UDC 517.555

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GROWTH OF ENTIRE FUNCTIONS OF BOUNDED L-INDEX IN DIRECTION

A problem on growth estimation for the maximum modulus of an entire function of bounded L-index in direction is solved. Some generalizations of the earlier one-dimensional results are obtained.

Let $\mathcal{E}_n$ be a class of entire functions $F:\mathbb{C}^n \to \mathbb{C}$, $n \ge 1$, and $\mathbb{L}_n$ be a class of positive continuous functions $L:\mathbb{C}^n \to \mathbb{R}_+ := (0,+\infty)$. For $L \in \mathbb{L}_n$, an entire function $F \in \mathcal{E}_n$ is called [1, 6] a *function of bounded L-index in a direction* $\mathbf{b} = (b_1,\ldots,b_n) \in \mathbb{C}^n \setminus \{0\}$ if there exists $m_0 \in \mathbb{Z}_+$ such that $(\forall m \in \mathbb{Z}_+)$ $(\forall z \in \mathbb{C}^n)$

$$\frac{1}{m!L^m(z)}\left|\frac{\partial^m F(z)}{\partial \mathbf{b}^m}\right| \le \max\left\{\frac{1}{k!L^k(z)}\left|\frac{\partial^k F(z)}{\partial \mathbf{b}^k}\right| : 0 \le k \le m_0\right\}, \tag{1}$$

where

$$\frac{\partial F(z)}{\partial \mathbf{b}} = \sum_{j=1}^{n} \frac{\partial F(z)}{\partial z_j} b_j = \left\langle \operatorname{grad} F, \overline{\mathbf{b}} \right\rangle,$$

$$\frac{\partial^k F(z)}{\partial \mathbf{b}^k} = \frac{\partial}{\partial \mathbf{b}}\left(\frac{\partial^{k-1}F(z)}{\partial \mathbf{b}^{k-1}}\right), \qquad k \ge 1,$$

$$\frac{\partial^0 F(z)}{\partial \mathbf{b}^0} = F(z).$$

The least such an integer m_0 is called *L-index in the direction* $\mathbf{b}$ of the function F and is denoted by $N(F,L) = N_{\mathbf{b}}(F,L)$. Hence, for $n = 1$, $\mathbf{b} = 1$ and $L(z) \equiv \ell(|z|)$, $\ell \in \mathbb{L}_1$, we obtain [3] the definition of *an entire function $f:\mathbb{C} \to \mathbb{C}$ of bounded ℓ-index*. For $n = 1$, $\mathbf{b} = 1$ and $\ell(r) \equiv 1$, we have the definition *of an entire function of bounded index* [7]. As $n = 1$, it is known (in the general case [3] and in case $\ell(r) \equiv 1$ [10]) that if function $L \in \mathbb{L}_1$, $L(z) \equiv \ell(|z|)$, meets the condition

$$\varliminf_{r\to+\infty} \frac{1}{\ell(r)} \min\left\{\ell(t) : \frac{r}{1+\delta} \le t \le r\right\} = \lambda(\delta) \to 1, \qquad \delta \to +0, \tag{2}$$

and $f:\mathbb{C} \to \mathbb{C}$ is an entire function of bounded ℓ-index $N(f,\ell) = N_1(f,\ell)$, then

$$\varlimsup_{r\to+\infty} \frac{\ln M_f(r)}{\ell_0(r)} \le N(f,\ell) + 1, \tag{3}$$

where $M_f(r) = \max\{|f(z)| : \ |z| = r\}$, $\ell_0(r) = \int_0^r \ell(t)\,dt$. In 2006 M. M. Sheremeta raised the following question: *what is the growth estimate for the maximum modulus of entire functions $F \in \mathcal{E}_n$ $(n > 1)$ of bounded index in direction?* It follows from (2), (3) that if the function $\ell(r) = L(z_0 + re^{i\theta}\mathbf{b})$ satisfies condition

(2) for fixed $z_0 \in \mathbb{C}^n$ and $\theta \in [0, 2\pi]$, then, for entire function $F \in \mathcal{E}_n$ of bounded L-index in the direction $\mathbf{b}$, we have

$$\overline{\lim_{r\to+\infty}} \frac{\ln M_{\mathbf{b}}(r, z^0, F)}{\int_0^r L(z_0 + te^{i\theta}\mathbf{b})\,dt} \le N_{\mathbf{b}}(F, L) + 1,$$

where $M_{\mathbf{b}}(r, z^0, F) := \max\{|F(z_0 + \tau\mathbf{b})| : |\tau| = r\}$.

Another definition of bounded index in $\mathbb{C}^2$ was considered by Nuray and Patterson [8, 9]. They used the partial derivatives instead of the directional ones and found the relationship between the concept of bounded index and the radius of univalence, respectively p-valence, of entire bivariate functions and their partial derivatives at arbitrary points in $\mathbb{C}^2$.

We need some standard notations. Let $F \in \mathcal{E}_n$. For a given $z^0 \in \mathbb{C}^n$, we expand a function $f(w) := F(z^0 + w\mathbf{b})$ in a power series in $w \in \mathbb{C}$:

$$F(z^0 + w\mathbf{b}) = \sum_{m=0}^{\infty} b_m(z^0) w^m, \qquad b_m(z^0) = \frac{1}{m!}\frac{\partial^m F(z^0)}{\partial \mathbf{b}^m}.$$

Denote

$$M_{\mathbf{b},j}(r, z^0, F) = \max\left\{\left|\frac{\partial^j F(z^0 + w\mathbf{b})}{\partial \mathbf{b}^j}\right| : |w| = r\right\}, \quad j \in \mathbb{N}, \quad r > 0,$$

$$\mu_{\mathbf{b}}(r, z^0, F) = \max\{|b_m(z^0)| r^m : m \ge 0\},$$

$$\nu_{\mathbf{b}}(r, z^0, F) = \max\{m : |b_m(z^0)| r^m = \mu_{\mathbf{b}}(r, z^0, F)\}.$$

We put

$$\varphi(\delta, \theta, z^0) := \underline{\lim_{r\to+\infty}} \min\left\{L(z^0 + te^{i\theta}\mathbf{b}) : \frac{r}{1+\delta} \le t \le r\right\} \frac{1}{L(z^0 + re^{i\theta}\mathbf{b})}.$$

It easy to see that $\exists \lim_{\delta\to+0} \varphi(\delta, \theta, z^0) := \varphi_0(\theta, z^0)$, $\varphi(\delta, \theta, z^0) \le \varphi_0(\theta, z^0) \in [0, 1]$.

Theorem 1. *Let $L \in \mathbb{L}_n$ be such that $rL(z^0 + re^{i\theta}\mathbf{b}) \to +\infty$, $r \to +\infty$, and*

$$\varphi_0(\theta, z^0) \in (0, 1] \tag{4}$$

for some $z^0 \in \mathbb{C}^n$, $\mathbf{b} \in \mathbb{C}^n \setminus \{0\}$ and $\theta \in [0, 2\pi]$. If an entire transcendental function $F \in \mathcal{E}_n$ is of bounded L-index $N_{\mathbf{b}}(F, L)$ in the direction $\mathbf{b}$, then

$$\tau(z^0, \theta) := \overline{\lim_{r\to+\infty}} \frac{\nu_{\mathbf{b}}(r, z^0, F)}{rL(z^0 + re^{i\theta}\mathbf{b})} \le \frac{N_{\mathbf{b}}(F, L) + 1}{\varphi_0(\theta, z^0)}, \tag{5}$$

$$\overline{\lim_{r\to+\infty}} \frac{\ln M_{\mathbf{b}}(r, z^0, F)}{\int_0^r L(z^0 + te^{i\theta}\mathbf{b})\,dt} \le \frac{N_{\mathbf{b}}(F, L) + 1}{\varphi_0^2(\theta, z^0)}. \tag{6}$$

P r o o f. Our proof is based on ideas of Sheremeta and Kuzyk [3, 11].

1°. *Proof of* (5). For simplicity, we denote $\tau = \tau(z^0, \theta)$. If $\tau = 0$, then (6) is obvious. If $f(w) := F(z^0 + w\mathbf{b})$ is polynomial, then $\nu_{\mathbf{b}}(r, z^0, F) \equiv \nu_0 = \deg f \ge 1$ $(r \ge r_0)$ and $\tau = 0$ again. Therefore, if $\tau > 0$, then $f(w) := F(z^0 + w\mathbf{b})$ is entire transcendental function. By definition of limit superior for every $\delta\in(0,\tau)$ (for any $\tau\in(0,+\infty)$ in the case $\tau(z^0,\theta)=+\infty$), there exists an increasing to $+\infty$ sequence (r_n) such that $(1+\delta)r_n<r_{n+1}$ and $\nu(r_n,z^0,F)>(\tau-\delta)r_nL(z^0+r_ne^{i\theta}\mathbf{b})$ For $r\in[r_n,(1+\delta)r_n]$ and $n\ge n_0(\delta)$, in view of (4), we have

$$r_nL(z^0+r_ne^{i\theta}\mathbf{b})\ge\frac{1}{1+\delta}r\min\left\{L(z^0+te^{i\theta}\mathbf{b}):\frac{r}{1+\delta}\le t\le r\right\}\ge$$

$$\ge\frac{1}{1+\delta}\,(\varphi(\delta,\theta,z^0)-\delta)\,r\,L(z^0+re^{i\theta}\mathbf{b}),$$

therefore

$$\nu(r,z^0,F)\ge\frac{\tau-\delta}{1+\delta}\big(\varphi(\delta,\theta,z^0)-\delta\big)\,r\,L(z^0+re^{i\theta}\mathbf{b}). \tag{7}$$

We put $U_\delta=\bigcup\limits_{n\ge n_0(\delta)}[r_n,(1+\delta)r_n]$. It is clear that the logarithmic measure of U_δ equals to infinity and for all $r\in U_\delta$ the inequality (7) is valid. But it is known [2, p. 26] that if $f(w)=\sum\limits_{p=0}^{\infty}a_pw^p$ is an entire transcendental function of one variable, then for every fixed $j\in\mathbb{N}$

$$M_j(r,f):=\max\{|f^{(j)}(w)|:|w|=r\}\sim\left(\frac{\nu(r,f)}{r}\right)^jM(r,f), \tag{8}$$

as $r\to+\infty$ beyond some set of finite logarithmic measure depending on j, where

$$\nu(r,f)=\max\{p:|a_p|r^p=\mu(r,f)\},$$

$$\mu(r,f)=\max\{|a_p|r^p:p\ge0\},$$

$$M(r,f):=\max\{|f(w)|:|w|=r\}.$$

By applying (8) to the function $f(w)=F(z^0+w\mathbf{b})$ as a function of variable $w\in\mathbb{C}$, we can conclude that there exists an increasing to $+\infty$ sequence (r_k^*) such that

$$\frac{M_{\mathbf{b},j}(r_k^*,z^0,F)}{M(r_k^*,z^0,f)}\sim\left(\frac{\nu(r_k^*,z^0,f)}{r_k^*}\right)^j,\qquad k\to\infty,$$

and (7) holds with $r=r_k^*$.

Therefore, for all $j\in\{0,1,2,\dots,N_b(F,L)\}$ and for all $k\ge k_0(\delta)$,

$$\frac{M_{\mathbf{b},j+1}(r_k^*,z^0,F)}{M_{\mathbf{b},j}(r_k^*,z^0,F)}>(1-\delta)\frac{\nu_{\mathbf{b}}(r_k^*,z^0,F)}{r_k^*}>$$

$$>\frac{\tau-\delta}{1+\delta}(1-\delta)(\varphi(\delta,\theta,z^0)-\delta)L(z^0+r_k^*e^{i\theta}\mathbf{b}). \tag{9}$$

We put

$$\tau_0(\delta,\theta,z^0):=\frac{1-\delta}{1+\delta}(\tau-\delta)(\varphi(\delta,\theta,z^0)-\delta)$$

and assume that $\tau_0(\delta,\theta,z^0)>N_{\mathbf{b}}(F,L)+1$. Then (9) implies

$$\frac{M_{\mathbf{b},j+1}(r_k^*, z^0, F)}{(j+1)!L^{j+1}(z^0 + r_k^* e^{i\theta}\mathbf{b})} >$$

$$> \frac{\tau_0(\delta,\theta,z^0)}{j+1} \frac{M_{\mathbf{b},j}(r_k^*, z^0, F)}{j!L^j(z^0 + r_k^* e^{i\theta}\mathbf{b})} > \frac{M_{\mathbf{b},j}(r_k^*, z^0, F)}{j!L^j(z^0 + r_k^* e^{i\theta}\mathbf{b})}$$

for all $j \in \{0, 1, 2, \ldots, N_{\mathbf{b}}(F, L)\}$ and for all $k \geq k_0(\delta)$. Thus,

$$\frac{M_{\mathbf{b},N+1}(r_k^*, z^0, f)}{(N+1)!L^{n+1}(z^0 + r_k^* e^{i\theta}\mathbf{b})} > \max_{0 \le q \le N} \left\{ \frac{M_{\mathbf{b},q}(r_k^*, z^0, f)}{q!L^q(z^0 + r_k^* e^{i\theta}\mathbf{b})} \right\}, \quad N = N_{\mathbf{b}}(F, L).$$

This inequality contradicts inequality (1). Thus, $\tau_0(\delta,\theta,z^0) \le N_{\mathbf{b}}(f,\ell) + 1$ and due to the arbitrarity of δ, we have $\tau\varphi_0(\theta,z^0) \le N_{\mathbf{b}}(f,\ell)+1$, i. e. $\tau \le \dfrac{N_{\mathbf{b}}(f,\ell)+1}{\varphi_0(\theta,z^0)}$.

2°. *Proof of* (6). Let

$$F(z^0 + w\mathbf{b}) = \sum_{m=0}^{\infty} b_m(z^0) w^m .$$

Then for any $\delta > 0$, we have

$$M_{\mathbf{b}}(r, z^0, F) \le \sum_{m=0}^{\infty} \left| b_m(z^0) \right| r^m =$$

$$= \sum_{m=0}^{\infty} \frac{\left| b_m(z^0) \right| ((1+\delta)r)^m}{(1+\delta)^m} \le \frac{1+\delta}{\delta} \mu_{\mathbf{b}}((1+\delta)r, z^0, F).$$

But due to inequality (5)

$$\ln \mu_{\mathbf{b}}(r, z^0, F) = \ln \mu_{\mathbf{b}}(0, z^0, F) + \int_0^r \frac{\nu_b(t, z^0, f)}{t} dt \le$$

$$\le (1 + o(1)) \frac{N_{\mathbf{b}}(F, L) + 1}{\varphi_0(\theta, z^0)} \int_0^r L(z^0 + te^{i\theta}\mathbf{b})\, dt$$

as $r \to +\infty$. Since $\int_0^{+\infty} L(z^0 + te^{i\theta}\mathbf{b})\, dt = +\infty$, we obtain

$$\overline{\lim_{r\to+\infty}} \frac{\ln M_{\mathbf{b}}(r, z^0, F)}{\int_0^r L(z^0 + te^{i\theta}\mathbf{b})\, dt} \le \overline{\lim_{r\to+\infty}} \frac{\ln \mu_{\mathbf{b}}((1+\delta)r, z^0, F)}{\int_0^r L(z^0 + te^{i\theta}\mathbf{b})\, dt} \le$$

$$\le \overline{\lim_{r\to+\infty}} \frac{\ln \mu_{\mathbf{b}}((1+\delta)r, z^0, F)}{\int_0^{(1+\delta)r} L(z^0 + te^{i\theta}\mathbf{b})\, dt} \overline{\lim_{r\to+\infty}} \frac{\int_0^{(1+\delta)r} L(z^0 + te^{i\theta}\mathbf{b})\, dt}{\int_0^r L(z^0 + te^{i\theta}\mathbf{b})\, dt} \le$$

$$\le \frac{N_{\mathbf{b}}(F, L) + 1}{\varphi_0(\theta, z^0)} \overline{\lim_{r\to+\infty}} \frac{\int_0^{(1+\delta)r} L(z^0 + te^{i\theta}\mathbf{b})\, dt}{\int_0^r L(z^0 + te^{i\theta}\mathbf{b})\, dt}.$$

Then using L'Hospital's rule by condition (4), we deduce

$$\overline{\lim_{r\to+\infty}} \frac{\int_0^{(1+\delta)r} L(z^0 + te^{i\theta}\mathbf{b})\,dt}{\int_0^r L(z^0 + te^{i\theta}\mathbf{b})\,dt} \le \overline{\lim_{r\to+\infty}} \frac{(1+\delta)L(z^0 + r(1+\delta)e^{i\theta}\mathbf{b})}{L(z^0 + re^{i\theta}\mathbf{b})} =$$

$$= (1+\delta)\overline{\lim_{r\to+\infty}} \frac{L(z^0 + re^{i\theta}\mathbf{b})}{L\left(z^0 + \frac{r}{1+\delta}e^{i\theta}\mathbf{b}\right)} \le \frac{1+\delta}{\varphi(\delta,\theta,z^0)}.$$

Thus, $\overline{\lim_{r\to+\infty}} \frac{\ln M_{\mathbf{b}}(r,z^0,F)}{\int_0^r L(z^0 + te^{i\theta}\mathbf{b})\,dt} \le \frac{N_{\mathbf{b}}(F,L)+1}{\varphi_0(\theta,z^0)}\cdot\frac{1+\delta}{\varphi(\delta,\theta,z^0)}$ and, as $\delta\to+0$, we obtain (6). The proof of Theorem 1 is completed. ♦

Remark 1. *If* $L(z) = L_0(|z|)$ *be a nondecreasing function of variable* $|z|$ *then* L *satisfies* (4).

Corollary 1. *Let* $\mathbf{b}\in\mathbb{C}^n\setminus\{0\}$, *and* $L\in\mathbb{L}_n$ *be a function such that* $(\forall z^0 \in \mathbb{C}^n)$ $(\forall\theta\in[0,2\pi]) : rL(z^0 + re^{i\theta}\mathbf{b}) \to +\infty$, $r\to+\infty$, *and*

$$\inf_{z^0\in\mathbb{C}^n}\min_{\theta\in[0,2\pi]}\varphi_0(\theta,z^0) = \varphi_0 \in (0,1]. \tag{10}$$

If an entire transcendental function $F\in\mathcal{E}_n$ *has bounded* L*-index* $N_{\mathbf{b}}(F,L)$ *in the direction* $\mathbf{b}$, *then*

$$\sup_{z^0\in\mathbb{C}^n}\max_{\theta\in[0,2\pi]}\overline{\lim_{r\to+\infty}}\frac{\nu_{\mathbf{b}}(r,z^0,F)}{rL(z^0 + re^{i\theta}\mathbf{b})} \le \frac{N_{\mathbf{b}}(F,L)+1}{\varphi_0}.$$

Corollary 2. *Let* $\mathbf{b}\in\mathbb{C}^n\setminus\{0\}$, *and* $L\in\mathbb{L}_n$ *be a function such that* $(\forall z^0 \in \mathbb{C}^n)$ $(\forall\theta\in[0,2\pi]) : rL(z^0 + re^{i\theta}\mathbf{b}) \to +\infty$, $r\to+\infty$, *and* (10) *holds. If an entire transcendental function* $F\in\mathcal{E}_n$ *is of bounded* L*-index* $N_{\mathbf{b}}(F,L)$ *in the direction* $\mathbf{b}$, *then*

$$\sup_{z^0\in\mathbb{C}^n}\max_{\theta\in[0,2\pi]}\overline{\lim_{r\to+\infty}}\frac{\ln M_{\mathbf{b}}(r,z^0,F)}{\int_0^r L(z^0 + te^{i\theta}\mathbf{b})\,dt} \le \frac{N_{\mathbf{b}}(F,L)+1}{\varphi_0^2}. \tag{11}$$

For $n=1$, $L=\ell$, $\mathbb{L}=\mathbb{L}_1$, $F=f$, $\mathbf{b}=1$, $z^0=0$, $\nu(r,f)=\nu_1(r,0,F)$, $M(r,f)=M_1(r,0,F)$, $N(f,\ell)=N_1(F,L)$, Theorem 1 implies the following one-dimensional corollary.

Corollary 3. *Let* $\ell\in\mathbb{L}$ *and*

$$\min_{\theta\in[0,2\pi]}\underline{\lim_{r\to+\infty}}\frac{\min\{\ell(te^{i\theta}) : r/(1+\delta)\le t\le r\}}{\ell(re^{i\theta})} = \varphi(\delta)\to\varphi_0\in(0,1] \tag{12}$$

as $\delta\to+0$. *If an entire transcendental function* f *has bounded* ℓ*-index* $N(f,\ell)$, *then*

$$\max_{\theta\in[0,2\pi]}\overline{\lim_{r\to+\infty}}\frac{\nu(r,f)}{r\ell(re^{i\theta})} \le \frac{N(f,\ell)+1}{\varphi_0},$$

$$\max_{\theta\in[0,2\pi]} \overline{\lim_{r\to+\infty}} \frac{\ln M(r,f)}{\int_0^r \ell(te^{i\theta})\,dt} \le \frac{N(f,\ell)+1}{\varphi_0^2}. \tag{13}$$

Remark 2. *Corollary 3 is a generalization of corresponding Sheremeta and Kuzyk's results* [3] *in two directions:* ***1°***) *we do not assume the function* ℓ *to be of the form* $\ell(|z|)$; ***2°***) *we do not assume that* $\varphi_0 = 1$ (*our results are valid for* $\varphi_0 \in (0,1]$).

***Example* 1.** For $n=1$ a function $\ell(z) = 2 + \frac{|\sin|z||}{|z|}$ satisfies condition (12) with $\varphi_0 = 1$.

Indeed, we can choose r_0 so that for every $r \ge r_0$ there exists $n \in \mathbb{N}$ such that $[2\pi n, 2\pi(n+1)] \subset \left[\frac{r}{1+\delta}, r\right]$. Then $\min\left\{\ell(te^{i\theta}) : \frac{r}{1+\delta} \le t \le r\right\} = 2$, and we have $\lim\limits_{r\to+\infty} \frac{2}{\ell(re^{i\theta})} = \lim\limits_{r\to+\infty} \frac{2}{2+|\sin r|/r} = 1 = \varphi_0$. Let $f(z) = e^{2z}$, then [4] $N(f,\ell) = 0$, $\ln M(r,f) = 2r$, $\nu(r,f) = [2r]$ and

$$\int_0^r \ell(t)\,dt = \int_0^r \left(2 + \frac{|\sin t|}{t}\right) dt = 2r + o(r).$$

Thus, $\overline{\lim\limits_{r\to+\infty}} \frac{\nu(r,f)}{r\ell(r)} = 1 = \frac{N(f,\ell)+1}{\varphi_0}$, $\overline{\lim\limits_{r\to+\infty}} \frac{\ln M(r,f)}{\int_0^r \ell(t)\,dt} = 1 = \frac{N(f,\ell)+1}{\varphi_0^2}$.

***Example* 2.** For $n=1$, a function $\ell(z) = \sin|z| + 2$ satisfies condition (12) with $\varphi_0 = 1/3$.

Let $f(z) = e^z$, then $N(f,\ell) = 0$, $\ln M(r,f) = r$, $\nu(r,f) = [r]$ and $\int_0^r \ell(t)\,dt = 2r + O(1)$. Thus,

$$\overline{\lim_{r\to+\infty}} \frac{\ln M(r,f)}{\int_0^r \ell(t)\,dt} = \frac{1}{2} < \frac{N(f,\ell)+1}{\varphi_0^2} = 9,$$

$$\overline{\lim_{r\to+\infty}} \frac{\nu(r,f)}{r\ell(r)} = 1 < \frac{N(f,\ell)+1}{\varphi_0} = 3.$$

***Example* 3**. Similarly, we can prove that an unbounded function $\ell(z) = (|z|+1)(\sin|z|+2)$ satisfies (12) also with $\varphi_0 = 1/3$.

Remark 3. *Generally speaking, if* $\varphi_0 \in (0,1)$, *then we do not know whether estimate* (11) *is sharp. For* $\varphi_0 = 1$, *Sheremeta and Kuzyk* [3] *proved the sharpness of the second inequality in* (13).

Theorem 2. *Let* $L \in \mathbb{L}_n$ *and for every* $z^0 \in \mathbb{C}^n$ *and* $\theta \in [0,2\pi]$ *the function* $u(r) = u(z^0,\theta,r) := L(z^0 + re^{i\theta}\mathbf{b})$ *be a continuously differentiable function of real variable* $r \ge 0$. *If an entire function* F *is of bounded* L*-index in the direction* $\mathbf{b} \in \mathbb{C}^n \setminus \{0\}$, *then for every* $z^0 \in \mathbb{C}^n$, $\theta \in [0,2\pi]$, $r \in [0,+\infty)$ *and every integer* $p \ge 0$

$$\frac{1}{p!L^p(z^0+re^{i\theta}\mathbf{b})}\left|\frac{\partial^p F(z^0+re^{i\theta}\mathbf{b})}{\partial \mathbf{b}^p}\right| \le$$

$$\le A_k \exp\int_0^r \left((N+1)L(z^0+te^{i\theta}\mathbf{b}) + N\frac{(-u_t'(z^0,\theta,t))^+}{L(z^0+te^{i\theta}\mathbf{b})}\right)dt\,, \qquad (14)$$

where $A_k = \max\left\{\frac{1}{k!\,L^k(z^0)}\left|\frac{\partial^k F(z^0)}{\partial \mathbf{b}^k}\right| : 0 \le k \le N\right\}$. *In particular,*

$$\left|F(z^0+re^{i\theta}\mathbf{b})\right| \le A_k \exp\int_0^r \left((N+1)L(z^0+te^{i\theta}\mathbf{b}) + N\frac{(-u_t'(z^0,\theta,t))^+}{L(z^0+te^{i\theta}\mathbf{b})}\right)dt\,.$$

If, in addition, $\frac{(-u'(r))^+}{L^2(z^0+re^{i\theta}\mathbf{b})}$ *tends to zero uniformly with* $r \to +\infty$ *for* $z^0 \in \mathbb{C}^n$, $\theta \in [0,2\pi]$, *then*

$$\sup_{z^0\in\mathbb{C}^n}\varlimsup_{r\to+\infty}\max\left\{\frac{\ln\left|F(z^0+re^{i\theta}\mathbf{b})\right|}{\int_0^r L(z^0+te^{i\theta}\mathbf{b})\,dt} : \theta\in[0,2\pi]\right\} \le N_{\mathbf{b}}(F,L)+1\,.$$

P r o o f. Denote $N = N_{\mathbf{b}}(F,L)$. For fixed $z^0 \in \mathbb{C}^n$ and $\theta \in [0,2\pi]$, we consider the function

$$g(r) = \max\left\{\frac{1}{k!\,L^k(z^0+re^{i\theta}\mathbf{b})}\left|\frac{\partial^k F(z^0+re^{i\theta}\mathbf{b})}{\partial \mathbf{b}^k}\right| : 0 \le k \le N\right\}. \qquad (15)$$

Since the function $\frac{1}{k!\,L^k(z^0+re^{i\theta}\mathbf{b})}\left|\frac{\partial^k F(z^0+re^{i\theta}\mathbf{b})}{\partial \mathbf{b}^k}\right|$ is continuously differentiable by real variable $r \in [0,+\infty)$, outside of a zero set of function $\frac{\partial^k F(z^0+re^{i\theta}\mathbf{b})}{\partial \mathbf{b}^k}$, the function g is continuously differentiable on $[0,+\infty)$ except, perhaps, a countable set of points. Since for every continuosly differentiable function g of real variable r the inequality $\frac{d}{dr}|g(r)| \le |g'(r)|$ holds with the exception of the points $\{r = t : g(t) = 0\}$, we estimate the derivative

$$\frac{d}{dr}\left(\frac{1}{k!\,u^k(r)}\left|\frac{\partial^k F(z^0+re^{i\theta}\mathbf{b})}{\partial \mathbf{b}^k}\right|\right) = \frac{1}{k!\,u^k(r)}\frac{d}{dr}\left|\frac{\partial^k F(z^0+re^{i\theta}\mathbf{b})}{\partial \mathbf{b}^k}\right| +$$

$$+\left|\frac{\partial^k F(z^0+re^{i\theta}\mathbf{b})}{\partial \mathbf{b}^k}\right|\frac{d}{dr}\frac{1}{k!\,u^k(r)} \le$$

$$\le \frac{1}{k!\,u^k(r)}\left|\sum_{j=1}^n \frac{\partial}{\partial z_j}\frac{\partial^k F(z^0+re^{i\theta}\mathbf{b})}{\partial \mathbf{b}^k}e^{i\theta}b_j\right| -$$

$$-\left|\frac{\partial^k F(z^0+re^{i\theta}\mathbf{b})}{\partial \mathbf{b}^k}\right|\frac{ku'(r)}{k!\,u^{k+1}(r)} \le$$

$$\le \frac{1}{k!\,u^k(r)}\left|\frac{\partial^{k+1} F(z^0+re^{i\theta}\mathbf{b})}{\partial \mathbf{b}^{k+1}}\right| +$$

$$+\frac{1}{k!\,u^k(r)}\left|\frac{\partial^k F(z^0+re^{i\theta}\mathbf{b})}{\partial \mathbf{b}^k}\right|\frac{k(-u'(r))^+}{u(r)}\,. \qquad (16)$$

For absolutely continuous functions $h_1, h_2, \dots, h_k$ and $h(x) := \max\{h_j(x) : 1 \le j \le k\}$, we have $h'(x) \le \max\{h_j'(x) : 1 \le j \le k\}$, $x \in [a,b]$ (see [11, p. 81]). The function g is absolutely continuous, therefore, it follows from (16) that

$$g'(r) \le \max\left\{\frac{d}{dr}\left(\frac{1}{k!\,L^k(z^0 + re^{i\theta}\mathbf{b})}\left|\frac{\partial^k F(z^0 + re^{i\theta}\mathbf{b})}{\partial \mathbf{b}^k}\right|\right) : 0 \le k \le N\right\} \le$$

$$\le \max\left\{\frac{1}{(k+1)!\,L^{k+1}(z^0 + re^{i\theta}\mathbf{b})}\left|\frac{\partial^{k+1} F(z^0 + re^{i\theta}\mathbf{b})}{\partial \mathbf{b}^{k+1}}\right| \times\right.$$

$$\times (k+1)L(z^0 + re^{i\theta}\mathbf{b}) + \frac{1}{k!\,L^k(z^0 + re^{i\theta}\mathbf{b})} \times$$

$$\left.\times \left|\frac{\partial^k F(z^0 + re^{i\theta}\mathbf{b})}{\partial \mathbf{b}^k}\right| k \frac{(-u_r'(z^0, \theta, r))^+}{L(z^0 + re^{i\theta}\mathbf{b})} : 0 \le k \le N\right\} \le$$

$$\le g(r)\left((N+1)L(z^0 + re^{i\theta}\mathbf{b}) + N\frac{(-u_r'(z^0, \theta, r))^+}{L(z^0 + re^{i\theta}\mathbf{b})}\right).$$

Thus, $\frac{d}{dr}\ln g(r) \le (N+1)L(z^0 + re^{i\theta}\mathbf{b}) + N\frac{(-u_r'(z^0, \theta, t))^+}{L(z^0 + re^{i\theta}\mathbf{b})}$ for every $z^0 \in \mathbb{C}^n$, $\theta \in [0, 2\pi]$ and $r > 0$ except, perhaps, a countable set of points.

Since F has bounded L-index in the direction $\mathbf{b}$, then $g(0) \ne 0$ and for each $r > 0$

$$\ln g(r) \le \ln g(0) + \int_0^r \left((N+1)L(z^0 + te^{i\theta}\mathbf{b}) + N\frac{(-u_t'(z^0, \theta, t))^+}{L(z^0 + te^{i\theta}\mathbf{b})}\right)dt.$$

Using (15), we obtain (14).

If, in addition, $\frac{(-u_r'(z^0, \theta, r))^+}{L^2(z^0 + re^{i\theta}\mathbf{b})} \to 0$ as $r \to \infty$ uniformly in $z^0 \in \mathbb{C}^n$ and $\theta \in [0, 2\pi]$, then

$$\ln g(r) \le \ln g(0) + (N+1)\int_0^r \left(L(z^0 + te^{i\theta}\mathbf{b}) + \frac{(-u_t'(z^0, \theta, t))^+}{L(z^0 + te^{i\theta}\mathbf{b})}\right)dt =$$

$$= \ln g(0) + (N+1)(1 + o(1))\int_0^r L(z^0 + te^{i\theta}\mathbf{b})\,dt,$$

as $r \to \infty$ uniformly in $z^0 \in \mathbb{C}^n$ and $\theta \in [0, 2\pi]$, so that for all $\theta \in [0, 2\pi]$, $z^0 \in \mathbb{C}^n$

$$|F(z^0 + re^{i\theta}\mathbf{b})| \le g(r) \le g(0)\exp\left\{(N+1)(1 + o(1))\int_0^r L(z^0 + te^{i\theta}\mathbf{b})\,dt\right\},$$

as $r \to \infty$, and we obtain that for every $z^0 \in \mathbb{C}^n$

$$\overline{\lim_{r\to+\infty}}\max\left\{\frac{\ln|F(z^0 + re^{i\theta}\mathbf{b})|}{\int_0^r L(z^0 + te^{i\theta}\mathbf{b})\,dt} : \theta \in [0, 2\pi]\right\} \le N_{\mathbf{b}}(F, L) + 1.$$

The proof of Theorem 2 is completed. ♦

Remark 4. *Note that* (14) *can be written in a more convenient form*

$$\max_{|\tau|=r}\left(\frac{1}{p!\,L^p(z^0+\tau\mathbf{b})}\left|\frac{\partial^p F(z^0+\tau\mathbf{b})}{\partial\mathbf{b}^p}\right|\right)\le$$

$$\le A_k\exp\max_{\theta\in[0,2\pi]}\int_0^r\left\{(N+1)L(z^0+te^{i\theta}\mathbf{b})+\right.$$

$$\left.+N\frac{(-u_t'(z^0,\theta,t))^+}{L(z^0+te^{i\theta}\mathbf{b})}\right\}dt\,. \tag{17}$$

For $n=1$ we deduce the following

Corollary 4. *Let* $\ell:\mathbb{C}\to\mathbb{R}_+$ *and, for* $\theta\in[0,2\pi]$, *a function* $\ell(re^{i\theta})$ *be a continuously differentiable function of real variable* $r\in[0,+\infty)$ *and* $u(r,\theta)=$ $=\ell(re^{i\theta})$. *If* $f(z)$ *is an entire function of bounded* ℓ*-index, then for every integer* $p\ge 0$

$$\frac{\left|f^{(p)}(re^{i\theta})\right|}{p!\,\ell^p(re^{i\theta})}\le B_k\exp\int_0^r\left\{(N+1)\ell(te^{i\theta})+N\frac{(-u_t'(\theta,t))^+}{\ell(te^{i\theta})}\right\}dt\,,$$

where $B_k=\max\left\{\dfrac{\left|f^{(k)}(0)\right|}{k!\,\ell^k(0)}:0\le k\le N\right\}$.

If, in addition, $\dfrac{(-u'(\theta,r))^+}{\ell^2(re^{i\theta})}\to 0$ *as* $r\to+\infty$ *uniformly in* $\theta\in[0,2\pi]$, *then*

$$\overline{\lim_{r\to+\infty}}\frac{\ln M(r,f)}{\max\limits_{\theta\in[0,2\pi]}\int_0^r\ell(te^{i\theta})\,dt}\le\overline{\lim_{r\to+\infty}}\max_{\theta\in[0,2\pi]}\frac{\ln\left|f(re^{i\theta})\right|}{\int_0^r\ell(te^{i\theta})\,dt}\le N(f,\ell)+1\,.$$

The Corollary 4 is a generalization of corresponding result of Sheremeta and Kuzyk [3] because we do not assume that $\ell(z)=\ell(|z|)$.

Corollary 5. *Let* $F:\mathbb{C}^n\to\mathbb{C}$ *be an entire function of bounded* L*-index in the direction* $\mathbf{b}\in\mathbb{C}^n\setminus\{0\}$, $N=N_{\mathbf{b}}(F,L)$, *and* z^0 *be a fixed point in* $\mathbb{C}^n$, *such that* $F(z^0)=1$. *Then for every* $r>0$

$$\int_0^r\frac{n(t,z^0,0,1/F)}{t}dt\le\ln\max\left\{\left|F(z^0+t\mathbf{b})\right|:|t|=r\right\}\le$$

$$\le\ln\max\left\{\frac{1}{p!\,L^p(z^0)}\left|\frac{\partial^p F(z^0)}{\partial\mathbf{b}^p}\right|:0\le k\le N\right\}+$$

$$+\max_{\theta\in[0,2\pi]}\int_0^r\left\{(N+1)L(z^0+te^{i\theta}\mathbf{b})+N\frac{(-u_{t'}(z^0,\theta,t))^+}{L(z^0+te^{i\theta}\mathbf{b})}\right\}dt\,.$$

P r o o f. The first inequality follows from the classical Jensen Theorem for function $F(z^0+t\mathbf{b})$ as a function of variable $t\in\mathbb{C}$. The second inequality follows from (17) for $p=0$. ♦

The Corollary 5 is a generalization of the results [5] for entire functions of bounded L-index in direction.

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ЗРОСТАННЯ ЦІЛИХ ФУНКЦІЙ ОБМЕЖЕНОГО L-ІНДЕКСУ ЗА НАПРЯМКОМ

Розв'язано задачу про знаходження оцінки зростання максимуму модуля цілої функції обмеженого L-індексу за напрямом. Отримано деякі узагальнення одновимірних результатів, встановлених раніше.

РОСТ ЦЕЛЫХ ФУНКЦИЙ ОГРАНИЧЕННОГО L-ИНДЕКСА ПО НАПРАВЛЕНИЮ

Решена задача о нахождении оценки роста максимума модуля целой функции ограниченного L-индекса по направлению. Получены некоторые обобщения одномерных результатов, установленных раньше.

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Received
08.02.16