UDC 519.6

M.I. SCHLESINGER
International Research and Training Center for Information Technologies and Systems
of the NAS of Ukraine and MES of Ukraine, Kyiv, Ukraine, e-mail: *schles@irtc.org.ua*.

MINIMAX THEOREM FOR FUNCTIONS ON THE CARTHESIAN PRODUCT OF BRANCHING POLYLINES

Abstract. The paper proves the minimax theorem for a specific class of functions that are defined on branching polylines in a linear space, not on convex subsets of a linear space. The existence of a saddle point for such functions does not follow directly from the classical minimax theorem and needs individual consideration based both on convex analysis and on graph theory. The paper presents a self-sufficient analysis of the problem. It contains everything that enables plain understanding of the main result and its proof and avoids using concepts outside the scope of obligatory mathematical education of engineers. The paper is adressed to researchers in applied mechanics, engineering and other applied sciences as well as to mathematicians who lecture convex analysis and optimization methods to non-mathematicians.

Keywords: minimax, saddle point, convex analysis, optimization, branching polyline.

The minimax theorem, also known as the saddle point theorem, states one of the fundamental concepts in economics, mechanics, electrical engineering and other applied nonmathematical sciences. Classical version of the minimax theorem has been formulated and proven by J. von Neumann in 1928 [1], generalized by M. Sion in 1958 [2] and newly proven in [3–5].

This paper proves the minimax theorem for specific functions that do not satisfy the conditions of the minimax theorem in its commonly used formulation. These functions are defined on non-convex subsets of linear spaces referred to as branching polylines. The existence of a saddle point for such functions cannot be resolved by a mere reference to the classical minimax theorem and needs individual consideration. The paper presents a self-sufficient analysis of the problem. It contains everything that enables a plain understanding of the main result and its proof and avoids using concepts outside the scope of obligatory mathematical education of engineers.

1. DEFINITIONS AND FORMULATION OF THE MAIN RESULT

Let $\mathbb{R}$, $\mathbb{N}$, $\mathbb{N}^*$ be sets of real numbers, nonnegative and positive integers, respectively. Let $\mathbb{R}^k$, $k \in \mathbb{N}^*$, be a k-dimensional linear space with Euclidean metric $\Delta^k : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$, where value $\Delta^k(x, x')$ is the distance between points $x \in \mathbb{R}^k$ and $x' \in \mathbb{R}^k$. A line segment between points $x, x' \in \mathbb{R}^k$ is denoted as

$$[x, x'] = \{\alpha \cdot x + (1-\alpha) \cdot x' \mid 0 \le \alpha \le 1\}.$$

For a given set X and a function $f : X \to \mathbb{R}$ symbols “argmax” and “argmin” are used to express the sets

$$\arg\max_{x \in X} f(x) = \{u \in X \mid f(u) = \max_{x \in X} f(x)\},$$

$$\arg\min_{x \in X} f(x) = \{u \in X \mid f(u) = \min_{x \in X} f(x)\}.$$

Let $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$ be closed bounded sets and $f : X \times Y \to \mathbb{R}$ be a continuous function of two arguments. For any such function the inequality

$$\min_{x \in X} \max_{y \in Y} f(x, y) \ge \max_{y \in Y} \min_{x \in X} f(x, y) \tag{1}$$

holds because it follows from the chain of evident inequalities

$$\min_{x\in X}\max_{y\in Y} f(x, y) = \max_{y\in Y} f(x^*, y) \ge f(x^*, y^*) \ge \min_{x\in X} f(x, y^*) = \max_{y\in Y}\min_{x\in X} f(x, y),$$

where $x^* \in \arg\min_{x\in X}[\max_{y\in Y} f(x, y)]$, $y^* \in \arg\max_{y\in Y}[\min_{x\in X} f(x, y)]$.

The classical minimax theorem states that if X and Y are convex subsets in two finite-dimensional linear spaces and f belongs to a certain class of functions then inequality (1) turns into equality. This article formulates and proves a similar theorem for a case when X and Y are not convex subsets but subsets of a specific nature, which are called branching polylines and are defined in the following way.

Let $\Gamma = \langle V, E \subset V \times V\rangle$ be an undirected graph with a finite set V of vertices and a set E of edges. Each vertex $v \in V$ of the graph is a point in a finite-dimensional linear space and each edge $(v, v') \in E$ defines a straight-line segment $[v, v']$ that connects v and v'.

Let $\Gamma = \langle V \subset \mathbb{R}^m, E \subset V \times V\rangle$ be an unoriented tree.

Definition 1. The set $X = \bigcup_{(v,v')\in E} [v, v']$ is called a branching polyline without self-intersections if $([v, v'] \cap [u, u']) = \varnothing$ for all edges (v, v'), (u, u') such that $\{v, v'\} \cap \{u, u'\} = \varnothing$.

Hereinafter a short identifier “polyline” is used instead of “branching polyline without self-intersections”.

Definition 2. A sequence $v_0, v_1, \ldots, v_i, \ldots, v_l$ of vertices $v_i \in V$ is called a path in Γ between v_0 and v_l if $(v_{i-1}, v_i) \in E$ for all $0 < i \le l$.

Let X be a polyline defined by unoriented tree $\Gamma = \langle V \subset \mathbb{R}^m, E \subset V \times V\rangle$, and let $x, x' \in X$.

Definition 3. A path between x and x' in polyline X is

— either a straight-line segment $[x, x']$ if an edge $(v, v') \in E$ exists such that $x, x' \in [v, v']$; its length equals $\Delta^m(x, x')$;

— or a set $[x, v_1] \cup \bigcup_{i=2}^{l-1}[v_{i-1}, v_i] \cup [v_{l-1}, x']$ if a path $v_0 \ldots v_i \ldots v_l$ in Γ, $l > 1$, exists such that $x \in [v_0, v_1]$ and $x' \in [v_{l-1}, v_l]$; its length equals

$$\Delta^m(x, v_1) + \sum_{i=2}^{l-1} \Delta^m(v_{i-1}, v_i) + \Delta^m(v_{l-1}, x').$$

Figure 1 illustrates the difference between paths in Γ and X. A path in Γ is a finite sequence of vertices whereas a path in X is a set of continuum cardinality.

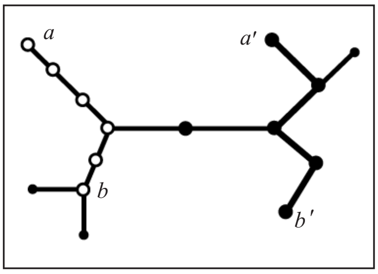

Fig. 1. Path between a and b in a graph (white circles); path between a' and b' in a polyline (bold lines)

Let us denote by $(x_1 \leftrightarrow x_2)_X$ a path in a polyline X between points $x_1, x_2 \in X$ and use simply $(x_1 \leftrightarrow x_2)$ if it is clear which polyline this path belongs to. Let us denote by $d_X(x_1, x_2)$ the length of a path $(x_1 \leftrightarrow x_2)_X$. It is easy to see that the function $d_X : X \times X \to \mathbb{R}$ forms a metric on X.

Let us consider the set $\{x \in X \mid a \in (x \leftrightarrow b)\}$ for some polyline X and points $a, b \in X$, $a \ne b$, and introduce short expressions $(\succ a \leftrightarrow b)$ or $(b \leftrightarrow a \prec)$ for such sets,

$$(\succ a \leftrightarrow b) = (b \leftrightarrow a \prec) = \{x \in X \mid a \in (x \leftrightarrow b)\}.$$

Unformally, expressions $(\succ a \leftrightarrow b)$ and $(b \leftrightarrow a \prec)$ represent the subset of points that the point a separates from the point b, expressions $(\succ b \leftrightarrow a)$ and $(a \leftrightarrow b \prec)$ represent the subset of points that b separates from a. Figure 2 illustrates the essence of such subsets.

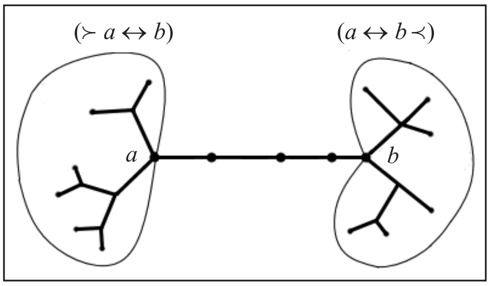

Fig. 2. Examples of sets $(\succ a \leftrightarrow b)$ and $(a \leftrightarrow b \prec)$

Definition 4. A function $\varphi: X \to \mathbb{R}$ is called uniformly continuous on a set X if for any $\varepsilon > 0$ a number $h(\varepsilon) > 0$ exists such that for any x_1, $x_2 \in X$ the inequality $d_X(x_1, x_2) \le h(\varepsilon)$ implies the inequality $|\varphi(x_1) - \varphi(x_2)| \le \varepsilon$.

Let X and Y be polylines with metrices $d_X : X \times X \to \mathbb{R}$ and $d_Y : Y \times Y \to \mathbb{R}$.

Definition 5. A function $f : X \times Y \to \mathbb{R}$ of two arguments is called uniformly continuous on a set $X \times Y$ if for any $\varepsilon > 0$ a number $h(\varepsilon) > 0$ exists such that for any x_1, $x_2 \in X$ and y_1, $y_2 \in Y$ the inequalities $d_X(x_1, x_2) \le h(\varepsilon)$ and $d_Y(y_1, y_2) \le h(\varepsilon)$ imply the inequality $|f(x_1, y_1) - f(x_2, y_2)| \le \varepsilon$.

Definition 6. A function $\varphi: X \to \mathbb{R}$ is called quasiconvex if for any x_1, x^*, x_2 such that $x^* \in (x_1 \leftrightarrow x_2)_X$ the inequality $\varphi(x^*) \le \max\{\varphi(x_1), \varphi(x_2)\}$ holds; a function $\varphi: X \to \mathbb{R}$ is called quasiconcave if for any x_1, x^*, x_2 such that $x^* \in (x_1 \leftrightarrow x_2)_X$ the inequality $\varphi(x^*) \ge \min\{\varphi(x_1), \varphi(x_2)\}$ holds.

Figure 3 shows examples of quasiconvex and quasiconcave functions for a case when X consists of one straight-line segment. The examples illustrate the difference between quasiconvex or quasiconcave functions and functions that are convex or concave in the accepted meaning of these words.

Hereinafter a prefix “quasi” in words “quasiconvex”, “-concave” is omitted.

Definition 7. A function $f : X \times Y \to \mathbb{R}$ is called convex on X if the function $\varphi_b : X \to \mathbb{R}$, $\varphi_b : x \mapsto f(x, b)$, is convex for any $b \in Y$; a function $f : X \times Y \to \mathbb{R}$ is called concave on Y if the function $\psi_a : Y \to \mathbb{R}$, $\psi_a : y \mapsto f(a, y)$, is concave for any $a \in X$.

The paper proves the following theorem.

Theorem 1. Let X and Y be two polylines. If $f : X \times Y \to \mathbb{R}$ is uniformly continuous on $X \times Y$, convex on X and concave on Y then

$$\min_{x \in X} \max_{y \in Y} f(x, y) = \max_{y \in Y} \min_{x \in X} f(x, y).$$

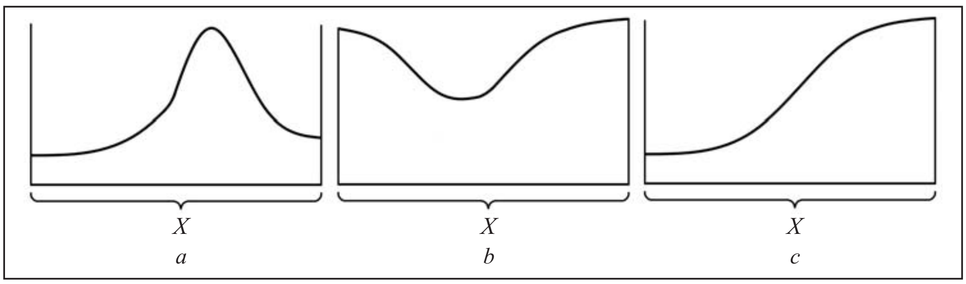

Fig. 3. Quasiconvex and quasiconcave functions: quasiconcave, non-concave function (*a*); quasiconvex, non-convex function (*b*); quasiconvex, quasiconcave, non-convex, non-concave function (*c*)

2. OUTLINE OF THE PROOF

Theorem 1 generalizes Lemma 7 in [6], the last being a special case of Theorem 1 when both polylines X and Y consist of a single straight-line segment. Naturally, there are several similar fragments in proofs of these two statements. However the generalization of the quoted lemma onto case of Theorem 1 is not quite straightforward because it needs interlaced consideration based both on convex analysis and on graph theory. The next Subsections 2.1, 2.2 and 2.3 annotate key points of the proof and enable to embrace the whole proof at one glance. A complete proof of the theorem consists of three parts described in Secs. 4, 5 and 6.

Everywhere in the following text, the conjunction "or" is used in a non-exclusive sense. So, for some conditions P, Q a proposition "P or Q" means that at least one of these conditions is satisfied and it is not excluded that both P and Q are satisfied. An expression $P \Rightarrow Q$ is used in a sense "if P then Q".

2.1. Lemmas about two points. Let X be a polyline, Y be a closed bounded set, $f : X \times Y \to \mathbb{R}$ be a continuous function, convex on X and not necessarily concave on Y. It is proved in Sec. 4 that in this case a point $y^* \in Y$ exists such that

$$\min_{x \in X} \max_{y \in Y} f(x, y) = \min_{x \in X} f(x, y^*) \tag{2}$$

or two points $y_1^*, y_2^* \in Y$, $y_1^* \neq y_2^*$, exist such that

$$\min_{x \in X} \max_{y \in Y} f(x, y) = \min_{x \in X} \max \{f(x, y_1^*), f(x, y_2^*)\} . \tag{3}$$

Evidently, if (2) fulfils then $\min\limits_{x \in X} \max\limits_{y \in Y} f(x, y) = \max\limits_{y \in Y} \min\limits_{x \in X} f(x, y)$. Section 4 proves that if (3) fulfils then points $x_1, x_2 \in X$ and $x^* \in (x_1 \leftrightarrow x_2)$ exist such that

$$f(u, y_1^*) \geq \min_{x \in X} \max_{y \in Y} f(x, y) \text{ for all } u \in (x_1 \leftrightarrow x^* \prec), \tag{4}$$

$$f(u, y_2^*) \geq \min_{x \in X} \max_{y \in Y} f(x, y) \text{ for all } u \in (\succ x^* \leftrightarrow x_2), \tag{5}$$

Similarly, if X and Y are polylines and $f : X \times Y \to \mathbb{R}$ is a continuos function concave on Y then $\max\limits_{y \in Y} \min\limits_{x \in X} f(x, y) = \min\limits_{x \in X} \max\limits_{y \in Y} f(x, y)$ or points $x_1^*, x_2^* \in X$, $x_1^* \neq x_2^*$, and points $y_1, y_2 \in Y$, $y^* \in (y_1 \leftrightarrow y_2)$ exist such that

$$\max_{y \in Y} \min_{x \in X} f(x, y) = \max_{y \in Y} \min \{f(x_1^*, y), f(x_2^*, y)\}, \tag{6}$$

$$f(x_1^*, v) \leq \max_{y \in Y} \min_{x \in X} f(x, y) \text{ for all } v \in (y_1 \leftrightarrow y^* \prec), \tag{7}$$

$$f(x_2^*, v) \leq \max_{y \in Y} \min_{x \in X} f(x, y) \text{ for all } v \in (\succ y^* \leftrightarrow y_2). \tag{8}$$

2.2. Polylines in the Carthesian product of polylines. Let X and Y be polylines, $x_1, x_2, x^* \in (x_1 \leftrightarrow x_2)$ be some three points in X, y_1^*, y_2^* be some two points in Y. The triple (x_1, x^*, x_2) does not necessarily satisfy conditions (4) and (5), the pair (y_1^*, y_2^*) does not necessarily satisfy condition (3). Section 5 defines how a certain polyline $XYX \subset X \times Y$ has to be composed of subsets $(x_1 \leftrightarrow x^* \prec)$, $(\succ x^* \leftrightarrow x_2)$ and a path $(y_1^* \leftrightarrow y_2^*)$.

Similarly, let $y_1, y_2 \in Y$, $y^* \in (y_1 \leftrightarrow y_2)$ and $x_1^*, x_2^* \in X$. It is defined in Sec. 5 how a certain polyline $YXY \subset X \times Y$ has to be composed of subsets $(y_1 \leftrightarrow y^* \prec)$, $(\succ y^* \leftrightarrow y_2)$ and a path $(x_1^* \leftrightarrow x_2^*) \subset X$.

It is proved in Sec. 5 that the mentioned polylines XYX and YXY have a non-empty intersection.

2.3. Proof completion. Let a function $f : X \times Y \to \mathbb{R}$ be convex on X and concave on Y. As it is mentioned in Subsec. 2.1, in this case the equality

$$\min_{x \in X} \max_{y \in Y} f(x, y) = \max_{y \in Y} \min_{x \in X} f(x, y)$$

holds or there exist

points $y_1^*, y_2^* \in Y$ that satisfy (3),

points $x_1, x_2 \in X$, $x^* \in (x_1 \leftrightarrow x_2)$ that satisfy (4) and (5),

points $x_1^*, x_2^* \in X$ that satisfy (6),

points $y_1, y_2 \in Y$, $y^* \in (y_1 \leftrightarrow y_2)$ that satisfy (7) and (8).

It is proved in Sec. 6 that conditions (3)–(5) imply inequalities

$$f(u, v) \geq \min_{x \in X} \max_{y \in Y} f(x, y) \text{ for all } (u, v) \in XYX, \tag{9}$$

and conditions (6)–(8) imply that

$$f(u, v) \leq \max_{y \in Y} \min_{x \in X} f(x, y) \text{ for all } (u, v) \in YXY. \tag{10}$$

As it has been mentioned in Subsec. 2.2, $XYX \cap YXY \neq \varnothing$. Due to (9) and (10) any point $(x', y') \in XYX \cap YXY$ satisfies inequalities

$$\min_{x \in X} \max_{y \in Y} f(x, y) \leq f(x', y') \leq \max_{y \in Y} \min_{x \in X} f(x, y)$$

and consequently, due to (1), $\min_{x \in X} \max_{y \in Y} f(x, y) = \max_{y \in Y} \min_{x \in X} f(x, y)$.

3. PRELIMINARY STAGE OF THE PROOF

This section formulates formal properties of polylines and functions defined on polylines that are used in next stages of the proof. With the exception of Lemma 1, all these properties follow rather straightforwardly from definitions and are formulated without proofs. As for Lemma 1, it formulates a special case of a known and thoroughly researched property of set-valued functions (see, for example, a Theorem 1.4.16 in [7]). The Lemma 1 is formulated and proved below in a form that is appropriate to the context of this paper.

Property 1. Let X be a polyline, $x_1, x_2 \in X$, $x_1 \neq x_2$, $x^* \in (x_1 \leftrightarrow x_2)$. In this case $X = (x_1 \leftrightarrow x^* \prec) \cup (\succ x^* \leftrightarrow x_2)$.

Figure 4 shows two examples of different coverings of X with $X_1 = (x_1 \leftrightarrow x^* \prec)$ and $X_2 = (\succ x^* \leftrightarrow x_2)$. If x^* is not a branching point in X then the subsets X_1 and X_2 intersect in a single point x^*, as depicted by Fig. 4, *a*. Another case is shown on Fig. 4, *b* when x^* is a branching point in X and the intersection $X_1 \cap X_2$ contains not only x^*.

Property 2. Let $x_1, x^*, x_2 \in X$ be three points such that $x^* \in (x_1 \leftrightarrow x_2)$, let $X_1 = (x_1 \leftrightarrow x^* \prec)$, $X_2 = (\succ x^* \leftrightarrow x_2)$ and let $x_1^*, x_2^* \in X$ be some points.

If $x^* \notin (x_1^* \leftrightarrow x_2^*)$ then $\{x_1^*, x_2^*\} \subset X_1$ or $\{x_1^*, x_2^*\} \subset X_2$.

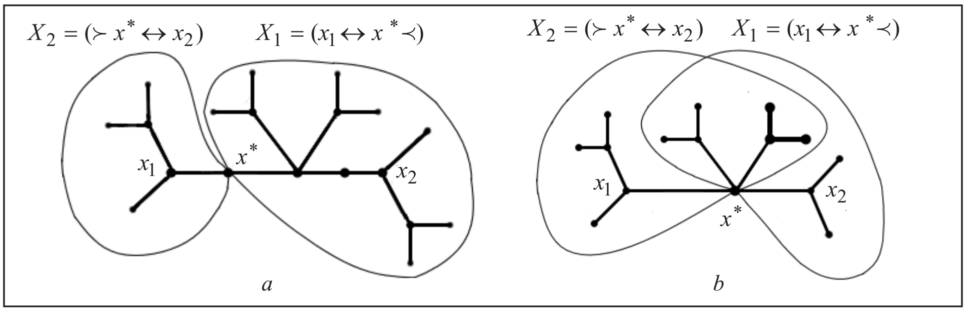

Fig. 4. Two forms of covering X by X_1 and X_2: first case $(X_1 \cap X_2) = \{x^*\}$ (*a*); second case $(X_1 \cap X_2) \neq \{x^*\}$ (*b*)

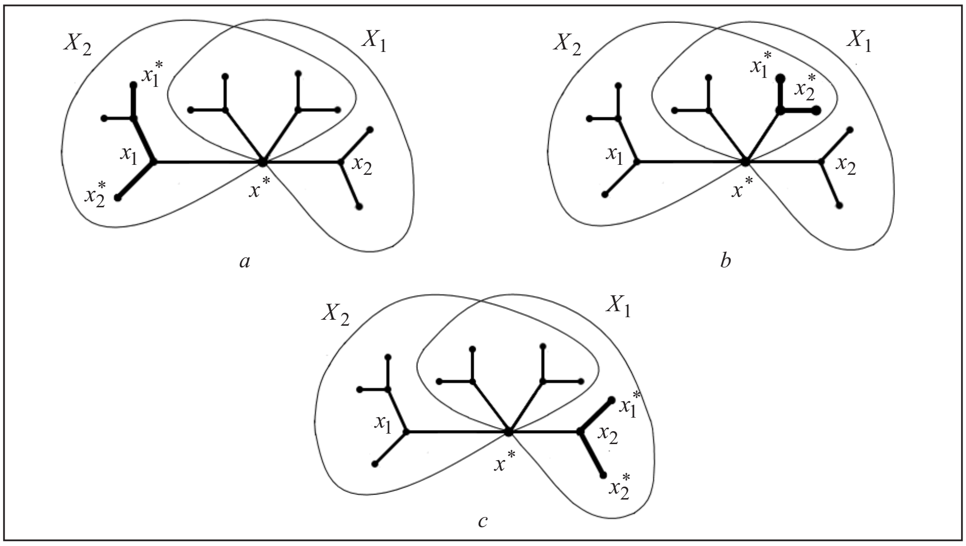

Fig. 5. Illustration of Property 2: $x^* \notin (x_1^* \leftrightarrow x_2^*)$, $\{x_1^*, x_2^*\} \subset X_2$ (*a*); $x^* \notin (x_1^* \leftrightarrow x_2^*)$, $\{x_1^*, x_2^*\} \subset \subset X_1 \cap X_2$ (*b*); $x^* \notin (x_1^* \leftrightarrow x_2^*)$, $\{x_1^*, x_2^*\} \subset X_1$ (*c*)

Figure 5 illustrates Property 2 for different locations of points x_1^* and x_2^*.

Property 3. Let X and Y be polylines. If a function $f : X \times Y \to \mathbb{R}$ is convex on X then the function $\varphi : X \to \mathbb{R}$, $x \mapsto \max_{y \in Y} f(x, y)$, is convex as well.

Property 4. If $\varphi : X \to \mathbb{R}$ is a convex function and $a, b \in X$ are points such that $\varphi(b) > \varphi(a)$ then $\varphi(x) \geq \varphi(b)$ for any $x \in (\succ b \leftrightarrow a)$.

Lemma 1. If a function $f : X \times Y \to \mathbb{R}$ is uniformly continuous on $X \times Y$ then the function $\varphi : X \to \mathbb{R}$, $x \mapsto \max_{y \in Y} f(x, y)$, is uniformly continuous on X.

Proof. Since $f : X \times Y \to \mathbb{R}$ is uniformly continuous on $X \times Y$, there exists a function $h : \mathbb{R} \to \mathbb{R}$ such that the implication

$$[\max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} \leq h(\varepsilon)] \Rightarrow \tag{11}$$

$$\Rightarrow [-\varepsilon \leq f(x_1, y_1) - f(x_2, y_2) \leq \varepsilon] \tag{12}$$

holds for all $\varepsilon > 0$, all x_1, $x_2 \in X$, and all y_1, $y_2 \in Y$ including all cases when $y_1 = y_2$. For these cases, condition (11) takes the form $d_X(x_1, x_2) \leq h(\varepsilon)$ because $d_Y(y, y) = 0$ for any $y \in Y$. So, for these cases implication (11) and (12) takes the form

$$[d_X(x_1, x_2) \leq h(\varepsilon)] \Rightarrow [-\varepsilon \leq f(x_1, y) - f(x_2, y) \leq \varepsilon] \tag{13}$$

that is valid for all $x_1, x_2 \in X$ and all $y \in Y$, and consequently, is valid for values $\upsilon_1 \in \arg\max_{y\in Y} f(x_1, y) \subset Y$ and $\upsilon_2 \in \arg\max_{y\in Y} f(x_2, y) \subset Y$ as well. For the value $\upsilon_1 \in \arg\max_{y\in Y} f(x_1, y)$ it follows from (13) that

$$[d_X(x_1, x_2) \le h(\varepsilon)] \Rightarrow [f(x_1, \upsilon_1) - f(x_2, \upsilon_1) \le \varepsilon] \Leftrightarrow$$

$$\Leftrightarrow [\max_{y\in Y} f(x_1, y) - f(x_2, \upsilon_1) \le \varepsilon] \Rightarrow [\max_{y\in Y} f(x_1, y) - \max_{y\in Y} f(x_2, y) \le \varepsilon]. \quad (14)$$

Similarly, for the value $\upsilon_2 \in \arg\max_{y\in Y} f(x_2, y)$ it follows from (13) that

$$[d_X(x_1, x_2) \le h(\varepsilon)] \Rightarrow [-\varepsilon \le f(x_1, \upsilon_2) - f(x_2, \upsilon_2)] \Leftrightarrow$$

$$\Leftrightarrow [-\varepsilon \le f(x_1, \upsilon_2)] - \max_{y\in Y}(x_2, y) \Rightarrow [-\varepsilon \le \max_{y\in Y} f(x_1, y) - \max_{y\in Y}(x_2, y)]. \quad (15)$$

It follows from (14) and (15) that

$$[d_X(x_1, x_2) \le h(\varepsilon)] \Rightarrow [-\varepsilon \le \max_{y\in Y} f(x_1, y) - \max_{y\in Y} f(x_2, y) \le \varepsilon]. \qquad \blacksquare$$

4. LEMMAS ABOUT TWO POINTS

Lemma 2. Let X be a polyline and $\varphi_1 : X \to \mathbb{R}$, $\varphi_2 : X \to \mathbb{R}$ be convex continuous functions such that

$$\min_{x\in X} \varphi_1(x) < \min_{x\in X} \max\{\varphi_1(x), \varphi_2(x)\}, \quad (16)$$

$$\min_{x\in X} \varphi_2(x) < \min_{x\in X} \max\{\varphi_1(x), \varphi_2(x)\}. \quad (17)$$

In this case points x_1, x_2 and x^* exist such that

$$x_1 \in \arg\min_{x\in X} \varphi_1(x),\ x_2 \in \arg\min_{\in X} \varphi_2(x),\ x_1 \ne x_2,\ x^* \in (x_1 \leftrightarrow x_2), \quad (18)$$

$$\varphi_1(x^*) = \varphi_2(x^*) = \min_{x\in X} \max\{\varphi_1(x), \varphi_2(x)\}. \quad (19)$$

Proof. Since the functions φ_1, φ_2 are continuous, the sets $\arg\min_{x\in X} \varphi_1(x)$ and $\arg\min_{x\in X} \varphi_2(x)$ are not empty. Let $x_1 \in \arg\min_{x\in X} \varphi_1(x)$ and $x_2 \in \arg\min_{x\in X} \varphi_2(x)$ be some points. Due to condition (16) the chain

$$\varphi_1(x_1) = \min_{x\in X} \varphi_1(x) < \min_{x\in X} \max\{\varphi_1(x), \varphi_2(x)\} \le \max\{\varphi_1(x_1), \varphi_2(x_1)\} \quad (20)$$

is valid that results in inequality $\varphi_1(x_1) < \max\{\varphi_1(x_1), \varphi_2(x_1)\}$ and so in inequality $\varphi_1(x_1) < \varphi_2(x_1)$. Similarly, condition (17) implies the chain

$$\varphi_2(x_2) = \min_{x\in X} \varphi_2(x) < \min_{x\in X} \max\{\varphi_1(x), \varphi_2(x)\} \le \max\{\varphi_1(x_2), \varphi_2(x_2)\} \quad (21)$$

and, consequently, implies the inequality $\varphi_2(x_2) < \varphi_1(x_2)$ and so

$$\varphi_1(x_1) - \varphi_2(x_1) < 0 < \varphi_1(x_2) - \varphi_2(x_2). \quad (22)$$

It follows from (22) that $x_1 \ne x_2$. Due to continuity of φ_1 and φ_2 it follows also from (22) that a point $x^* \in (x_1 \leftrightarrow x_2)$ exists such that $\varphi_1(x^*) = \varphi_2(x^*)$.

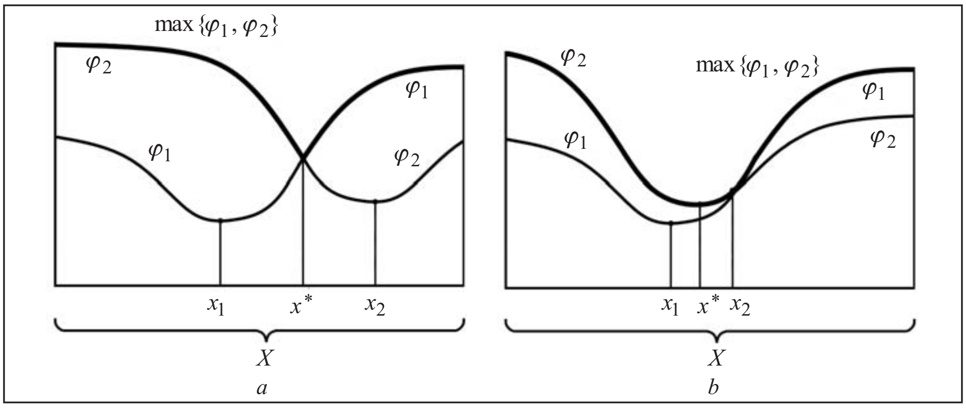

Fig. 6. Illustration of Lemma 2: conditions (16), (17) are satisfied, conditions (18), (19) are consistent (*a*); conditions (16), (17) are violated, conditions (18), (19) are inconsistent (*b*)

Condition (16) and several evident equalities and inequalities form the chain

$$\varphi_1(x^*) = \varphi_2(x^*) = \max\{\varphi_1(x^*), \varphi_2(x^*)\} \geq$$

$$\geq \min_{x \in X} \max\{\varphi_1(x), \varphi_2(x)\} > \min_{x \in X} \varphi_1(x) = \varphi_1(x_1)$$

that results in inequality $\varphi_1(x^*) > \varphi_1(x_1)$. Due to Property 4 inequality $\varphi_1(x^*) > \varphi_1(x_1)$ implies inequalities

$$\varphi_1(x) \geq \varphi_1(x^*) = \varphi_2(x^*) \text{ for all } x \in (x_1 \leftrightarrow x^* \prec)$$

and, consequently,

$$\max\{\varphi_1(x), \varphi_2(x)\} \geq \varphi_1(x^*) = \varphi_2(x^*) \text{ for all } x \in (x_1 \leftrightarrow x^* \prec). \tag{23}$$

In a similar way condition (17) validates the chain

$$\varphi_2(x^*) = \max\{\varphi_1(x^*), \varphi_2(x^*)\} \geq \min_{x \in X} \max\{\varphi_1(x), \varphi_2(x)\} > \min_{x \in X} \varphi_2(x) = \varphi_2(x_2)$$

that results in inequality $\varphi_2(x^*) > \varphi_2(x_2)$. Due to Property 4 the last inequality implies that $\varphi_2(x) \geq \varphi_1(x^*) = \varphi_2(x^*)$ for all $x \in (\succ x^* \leftrightarrow x_2)$ and, consequently,

$$\max\{\varphi_1(x), \varphi_2(x)\} \geq \varphi_1(x^*) = \varphi_2(x^*) \text{ for all } x \in (\succ x^* \leftrightarrow x_2). \tag{24}$$

Expressions (23) and (24) state that the certain condition holds on two subsets of X whose union is X (see Property 1). Thus the expressions may be represented in the unified form $\max\{\varphi_1(x), \varphi_2(x)\} \geq \varphi_1(x^*) = \varphi_2(x^*)$ for all $x \in X$ or, equivalently,

$$\varphi_1(x^*) = \varphi_2(x^*) = \min_{x \in X} \max\{\varphi_1(x), \varphi_2(x)\}. \qquad \blacksquare$$

See Fig. 6 that illustrates the lemma for a special case when X consists of a single line segment.

Let $P: Y \mapsto \{Y', Y''\}$ be a function that for each polyline Y defines two connected subsets Y' and Y'' such that $Y' \cup Y'' = Y$.

Definition 8. A function P is called a contracting one if

$$\lim_{i \to \infty} \max_{y, y' \in Y_i} d_Y(y, y') = 0 \tag{25}$$

for any sequence $(Y_i \mid i \in \mathbb{N})$ such that $Y_i \in P(Y_{i-1})$ for all $i \in \mathbb{N}^*$.

Let us choose some contracting function P and fix it for subsequent consideration.

Definition 9. A sequence $(Y_i \mid 0 \le i \le m)$ is called a finite nesting for a function $f : X \times Y \to \mathbb{N}$ if conditions

$$Y_0 = Y, \; Y_i \in P(Y_{i-1}), \quad \min_{x \in X} \max_{y \in Y_i} f(x, y) = \min_{x \in X} \max_{y \in Y} f(x, y) \tag{26}$$

are satisfied for all i, $0 < i \le m$; an infinite sequence $(Y_i \mid i \in \mathbb{N})$ is called an infinite nesting for f if conditions (26) are satisfied for all $i \in \mathbb{N}^*$.

Lemma 3. Let $f : X \times Y \to \mathbb{R}$ be a continuous function convex on X and let a finite nesting $(Y_i \mid 0 \le i \le m)$ for the function f exist such that

$$\min_{x \in X} \max_{y \in Y'} f(x, y) < \min_{x \in X} \max_{y \in Y} f(x, y), \; \min_{x \in X} \max_{y \in Y''} f(x, y) < \min_{x \in X} \max_{y \in Y} f(x, y), \tag{27}$$

where $\{Y', Y''\} = P(Y_m)$.

In this case points $x_1, x_2 \in X$, $x_1 \ne x_2$, $x^* \in (x_1 \leftrightarrow x_2)$, $y_1^* \in Y'$, $y_2^* \in Y''$, exist such that

$$f(x^*, y_1^*) = f(x^*, y_2^*) = \min_{x \in X} \max_{y \in Y} f(x, y), \tag{28}$$

$$f(u, y_1^*) \ge \min_{x \in X} \max_{y \in Y} f(x, y) \text{ for all } u \in (x_1 \leftrightarrow x^* \prec), \tag{29}$$

$$f(u, y_2^*) \ge \min_{x \in X} \max_{y \in Y} f(x, y) \text{ for all } u \in (x_1 \leftrightarrow x^* \prec). \tag{30}$$

Proof. The equalities

$$\min_{x \in X} \max \{\max_{y \in Y'} f(x, y), \max_{y \in Y''} f(x, y)\} = \min_{x \in X} \max_{y \in Y_m} f(x, y) = \min_{x \in X} \max_{y \in Y} f(x, y) \tag{31}$$

are valid. The first one is valid because $Y_m = Y' \cup Y''$ and the second one is valid due to Definition 9.

Functions $\varphi_1 : X \to \mathbb{R}$, $x \mapsto \max_{y \in Y'} f(x, y)$, and $\varphi_2 : X \to \mathbb{R}$, $x \mapsto \max_{y \in Y''} f(x, y)$, satisfy conditions of Lemma 2. They are continuous due to Lemma 1, convex due to Property 1 and satisfy conditions (16) and (17) because due to (31) conditions (27) may be represented as

$$\min_{x \in X} \varphi_1(x) < \min_{x \in X} \max \{\varphi_1(x), \varphi_2(x)\}, \; \min_{x \in X} \varphi_2(x) < \min_{x \in X} \max \{\varphi_1(x), \varphi_2(x)\}.$$

It follows from Lemma 2 that there exist points

$$x_1 \in \arg \min_{x \in X} \max_{y \in Y'} f(x, y), \; x_2 \in \arg \min_{x \in X} \max_{y \in Y''} f(x, y), \; x_1 \ne x_2, \; x^* \in (x_1 \leftrightarrow x_2),$$

that satisfy equalities

$$\max_{y \in Y'} f(x^*, y) = \max_{y \in Y''} f(x^*, y) = \min_{x \in X} \max \{\max_{y \in Y'} f(x, y), \max_{y \in Y''} f(x, y)\},$$

and due to (31) equalities

$$\max_{y \in Y'} f(x^*, y) = \max_{y \in Y''} f(x^*, y) = \min_{x \in X} \max_{y \in Y} f(x, y)$$

as well. Consequently, points $y_1^* \in Y'$ and $y_2^* \in Y''$ exist such that

$$f(x^*, y_1^*) = f(x^*, y_2^*) = \min_{x\in X} \max_{y\in Y} f(x, y). \quad (32)$$

Let us prove that points $x_1, x_2, x^*, y_1^*, y_2^*$ satisfy inequalities (29) and (30). The chain of equalities and inequalities

$$f(x_1, y_1^*) \le \max_{y\in Y'} f(x_1, y) = \min_{x\in X} \max_{y\in Y'} f(x, y) < \min_{x\in X} \max_{y\in Y} f(x, y) = f(x^*, y_1^*)$$

is valid. The inequality at the first link of the chain is evident, the equality at the second link is valid by definition $x_1 \in \arg\min_{x\in X} \max_{y\in Y'} f(x, y)$, the strict inequality at the third link is valid by condition (27), the equality at the last link is valid due to (32). The chain results in inequality $f(x_1, y_1^*) < f(x^*, y_1^*)$ that due to Property 4 implies inequalities (29). Similarly, the chain

$$f(x_2, y_2^*) \le \max_{y\in Y''} f(x_2, y) = \min_{x\in X} \max_{y\in Y''} f(x, y) < \min_{x\in X} \max_{y\in Y} f(x, y) = f(x^*, y_2^*)$$

results in $f(x_2, y_2^*) < f(x^*, y_2^*)$ that due to Property 4 implies (30). ■

The next lemma relates to functions $f: X \times Y \to \mathbb{R}$, for which an infinite nesting $(Y_i \mid i \in \mathbb{N})$ exists. Let us recall that for such nestings

$$\min_{x\in X} \max_{y\in Y_i} f(x, y) = \min_{x\in X} \max_{y\in Y} f(x, y) \text{ for all } i \in \mathbb{N}, \quad (33)$$

$$\lim_{i\to\infty} \max_{y, y'\in Y_i} d_Y(y, y') = 0. \quad (34)$$

Lemma 4. If for a continuous function $f: X \times Y \to \mathbb{R}$ an infinite nesting $(Y_i \mid i \in \mathbb{N})$ exists then $\min_{x\in X} \max_{y\in Y} f(x, y) = \max_{y\in Y} \min_{x\in X} f(x, y)$.

Proof. Let for all $x \in X$ a sequence $(\max_{y\in Y_i} f(x, y) \mid i \in \mathbb{N})$ be defined. Any such sequence has a lower bound $\min_{u\in X} \max_{y\in Y} f(u, y)$ because

$$\max_{y\in Y_i} f(x, y) \ge \min_{u\in X} \max_{y\in Y_i} f(u, y) = \min_{u\in X} \max_{y\in Y} f(u, y) \text{ for all } i \in \mathbb{N}.$$

The inequality in this expression is evident, the equality is valid due to condition (33). Any such sequence does not increase because $Y_i \subset Y_{i-1}$ and, consequently, it has a limit such that

$$\lim_{i\to\infty} \max_{y\in Y_i} f(x, y) \ge \min_{x\in X} \max_{y\in Y} f(x, y) \text{ for all } x \in X. \quad (35)$$

Due to (34) all sequences $(y_i \in Y_i \mid i \in \mathbb{N})$ have the same limit $y^* = \lim_{i\to\infty} y_i$ and it holds for the sequence $(y_i \in \arg\max_{y\in Y_i} f(x, y) \mid i \in \mathbb{N})$ as well. For all sequences of the form $(y_i \in \arg\max_{y\in Y_i} f(x, y) \mid i \in \mathbb{N})$ and all $x \in X$ the chain

$$\min_{u\in X} \max_{y\in Y} f(u, y) \le \lim_{i\to\infty} \max_{y\in Y_i} f(x, y) = \lim_{i\to\infty} f(x, y_i) = f(x, \lim_{i\to\infty} y_i) = f(x, y^*)$$

is valid. The inequality at the first link of the chain is valid due to (35), the equality at the second link is valid because $y_i \in \arg\max_{y \in Y_i} f(x, y)$, the third link is valid because f is a continuous function. It follows from the chain that

$$\min_{u \in X} \max_{y \in Y} f(u, y) \le f(x, y^*) \text{ for all } x \in X$$

x_2

and, consequently, for all $x \in \arg\min_{x \in X} f(x, y^*) \subset X$ as well. So, for a value $x' \in \arg\min_{x \in X} f(x, y^*)$ the chain

$$\min_{u \in X} \max_{y \in Y} f(u, y) \le f(x', y^*) = \min_{x \in X} f(x, y^*) \le \max_{y \in Y} \min_{x \in X} f(x, y)$$

is valid that implies $\min_{x \in X} \max_{y \in Y} f(x, y) \le \max_{y \in Y} \min_{x \in X} f(x, y)$ and due to (1)

$$\min_{x \in X} \max_{y \in Y} f(x, y) = \max_{y \in Y} \min_{x \in X} f(x, y). \qquad \blacksquare$$

Since any function $f : X \times Y \to \mathbb{R}$ has either a finite nesting that satisfies condition (27) or an infinite nesting, Lemmas 3 and 4 may be formulated in the following unified form.

Lemma 5. Let function $f : X \times Y$ be convex on X and continuous on $X \times Y$. In this case the equality $\min_{x \in X} \max_{y \in Y} f(x, y) = \max_{y \in Y} \min_{x \in X} f(x, y)$ is valid or points

$$y_1^*, y_2^* \in Y, \; x_1, x_2 \in X, \; x_1 \ne x_2, \; x^* \in (x_1 \leftrightarrow x_2),$$

exist such that

$$f(x^*, y_1^*) = f(x^*, y_2^*) = \min_{x \in X} \max_{y \in Y} f(x, y),$$

$$f(u, y_1^*) \ge \min_{x \in X} \max_{y \in Y} f(x, y) \text{ for all } u \in (x_1 \leftrightarrow x^* \prec),$$

$$f(u, y_2^*) \ge \min_{x \in X} \max_{y \in Y} f(x, y) \text{ for all } u \in (\succ x^* \leftrightarrow x_2).$$

Evidently, the following dual version of Lemma 5 is correct as well.

Lemma 6. Let function $f : X \times Y$ be concave on Y and continuous on $X \times Y$. In this case the equality $\min_{x \in X} \max_{y \in Y} f(x, y) = \max_{y \in Y} \min_{x \in X} f(x, y)$ is valid or points

$$x_1^*, x_2^* \in X, \; y_1, y_2 \in Y, \; y_1 \ne y_2, \; y^* \in (y_1 \# y_2),$$

exist such that

$$f(x_1^*, y^*) = f(x_2^*, y^*) = \max_{y \in Y} \min_{x \in X} f(x, y),$$

$$f(x_1^*, v) \le \max_{y \in Y} \min_{x \in X} f(x, y) \text{ for all } v \in (y_1 \leftrightarrow y^* \prec),$$

$$f(x_2^*, v) \le \max_{y \in Y} \min_{x \in X} f(x, y) \text{ for all } v \in (\succ y^* \leftrightarrow y_2).$$

5. POLYLINES IN THE CARTHESIAN PRODUCT OF POLYLINES

Let $X \subset \mathbb{R}^m$ be a polyline and points $x_1, x^*, x_2 \in X$, $x_1 \ne x_2$, $x^* \in (x_1 \leftrightarrow x_2)$, define subsets $X_1 = (x_1 \leftrightarrow x^* \prec) \subset X$, $X_2 = (\succ x^* \leftrightarrow x_2) \subset X$.

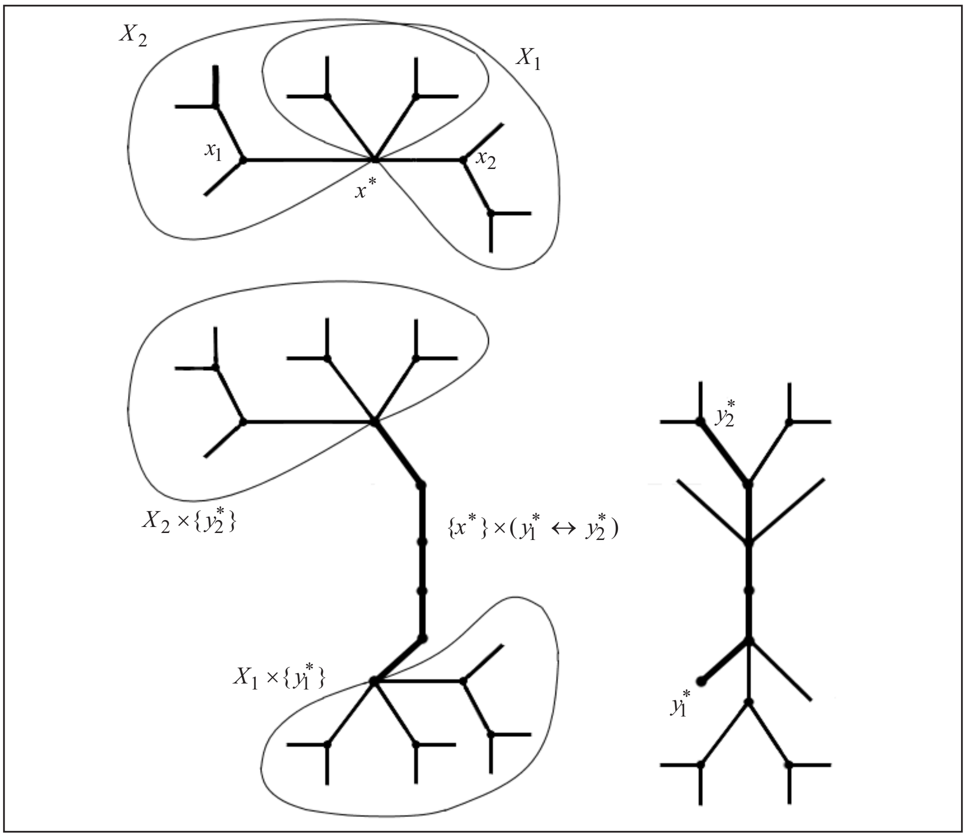

Fig. 7. Composing polyline XYX of subsets X_1, X_2 and a path $(y_1^* \leftrightarrow y_2^*)$

Let $Y \subset \mathbb{R}^n$ be a polyline and points $y_1, y^*, y_2 \in Y$, $y_1 \neq y_2$, $y^* \in (y_1 \leftrightarrow y_2)$, define subsets $Y_1 = (y_1 \leftrightarrow y^* \prec) \subset Y$, $Y_2 = (\succ y^* \leftrightarrow y_2) \subset Y$.

Definition 10. For given points $y_1^* \in Y$, $y_2^* \in Y$, $x^* \in X$ and subsets $X_1 \subset X$, $X_2 \subset X$ a polyline XYX is a subset

$$XYX = [X_1 \times \{y_1^*\}] \cup [\{x^*\} \times (y_1^* \leftrightarrow y_2^*)] \cup [X_2 \times \{y_2^*\}] \subset X \times Y.$$

Figure 7 illustrates how a set XYX is composed of subsets $X_1 \subset X$, $X_2 \subset X$ and a path $(y_1^* \leftrightarrow y_2^*) \subset Y$.

Definition 11. For given points $x_1^* \in X$, $x_2^* \in X$, $y^* \in Y$ and subsets $Y_1 \subset Y$, $Y_2 \subset Y$ a polyline YXY is a subset

$$YXY = [\{x_1^*\} \times Y_1] \cup [(x_1^* \leftrightarrow x_2^*) \times \{y^*\}] \cup [\{x_2^*\} \times Y_2] \subset X \times Y.$$

Figure 8 illustrates how a set YXY is composed of subsets $Y_1 \subset Y$, $Y_2 \subset Y$ and a path $(x_1^* \leftrightarrow x_2^*) \subset X$.

As may be seen from Definitions 10 and 11 any point on polylines XYX and YXY represents some pair (x, y) of points $x \in X$ and $y \in Y$. The polyline X is a subset of some m-dimensional linear space, the polyline Y belongs to other linear space, n-dimesion al one, polylines XYX and YXY are two different subsets of a third space, an $(m+n)$-dimensional one.

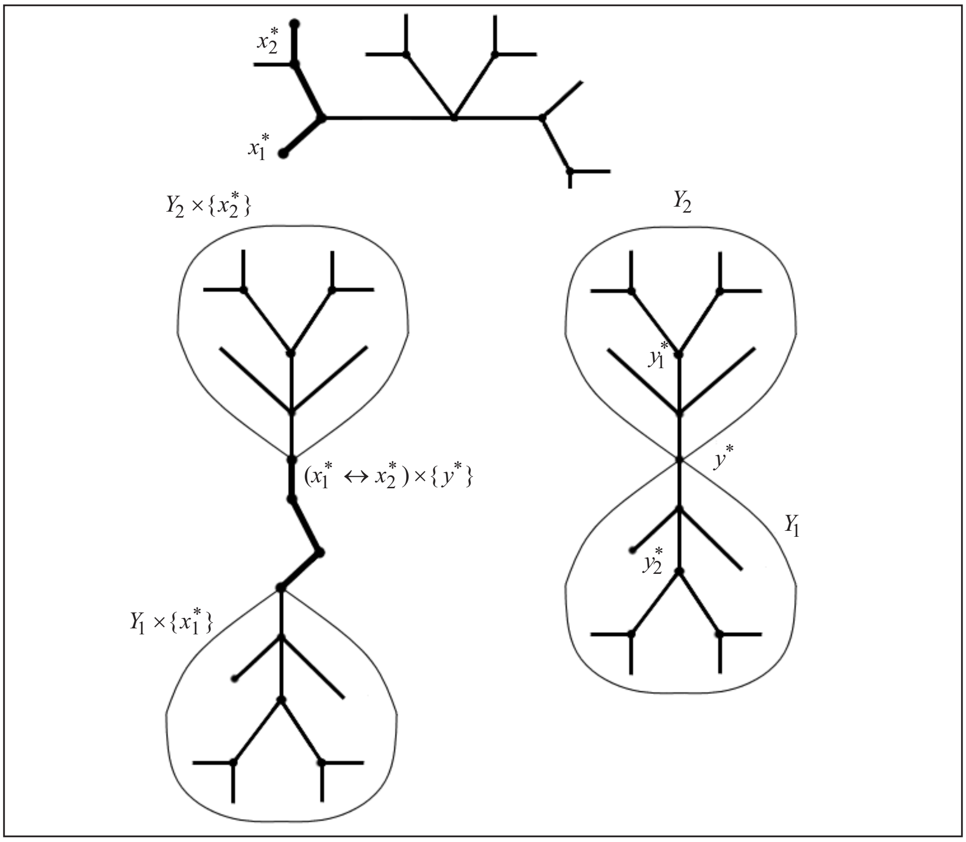

Fig. 8. Composing polyline *YXY* of subsets Y_1, Y_2 and a path $(x_1^* \leftrightarrow x_2^*)$

Lemma 7. Polylines XYX and YXY have a non-empty intersection.

Proof. The lemma is proved for a case when both conditions

$$x^* \in (x_1^* \leftrightarrow x_2^*), \ \ y^* \in (y_1^* \leftrightarrow y_2^*) \tag{36}$$

are satisfied as well as for a case when one of these conditions is not satisfied.

Let both conditions (36) be satisfied. It follows from $x^* \in (x_1^* \leftrightarrow x_2^*)$ and Definition 11 that

$$(x^*, y^*) \in [(x_1^* \leftrightarrow x_2^*) \times \{y^*\}] \subset YXY.$$

It follows from $y^* \in (y_1^* \leftrightarrow y_2^*)$ and Definition 10 that

$$(x^*, y^*) \in [\{x^*\} \times (y_1^* \leftrightarrow y_2^*)] \subset XYX.$$

Thus, $(x^*, y^*) \in XYX \cap YXY \neq \varnothing$.

Let one of the conditions (36) be not fulfilled. If $x^* \notin (x_1^* \leftrightarrow x_2^*)$ then due to Property 2 at least one of the conditions $\{x_1^*, x_2^*\} \subset X_1$, $\{x_1^*, x_2^*\} \subset X_2$ fulfils. It follows from Definition 10 that

$$\text{if } \{x_1^*, x_2^*\} \subset X_1 \text{ then } [\{x_1^*, x_2^*\} \times \{y_1^*\}] \subset [X_1 \times \{y_1^*\}] \subset XYX,$$

$$\text{if } \{x_1^*, x_2^*\} \subset X_2 \text{ then } [\{x_1^*, x_2^*\} \times \{y_2^*\}] \subset [X_2 \times \{y_2^*\}] \subset XYX.$$

So, a point $y' \in \{y_1^*, y_2^*\}$ exists such that both points (x_1^*, y') and (x_2^*, y') belong to XYX. Let us show that at least one of them belongs to YXY.

Due to Property 1 the equality $Y = Y_1 \cup Y_2$ is valid and so at least one of inclusions $y' \in Y_1$ or $y' \in Y_2$ is valid. It follows from Definition 11 that

$$\text{if } y' \in Y_1 \text{ then } (x_1^*, y') \in [\{x_1^*\} \times Y_1] \subset YXY,$$

$$\text{if } y' \in Y_2 \text{ then } (x_2^*, y') \in [\{x_2^*\} \times Y_2] \subset YXY.$$

So, if $x^* \notin (x_1^* \leftrightarrow x_2^*)$ then a point y' exists such that both points (x_1^*, y') and (x_2^*, y') belong to XYX and at least one of them belongs to YXY that results in $XYX \cap YXY \neq \varnothing$.

In a similar way it may be proved that $XYX \cap YXY \neq \varnothing$ if $y^* \notin (y_1^* \leftrightarrow y_2^*)$ as well. In this case a point x' exists such that both points (x', y_1^*) and (x', y_2^*) belong to YXY and at least one of them belongs to XYX. ■

6. PROOF COMPLETION

This section completes the proof of Theorem 1 as a corollary of Lemmas 5–7. Theorem 1 states that if $f : X \times Y \to \mathbb{R}$ is a continuous, convex on X and concave on Y function then

$$\min_{x \in X} \max_{y \in Y} f(x, y) = \max_{y \in Y} \min_{x \in X} f(x, y). \tag{37}$$

Proof. It follows from Lemma 5 that equality (37) is valid when conditions

$$f(x^*, y_1^*) = f(x^*, y_2^*) = \min_{x \in X} \max_{y \in Y} f(x, y), \tag{38}$$

$$f(u, y_1^*) \geq \min_{x \in X} \max_{y \in Y} f(x, y) \text{ for all } u \in (x_1 \leftrightarrow x^* \prec), \tag{39}$$

$$f(u, y_2^*) \geq \min_{x \in X} \max_{y \in Y} f(x, y) \text{ for all } u \in (\succ x^* \leftrightarrow x_2) \tag{40}$$

are inconsistent. It follows from Lemma 6 that equality (37) is valid when conditions

$$f(x_1^*, y^*) = f(x_2^*, y^*) = \max_{y \in Y} \min_{x \in X} f(x, y), \tag{41}$$

$$f(x_1^*, v) \leq \max_{y \in Y} \min_{x \in X} f(x, y) \text{ for all } v \in (y_1 \leftrightarrow y^* \prec), \tag{42}$$

$$f(x_2^*, v) \leq \max_{y \in Y} \min_{x \in X} f(x, y) \text{ for all } v \in (\succ y^* \leftrightarrow y_2) \tag{43}$$

are inconsistent. So, the proof of Theorem 1 is reduced to the proof that equality (37) is valid when conditions (38)–(43) are consistent.

Let points $x_1, x_2 \in X$, $x^* \in (x_1 \leftrightarrow x_2)$ and points $y_1^*, y_2^* \in Y$ exist that satisfy conditions (38)—(40) and let points $y_1, y_2 \in Y$, $y^* \in (y_1 \leftrightarrow y_2)$ and points x_1^*, $x_2^* \in X$ exist that satisfy conditions (41)–(43). As before, let us denote

$$X_1 = (x_1 \leftrightarrow x^* \prec) \subset X, \; X_2 = (\succ x^* \leftrightarrow x_2) \subset X,$$

$$Y_1 = (y_1 \leftrightarrow y^* \prec) \subset Y, \; Y_2 = (\succ y^* \leftrightarrow x_2) \subset Y,$$

$$XYX = [X_1 \times \{y_1^*\}] \cup [\{x^*\} \times (y_1^* \leftrightarrow y_2^*)] \cup [X_2 \times \{y_2^*\}],$$

$$YXY = [\{x_1^*\} \times Y_1] \cup [(x_1^* \leftrightarrow x_2^*) \times \{y^*\}] \cup [\{x_2^*\} \times Y_2].$$

Due to (39) and (40)

$$f(x, y) \geq \min_{u \in X} \max_{v \in Y} f(u, v) \text{ for all } (x, y) \in [X_1 \times \{y_1^*\}] \cup [X_2 \times \{y_2^*\}],$$

due to (38) and concavity of f on Y

$$f(x, y) \geq \min_{u \in X} \max_{v \in Y} f(u, v) \text{ for all } (x, y) \in [\{x^*\} \times (y_1^* \leftrightarrow y_2^*)],$$

that results in

$$f(x, y) \geq \min_{u \in X} \max_{v \in Y} f(u, v) \text{ for all } (x, y) \in XYX. \tag{44}$$

Similarly, it follows from (41)–(43) and convexity of f on $x \in X$ that

$$f(x, y) \leq \max_{v \in Y} \min_{u \in X} f(u, v) \text{ for all } (x, y) \in YXY. \tag{45}$$

Due to Lemma 7 the sets XYX and YXY have a non-empty intersection. Due to (44) and (45) any point $(x' y') \in XYX \cap YXY$ satisfies inequalities

$$\min_{x \in X} \max_{y \in Y} f(x, y) \leq f(x' y') \leq \max_{y \in Y} \min_{x \in X} f(x, y)$$

that due to (1) results in $\min_{x \in X} \max_{y \in Y} f(x, y) = \max_{y \in Y} \min_{x \in X} f(x, y)$. ■

REFERENCES

1. von Neumann J. Zur Theorie der Gesselschaftsspiele. Mathematishe Annalen. 1928. Vol. 100. P. 295–320.
2. Sion M. On general minimax theorems. *Pacific Journal Math*. 1958. Vol. 8, N 1. P. 171–176. https://doi.org/10.2140/pjm.1958.8.171.
3. Komiya H. Elementary proof for Sion's minimax theorem. *Kodai Mathematical Journal*. 1988. Vol. 11, N 1. P. 5–7. https://doi.org/10.2996/kmj/1138038812.
4. Kindler J. A simple proof of Sion's minimax theorem. *The American Mathematical Monthly*. 2005. Vol. 112, Iss. 4. P. 356–358. https://doi.org/10.1080/00029890.2005.11920204.
5. Ben-el-Mechiels H., Dimand R.W. A simpler proof of the von Neumann minimax theorem. *The American Mathematical Monthly*. 2011. Vol. 118, Iss. 7. P. 636–641. https://doi.org/10.4169/amer.math.monthly.118.07.636.
6. Schlesinger M., Krygin V. Minimax theorem revisited. *The American Mathematical Monthly*. 2023. Vol. 130, Iss. 2. P. 145–147. https://doi.org/10.1080/00029890.2022.2144085.
7. Aubin J.-P., Frankowska H. Set-Valued Analysis. Boston, MA: Birkhauser, 2008. P. 482.

М.І. Шлезінгер

ТЕОРЕМА ПРО МІНІМАКС ФУНКЦІЙ НА ДЕКАРТОВОМУ ДОБУТКУ РОЗГАЛУЖЕНИХ ЛАМАНИХ ЛІНІЙ

Анотація. Доведено теорему про мінімакс для специфічного класу функцій, визначених не на опуклих підмножинах лінійного простору, а на ламаних лініях у лінійному просторі. Існування сідлової точки для таких функцій не випливає безпосередньо з класичної теореми про мінімакс і потребує індивідуального аналізу, що грунтується на спільному використанні методів опуклого аналізу та теорії графів. У статті виконано самодостатній аналіз задачі. Вона містить у собі все, що потрібно для ясного розуміння і доведення основного результату без залучення понять, що виходять за межі стандартної математичної освіти інженерів. Статтю адресовано дослідникам, які використовують методи оптимізації у прикладній механіці, електротехніці та інших прикладних науках, а також математикам-викладачам опуклого аналізу та методів оптимізації для інженерів.

Ключові слова: мінімакс, сідлова точка, опуклий аналіз, оптимізація, розгалужена ламана лінія.

Надійшла до редакції 18.01.2023