l$ and $n \times n$, respectively (n is the number of singular values of the matrix Σ_s). A rectangular matrix Σ_1 has a dimension $l \times n$ and can be written in block form

$$\Sigma_1 = \begin{bmatrix} \Sigma_n \\ 0 \end{bmatrix}. \tag{25}$$

where Σ_n is a square matrix of dimension n. As a result, the least squares solution of system (23) will be written as

$$(A)_i = V_1 \Sigma_n^{-1} Q_{1n}' (Q_s \Sigma_s)_{2i},\ i = \overline{1, n}. \tag{26}$$

In (26) the matrix Q_{1n} is composed of the first n columns of the matrix Q_1, and $(Q_s \Sigma_s)_{2i}$ is the ith column vector of the matrix $(Q_s \Sigma_s)_2$. All solutions (26) completely determine the matrix A of dimension n. Solution (26) is used when the matrix $(Q_s \Sigma_s)_1$ is well conditioned. This can be judged by the condition number

determined by the ratio of the first and last singular value of the matrix Σ_n. For an ill-conditioned matrix, regularization should be used. In the considered solution method based on SVD, regularization will be carried out using the stabilizer

$$\alpha\Sigma_r^q,\ q=1,2, \tag{27}$$

which is additively added to the matrix Σ_n in (26). In (27) α is the regularization parameter, and Σ_r is the matrix of the following form

$$\Sigma_r=\begin{bmatrix} \sigma_n/\sigma_1 & 0 & \dots & 0 & 0 \\ 0 & \sigma_n/\sigma_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \sigma_n/\sigma_{n-1} & 0 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix},$$

σ_i are matrix Σ_n singular values.

As a result, the regularized solution is written as

$$(A)_i=V_1(\Sigma_n+\alpha\Sigma_r^q)^{-1}Q_{1n}(Q_s\Sigma_s)_{2i},\ i=\overline{1,n},\ q=1,2. \tag{28}$$

In the absence of information about the properties of the errors, it is expedient to take its quasi-optimal value as the regularization parameter α in (28) (see [14]). As such, we take the minimizing element of the extremal problem

$$\inf_{\alpha\ge 0}\left\|\alpha\frac{d\theta_\alpha}{d\alpha}\right\|, \tag{29}$$

where $\theta_\alpha=(A)_i$. There may be different values α for different i. In addition, problem (29) may turn out to be multiextremal. Then the least of them is taken as α_{quasiopt}.

We will find α_{quasiopt} graphically, building a function $\left\|\alpha\frac{d\theta_\alpha}{d\alpha}\right\|$ depending on α.

Denote

$$\upsilon_\alpha=\alpha\frac{d\theta_\alpha}{d\alpha}$$

and using (29) υ_α can be found for different α from the equation

$$\upsilon_\alpha=-\alpha V_1(\Sigma_n+\alpha\Sigma_r^q)^{-1}\cdot\Sigma_r^q\cdot V_1\cdot(A)_i. \tag{30}$$

For different values α, first the vector $\theta_\alpha=(A)_i$ is found from (28), and then υ_α itself is found from (30). Based on the chart of $\|\upsilon_\alpha\|$, we find α_{quasiopt} for the selected n. This completes finding a regularized solution for (26) of dimension n.

SIMULATION RESULTS

The quality of identification essentially depends on how successfully the signal and noise subspaces are separated. This is a rather complicated and not entirely unambiguous procedure. It essentially depends on the following factors: what are the dynamic properties of the data generating system, what is the SNR of each of the modes in the output signal, what is the dimension of the system, what are the external disturbances, what is the duration of the experiment and the chosen values of l and K.

In fact, (21) can be considered as a filtering procedure, i.e. separating the useful signal from the noise. This is the easiest to do when the noise level is much lower than the signal of the worst observed mode. Then, according to the gap between the essential and non-essential singular values of the matrix Σ, it is easy to carry out such a separation.

However, in practice this does not happen often. Therefore, by means of numerical simulation, we will study the dependence of the singular numbers' behavior in (20) on the pointed above factors. To do this, we introduce the notion of a well-identifiable system. First of all, this is defined by informative input. As mentioned earlier, this is determined by the SNR for all modes of the system. For each of them, the SNR must be substantially greater than one. The excitation under consideration most effectively exploits the resonant properties of the system and, as will be shown below, makes it possible, with the help of certain manipulations, to influence the SNR of different modes.

Now, on the base of (12), we introduce the concept of a well-identifiable system. This means that invariants $g_{mq}, f_{mp}^{rc}, f_{mp}^{rs}$ and eigenvalues are such that all matrix Y_{relax} columns form a system of order n maximally independent vectors (n is the dimension of the system) i.e., matrix Y_{relax} should be well-conditioned of order n. Besides matrices C and B are such that system for some Jordan realization is close to independent blocks. In addition, all the main modes of the transition matrix have the same high SNR.

It should be noted that an increase of the system dimension leads to a deterioration in the independence of the eigenvectors, since the eigenvalues approach each other. As a result, Y_{relax} conditioning worsens. Figs. 2–7 present the results of modeling the identification process in the described active experiments with the determination of matrices A and C from data on relaxation intervals for well-identified systems of different dimensions. Figures 2–4 show the values of the singular values that are on the diagonal of the matrix Σ depending on the SNR, and Figs. 5–7 show eigenvalues of the system and model.

It can be seen that in the presence of noise in a certain SNR interval it is possible to establish the dimension of the model, which coincides with the data generating system's dimension. A discontinuity is clearly visible in the behavior of singular values, so it is possible to separate significant singular values from those that correspond to noise. However, as the dimension increases, the gap decreases, and this separation can be done at lower and lower noise levels; the larger dimension,

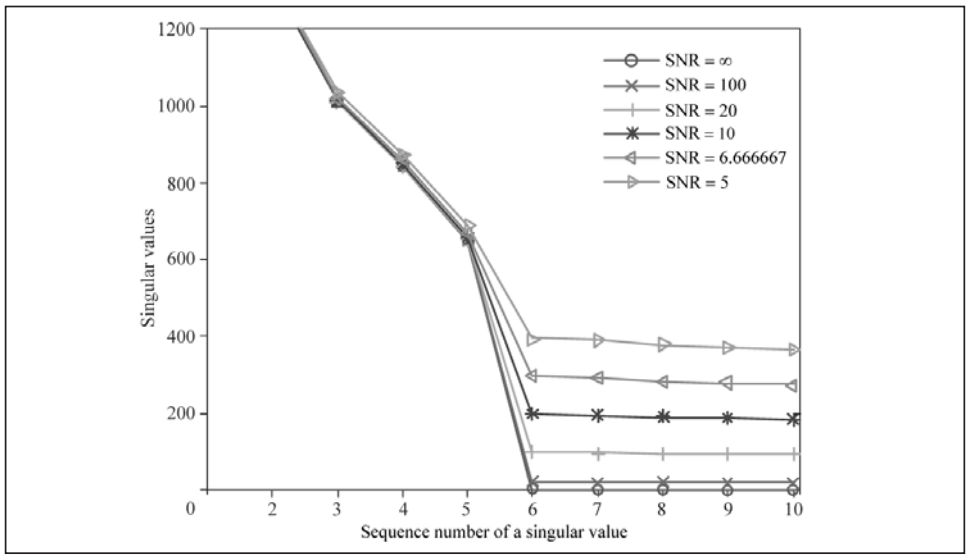

Fig. 2. Singular values for $n = 5$, $M = R = 3$

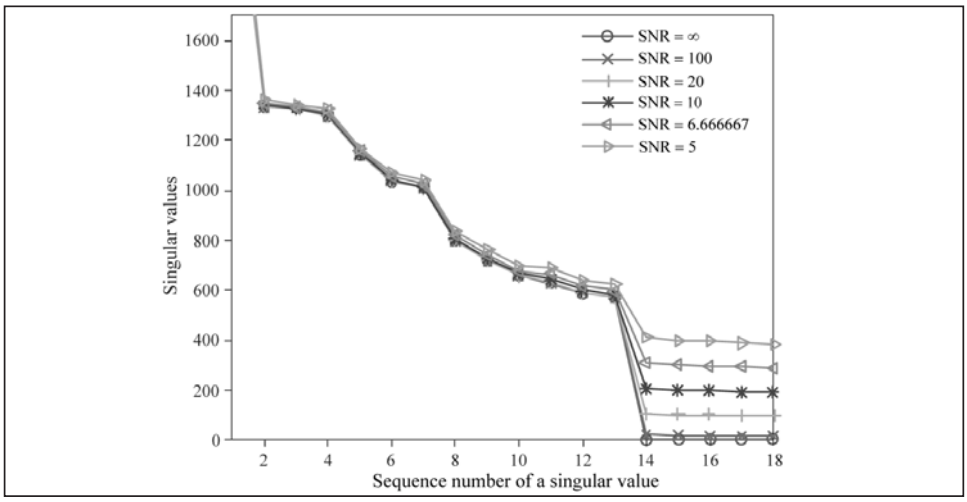

Fig. 3. Singular values for $n = 13$, $M = R = 4$

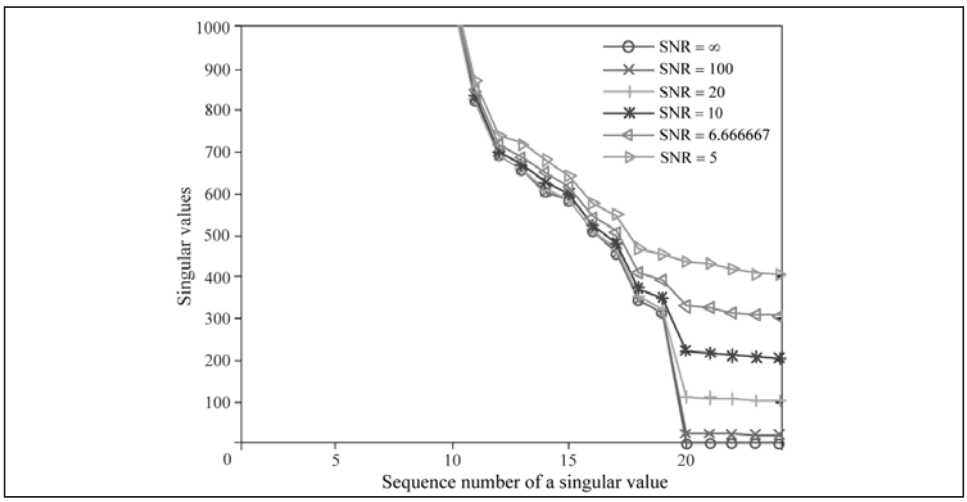

Fig. 4. Singular values for $n = 19$, $M = R = 5$

the stronger noise effect. In addition, knowing the exact dimension does not guarantee an acceptable solution of the identification problem with decreasing SNR. As it can be seen from Figs. 5–7, the accuracy of determining the matrix A eigenvalues for a given SNR deteriorates with increasing dimension. In any case, it is always effective to increase the SNR.

The greater the deviation of the system from a well-identifiable case, the worse the solution is. There is a system, which is maximally admissible in terms of identifiability, starting from which the identification problem becomes substantially ill-posed in the sense that even in the absence of noise only a surrogate model with a small forecast horizon can be built, or the identification problem loses any sense.

However, in practice there are often cases when it is possible to construct a reduced order model of satisfactory quality, i.e. having a dimension smaller than that of the generating system. The following figures and Table 1 show simulation results for one of such systems. The CM of the 16th order was considered, which has 10 outputs and 10 inputs. Singular numbers of the matrix Y_{relax} at different noises are shown in Fig. 8. It can be seen that, starting from dimension 13, the matrix Y_{relax} becomes ill-conditioned for exact data. However, under the influence of noise, we have an improvement in

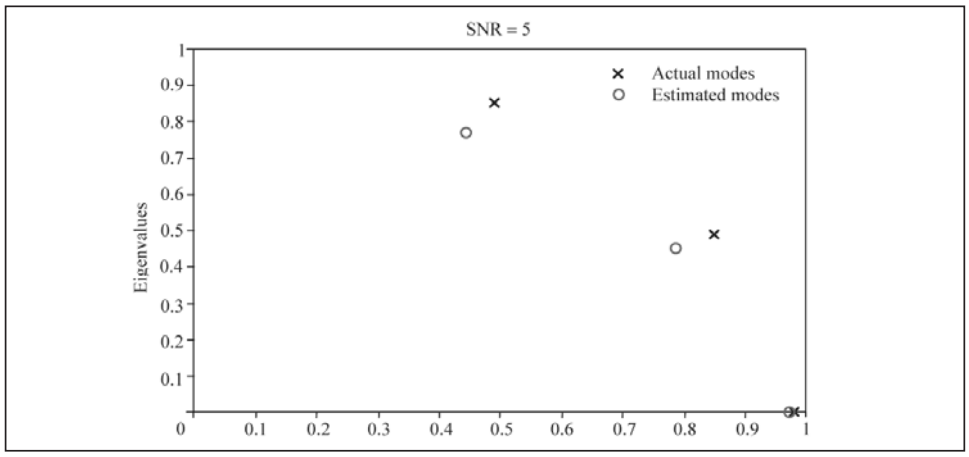

Fig. 5. Eigenvalues for $n = 5$

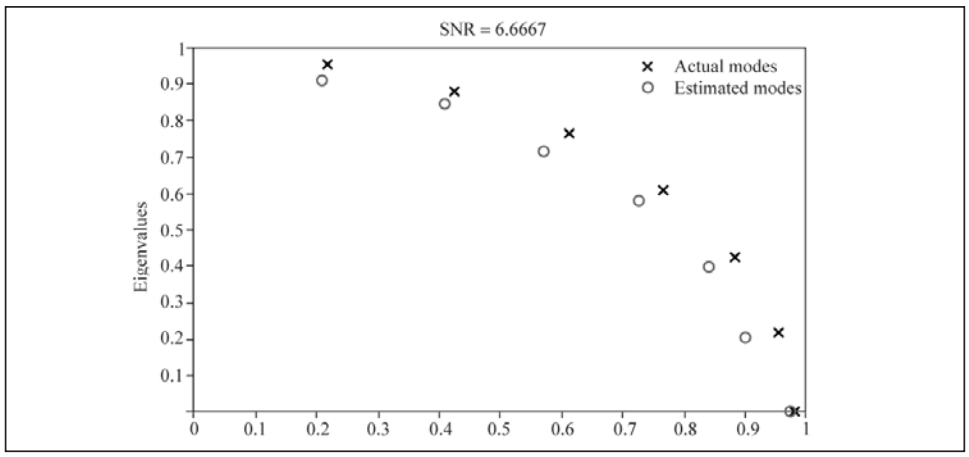

Fig. 6. Eigenvalues for $n = 13$

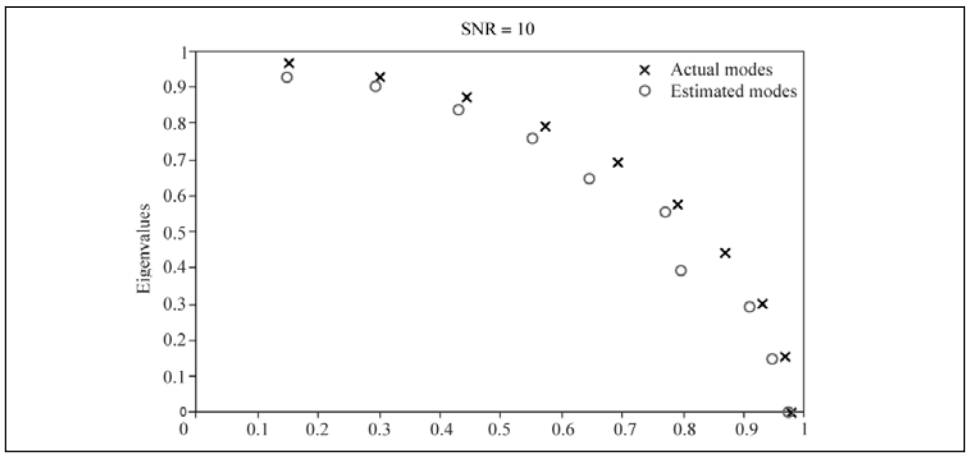

Fig. 7. Eigenvalues for $n = 19$

conditionality. But as the regularization parameter also increases with increasing noise, there is a need to use regularization. The proposed regularization procedure is such that for a given noise level, in the correct case, according to [14], the regularization parameter α in (27) will naturally tend to zero, i.e., instead of the regularized one, we get

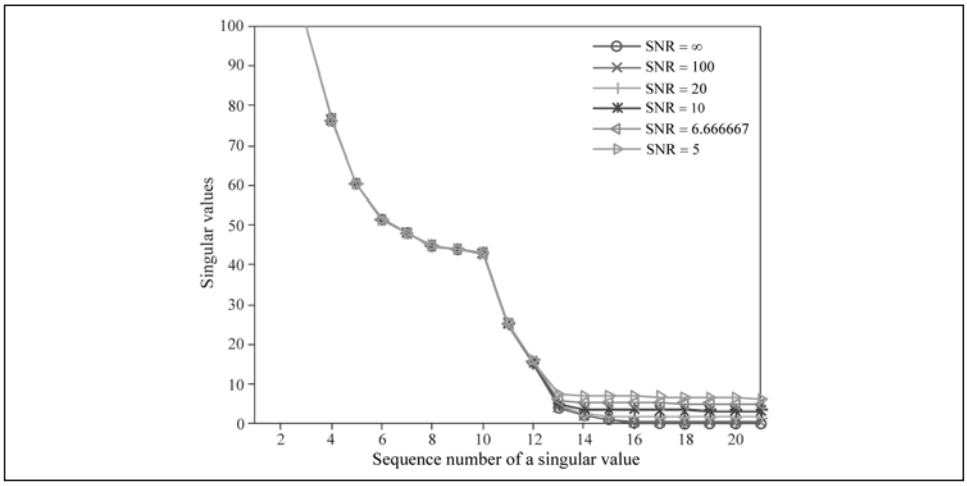

Fig. 8. Singular values for the CM

Table 1. Eigenvalues for the CM and its 12-dimensional equivalent

True eigenvalues	Estimated eigenvalues, $n = 16$	Estimated eigenvalues, $n = 12$
0.73	0.73	0.73
$0.29 + 0.55i$	$0.29 + 0.56i$	$0.28 + 0.55i$
$0.29 - 0.55i$	$0.29 - 0.56i$	$0.28 - 0.55i$
-0.6	-0.61	-0.61
0.5	0.5	0.5
$0.05 + 0.39i$	$-0.02 + 0.30i$	$-0.08 + 0.09i$
$0.05 - 0.39i$	$-0.02 - 0.30i$	$-0.08 - 0,09i$
$-0.28 + 0.21i$	$-0.04 + 0.18i$	$0.0004 + 0.18i$
$-0.28 - 0.21i$	$-0.04 - 0.18i$	$0.0004 - 0.18i$
-0.21	-0.18	-0.02
0.04	-0.4	0.008
0	$-0.10 + 0.03i$	-0.001
0	$-0.10 + 0.03i$	—
0	0	—
0	0	—
0	0	—

the usual solution. This is exactly what happened in this example. Comparison of identification results based on data on relaxation intervals and SNR = 10 is presented in Table 1. This table also gives the eigenvalues for the reduced order model equal to 12. A feature of the CM in this example was that it had a cluster of modes with eigenvalues close to zero. This led to appearing of very small singular numbers, starting from dimension 13, in Fig. 8. As a result, for dimensions 13 and more, the system of equations for finding the matrix A became ill-conditioned, but regularization was not required to find it. At the same time, the model of the reduced 12th order turned out to be quite acceptable as a solution to the identification problem. This is confirmed by Fig. 9, which shows the transient over relaxation intervals for all 10 output components of the model and the system.

EVALUATION OF THE MATRIX *B*

Let each excitation interval begin at the moment t_j, $j = \overline{1, K}$ $(t_j < t_i)$. From the Cauchy formula for the LTI system, one can obtain a matrix equation similar to (8) that is valid on the excitation interval. Its final form is written as

$$Y = \Gamma_l^{(1)} X_1 + \Phi_l^{(1)} U, \tag{31}$$

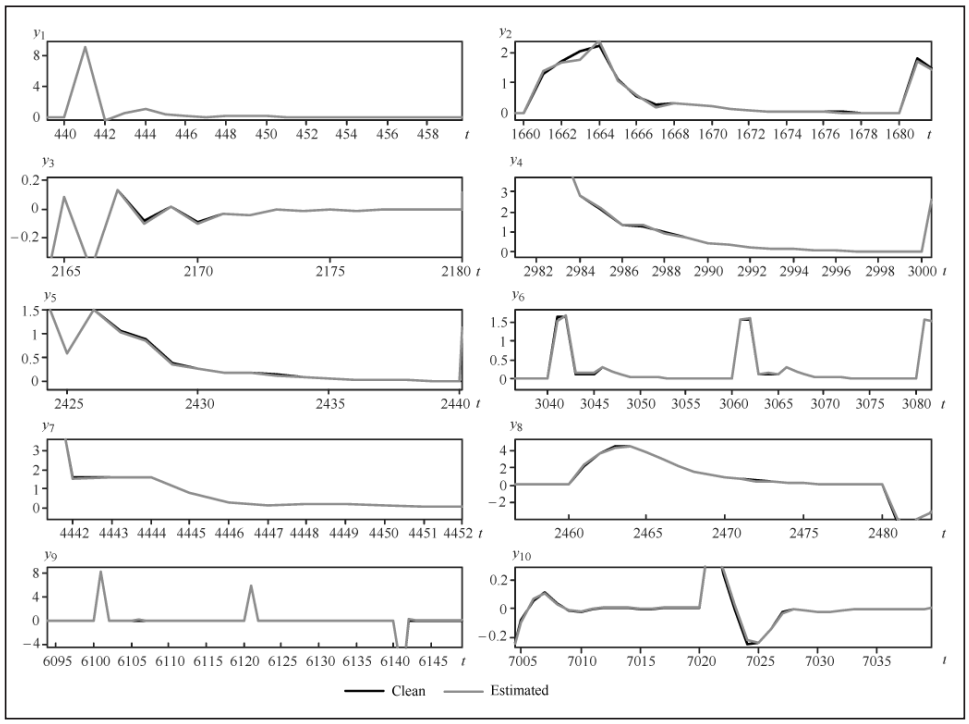

Fig. 9. Comparing outputs for real CM (without noise) and its 12-dimesional equivalent (random relaxation periods)

where $Y = [y(t_1 + 1, l)\ y(t_2 + 1, l) \dots\ y(t_K + 1, l)]$ is a matrix composed of a cascade output vector formed according to (9) on the excitation intervals;

$$\Gamma_l^{(1)} = \begin{bmatrix} A \\ \vdots \\ CA^{l-1} \end{bmatrix};\ X_1 = [x_{t_1}\ x_{t_2} \dots\ x_{t_K}];\ \Phi_l^{(1)} = \begin{bmatrix} CB & 0 & \dots & 0 \\ CAB & CB & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{l-2}B & CA^{l-3}B & \dots & CB \end{bmatrix};$$

$$U = [u(t_1, l)\ u(t_2, l) \dots\ u(t_K, l)].$$

The matrix U is formed by cascade vectors $u(t_i, l)$ formed according to the following rule:

$$u(t_i, l) = [u'(t_i)\ u'(t_{i+1}) \dots\ u'(t_i + l - 1)]'.$$

In (31) and the following notation $t_1, t_2 \dots$ correspond to the initial point of excitation t_j, in contrast to the points t_i that specify the initial point of the relaxation interval.

We write expression (31) as

$$\Phi_l^{(1)} U = Y_{\text{forced}}, \tag{32}$$

where $Y_{\text{forced}} = Y - \Gamma_l^{(1)} \cdot X_1$.

The right side of (32) contains only one unknown matrix X_1, i.e. values of the state vector at the start points t_j of the excitation intervals. If at the initial moment t_1 of excitation or at the beginning of the experiment the system was at rest ($x_1 = 0$), then the first column X_1 is zero and the first column of the matrix Y_{forced} coincides with measured $y(t_1 + 1, l)$ in the first excitation interval. If the initial state was non-zero and unknown, then in (32) it is necessary to form all the matrices, starting from the second excitation interval, discarding the data of the first interval. All other columns of the matrix X_1 are easily calculated from the known columns of the matrix $X_0 = V_s'$ for the previously established model dimension. All columns of the matrix X_1, starting

from the second one, i.e., matrices $X_1^{(1)}$, are calculated according to the rule

$$X_1^{(1)} = A^l X_0^{(K)}, \tag{33}$$

where $X_0^{(K)}$ is the matrix that is formed from X_0, with the last column deleted. As a result, the matrix Y_{forced} in the right side of (32) can be calculated.

As a result, matrix equation (32) can be used to find matrix B.

To do this, we multiply (32) on the right by the matrix U' and obtain

$$\Phi_l^{(1)} UU' = F, \tag{34}$$

where $F = Y_{\text{forced}} \cdot U'$.

When the matrix UU' is nondegenerate, we have

$$\Phi_l^{(1)} = F(UU')^{-1}. \tag{35}$$

With the considered method of the system excitation, it will be non-degenerate when $l \cdot r \le M$. When this condition is not met, it is always possible, when forming equation (31), to take a smaller value $l_1 \ge n$ instead of l, which is guaranteed to satisfy the specified condition. This can always be done at the stage when l and K are selected.

Besides one can use another algorithm for determining B. Here it should be noted that the matrix $\Phi_{l_1}^{(1)}$ is lower triangular, which is violated with a high probability in the approximate solution. Therefore, this should be taken into account when developing algorithms for finding B. Note also that the matrix $\Gamma_l^{(1)}$ can be directly extracted from the matrix Γ_l defined by relation (22) from the SVD decomposition. When calculating Y_{forced} on excitation intervals, it should be taken into account that the length of some rectangular excitation pulses is less than l. Then it is necessary to expand them using the relaxation intervals following them, on which we assume $u \equiv 0$. This is acceptable, since the length of the relaxation intervals is not less than l. Accordingly, the matrix U will have zero elements at the points of the included relaxation intervals.

Equations (35) should also be supplemented with relation (13), in which the matrices C' and B' correspond to the implementation represented by the graph (see Fig. 1). As a result, (35) and (13) form an overdetermined system of equations for finding the matrix B. In contrast to the problem of determining the matrix A for calculation B we usually have a well-posed problem. This is especially evident for the considered method of excitation of the system in active experiments. In essence, after finding the matrix B, we obtain a complete solution of the identification problem.

The described method of calculating a matrix B is not always efficient on accuracy. So can use another way for finding B. It comes down from the Jordan realization of the matrix A. For this we define the matrix A according to (26) or (28). After that we calculate the eigenvalues, which allow us to completely write down the eigenfunctions in (12). This is easily done in the absence of multiple roots. If they are present, it is necessary to modify (12) taking into account multiple roots. Then, from (12) or its modification, one can find the invariants $g_{mqr}, f_{mp}^{rc}, f_{mp}^{rs}$. To obtain them, an overdetermined linear system of equations is formed based on (12) at the excitation intervals.

Using the LSM or its generalization, including SVD, we solve the resulting system of equations and find these invariants. But before that, the outputs corresponding to free movement in each of the excitation intervals, i.e., $\Gamma_l^{(1)} \cdot X_1$ where X_1 is determined by relation (33), should be subtracted from $y_m(t)$. Then (12) will contain a forced motion with a zero initial state.

After finding the invariants for all outputs and inputs (with autonomous excitations), one can write down a system of equations from the relations connecting this invariants with the elements of the matrices C and B for the Jordan realization. Having solved it, we obtain the CM model (approximate) with its representation for the Jordan form. As computational experiments show, the eigenvalues of an approximate

model are only slightly different from the exact ones. This permits model tuning to maximize of the forecasting horizon.

TRANSITION TO GRAPH CM MODEL

CM model reconstructed by means of the realization-based approaches described above allow us to find an approximate model for either some uncertain or Jordanian implementation. However, the CM system represented as an oriented graph like Fig. 1 has a different realization. Therefore, it is desirable to reconstruct exactly the model that has an implementation represented by the CM graph. This can be done using the realization theory [12] and the features of CM modeling.

To do this, according to the theory of realizations, one should find a non-singular transformation T realizing such a transition.

Let A, B and C be the realization obtained from the solution of the identification problem, and A^g, B^g and C^g be one of the realizations (the solution may not be unique) corresponding to the graph. From the realization theory it follows

$$A^g T = TA,\ B^g = TB,\ C^g = CT^{-1}. \tag{36}$$

In the last two relations, the matrices B^g, B, C^g and C are known. Then relations (14) give $M \times R$ equations for finding A^g. If the number of zero connections on the graph is such that these equations are sufficient to calculate the non-zero components of the matrix A^g, then by solving them, we obtain a model for the implementation of the corresponding graph. Otherwise, T and the matrix A^g from all equations (36) should be found. In this case, by eliminating zero elements of the matrix A^g and using the equations of connection between the elements of the matrix T, which follow from the last two relations in (36), it is possible to reduce the dimension of the problem being solved. In the case of non-uniqueness of the remaining equations, we have a set of realizations corresponding to the CM graph. Then one of the possible realizations is found by solving underdetermined system of equations.

The performed investigations show that it is impossible to create a general method for identifying complex systems. Only systems close to well identifiable one which have large SNR for all modes allow us to solve the problem and obtain a qualitative model for it. In other cases, a surrogate model is found, or the identification becomes meaningless. In certain cases, using tuning algorithms, one can try to make transition from a surrogate model to a better description of the system. Each specific case will require its own approach, which depends on the behavior of singular values and the eigenvalues of the obtained surrogate model. Its parameters can be tuned, for example, on the basis of (12), taking into account that the invariants are close to exact values. The mathematical tools presented in the article allow to set and solve different problems of tuning and learning.

An important role in the process of model modification is played by actions aimed at increasing the SNR. In the considered method of excitation of the system and data acquisition, various manipulations with the signals received from different outputs but for the same inputs are permissible. If the noises are additive and consist of independent random realizations, then by adding or subtracting them, we obtain for some combinations an increase in the excitation signal or a decrease in noise, i.e., increase in SNR. When there are few inputs, it is possible to use data not only with the same, but also close input signals. Addition and subtraction of output signals is done point by point at relaxation intervals. As a result, we get a set of matrices Y_{relax} with different SNRs. For each such matrix, an SVD decomposition is made and charts of singular values are constructed similar to those shown in Fig. 2. Based on them we determine which singular values correspond to the signals of the system and which ones correspond to the noise. Based on the data with the highest SNR, we find a model. It also is possible to improve surrogate model by other tuning and training procedures. For example, by artificially adding noise to the measurement data at

relaxation intervals, one can clarify which of the singular numbers correspond to errors and which one correspond signals, i.e. to large SNR. Intermediate singular values between them correspond to the SNR close to 1 and for them, using tuning algorithms, one can try to extract the signal part.

In cases where tuning algorithms do not give a qualitative model, it is necessary, if possible, to change the experimental conditions. For example, one can replace the measuring and exciting system with a more efficient one, providing the closest proximity to a well-identified system.

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ОСОБЛИВОСТІ МОДЕЛЮВАННЯ ТА ІДЕНТИФІКАЦІЇ КОГНІТИВНИХ КАРТ В УМОВАХ НЕВИЗНАЧЕНОСТІ

Анотація. Розглянуто процес ідентифікації складних систем. Встановлено, що створити універсальний метод ідентифікації неможливо. Лише для добре ідентифікованої системи з високим відношенням сигнал–шум для кожної окремої моди системи можна реконструювати модель високої якості. В інших випадках, якщо існують моди з достатньо низьким відношенням сигнал–шум, можна отримати лише сурогатну модель. Для когнітивних карт розроблено теоретичні основи, які можна застосувати в підходах до пошуку сурогатної моделі, а потім для покращення результату з використанням різних алгоритмів налаштування та навчання. Для дослідження процесу ідентифікації застосовано числове моделювання.

Ключові слова: когнітивна карта, ідентифікація систем, метод виділеного підпростору, складна система, погана обумовленість, регуляризація.

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