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ON A SEMITOPOLOGICAL EXTENDED BICYCLIC SEMIGROUP WITH ADJOINED ZERO

In the paper it is shown that every Hausdorff locally compact semigroup topology on the extended bicyclic semigroup with adjoined zero $\mathcal{C}_{\mathbb{Z}}^0$ is discrete, but on $\mathcal{C}_{\mathbb{Z}}^0$ there exist $\mathfrak{c}$ many different Hausdorff locally compact shift-continuous topologies. Also, it is constructed on $\mathcal{C}_{\mathbb{Z}}^0$ the unique minimal shift-continuous topology and the unique minimal inverse semigroup topology.

Key words: *extended bicyclic semigroup, locally compact, semitopological semigroup, topological semigroup, minimal topological semigroup, discrete.*

Introduction and preliminaries. We follow the terminology of [13, 14, 17, 31]. In this paper all spaces are assumed to be Hausdorff. By $\mathbb{Z}$, $\mathbb{N}_0$ and $\mathbb{N}$ we denote the sets of all integers, non-negative integers and positive integers, respectively.

A *semigroup* is a non-empty set with a binary associative operation. A semigroup S is called *inverse* if every $a \in S$ possesses an unique inverse in S, i.e. if there exists a unique element $a^{-1} \in S$ such that

$$a \cdot a^{-1} \cdot a = a \qquad \text{and} \qquad a^{-1} \cdot a \cdot a^{-1} = a^{-1}.$$

A map that associates to any element of an inverse semigroup its inverse is called the *inversion*.

For a semigroup S, by $E(S)$ we denote the subset of all idempotents in S. If $E(S)$ is closed under multiplication, then we shall refer to $E(S)$ as the *band* of S. The semigroup operation on S determines the following partial order $\preceq$ on $E(S)$: $e \preceq f$ if and only if $ef = fe = e$. This order is called the *natural partial order* on $E(S)$. A *semilattice* is a commutative semigroup of idempotents. A semilattice E is called *linearly ordered* or a *chain* if its natural partial order is a linear order. A *maximal chain* of a semilattice E is a chain which is not properly contained in any other chain of E.

The Axiom of Choice implies the existence of maximal chains in every partially ordered set. According to [30, Definition II.5.12], a chain L is called an ω-chain if L is order-isomorphic to $\{0,-1,-2,-3,\ldots\}$ with the usual order $\leq$ or, equivalently, if L is isomorphic to $(\mathbb{N}_0, \max)$.

The bicyclic semigroup (or the *bicyclic monoid*) $\mathcal{C}(p,q)$ is the semigroup with the identity 1 that is generated by two elements p and q subjected only to the condition $pq = 1$. The bicyclic monoid $\mathcal{C}(p,q)$ is a combinatorial bisimple F-inverse semigroup (see [28]) and it plays an important role in the algebraic theory of semigroups and in the theory of topological semigroups. For example, the well-known Andersen's result [2] states that a (0-)simple semigroup is completely (0-)simple if and only if it does not contain the bicyclic semigroup. The bicyclic semigroup cannot be embedded into stable semigroups [27].

A (*semi*)*topological semigroup* is a topological space with a (separately) continuous semigroup operation. An inverse topological semigroup with the continuous inversion is called a *topological inverse semigroup*. A topology τ on a semigroup S is called:

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- *shift-continuous* if (S,τ) is a semitopological semigroup;
- *semigroup* if (S,τ) is a topological semigroup;
- *inverse semigroup* if (S,τ) is a topological inverse semigroup.

The bicyclic semigroup admits only the discrete semigroup topology and if a topological semigroup S contains it as a dense subsemigroup, then $\mathcal{C}(p,q)$ is an open subset of S [16]. Bertman and West in [12] extended this result for the case of Hausdorff semitopological semigroups. Stable and Γ-compact topological semigroups do not contain the bicyclic semigroup [3, 25]. The problem of embedding of the bicyclic monoid into compact-like topological semigroups was studied in [4, 5, 10, 24]. Also, in the paper [18] it was proved that the discrete topology is the unique topology on the extended bicyclic semigroup $\mathcal{C}_{\mathbb{Z}}$ such that the semigroup operation on $\mathcal{C}_{\mathbb{Z}}$ is separately continuous. A unexpected dichotomy for the bicyclic monoid with adjoined zero $\mathcal{C}^0 = \mathcal{C}(p,q) \sqcup \{0\}$ was found in [20]: every Hausdorff locally compact semitopological bicyclic semigroup with adjoined zero $\mathcal{C}^0$ is either compact or discrete.

The above dichotomy was extended by Bardyla in [7] to locally compact λ-polycyclic semitopological monoids, and in [8] to locally compact semitopological graph inverse semigroups and also by the authors in [21] to locally compact semitopological interassociates of the bicyclic monoid with an adjoined zero, and in [19] to locally compact semitopological 0-bisimple inverse ω-semigroups with compact maximal subgroups. The lattice of all weak shift-continuous topologies on $\mathcal{C}^0$ is described in [9].

On the Cartesian product $\mathcal{C}_{\mathbb{Z}} = \mathbb{Z} \times \mathbb{Z}$ we define the semigroup operation as follows:

$$(a,b)(c,d) = \begin{cases} (a-b+c,d), & b < c, \\ (a,d), & b = c, \\ (a,d+b-c), & b > c, \end{cases} \tag{1}$$

for $a,b,c,d \in \mathbb{Z}$. The set $\mathcal{C}_{\mathbb{Z}}$ with the operation defined above is called the *extended bicyclic semigroup* [33].

In [18] the algebraic properties of $\mathcal{C}_{\mathbb{Z}}$ were described. It was proved there that every non-trivial congruence $\mathfrak{C}$ on the semigroup $\mathcal{C}_{\mathbb{Z}}$ is a group congruence, and moreover, the quotient semigroup $\mathcal{C}_{\mathbb{Z}}/\mathfrak{C}$ is isomorphic to a cyclic group. It was shown that the semigroup $\mathcal{C}_{\mathbb{Z}}$ as a Hausdorff semitopological semigroup admits only the discrete topology and also the closure $\mathrm{cl}_T(\mathcal{C}_{\mathbb{Z}})$ of the semigroup $\mathcal{C}_{\mathbb{Z}}$ in a topological semigroup T was studied there.

In [22] we proved that the group $\mathrm{Aut}(\mathcal{C}_{\mathbb{Z}})$ of automorphisms of the extended bicyclic semigroup $\mathcal{C}_{\mathbb{Z}}$ is isomorphic to the additive group of integers.

By $\mathcal{C}_{\mathbb{Z}}^0$ we denote the extended bicyclic semigroup $\mathcal{C}_{\mathbb{Z}}$ with adjoined zero 0.

In this paper we show that every Hausdorff locally compact semigroup topology on the semigroup $\mathcal{C}_{\mathbb{Z}}^0$ is discrete, but on $\mathcal{C}_{\mathbb{Z}}^0$ there exist $\mathfrak{c}$ many different Hausdorff locally compact shift-continuous topologies. Also, we construct on $\mathcal{C}_{\mathbb{Z}}^0$ the unique minimal shift-continuous topology and the unique minimal inverse semigroup topology.

1. Locally compact shift-continuous topologies on the extended bicyclic semigroup. We need the following simple statement:

Proposition 1 [18, Proposition 2.1 (viii)**]**. *For every integer* n *the set*

$$\mathcal{C}_{\mathbb{Z}}[n]=\{(a,b): a\ge n \wedge b\ge n\}$$

is an inverse subsemigroup of $\mathcal{C}_{\mathbb{Z}}$ *that is isomorphic to the bicyclic semigroup* $\mathcal{C}(p,q)$ *by the map*

$$h:\mathcal{C}_{\mathbb{Z}}[n]\to\mathcal{C}(p,q), \qquad (a,b)\mapsto q^{a-n}p^{b-n}.$$

Proposition 1 implies the following

Corollary 1. *For every integer* n *the set* $\mathcal{C}^0_{\mathbb{Z}}[n]=\mathcal{C}_{\mathbb{Z}}[n]\sqcup\{0\}$ *is an inverse subsemigroup of* $\mathcal{C}^0_{\mathbb{Z}}$ *that is isomorphic to the bicyclic monoid* $\mathcal{C}^0$ *with adjoined zero by the map* $h:\mathcal{C}^0_{\mathbb{Z}}[n]\to\mathcal{C}^0$, $(a,b)\mapsto q^{a-n}p^{b-n}$ *and* $0\mapsto 0$.

Lemma 1. *Let* τ *be a non-discrete Hausdorff shift-continuous topology on* $\mathcal{C}^0_{\mathbb{Z}}$. *Then* $\mathcal{C}^0_{\mathbb{Z}}[n]$ *is a non-discrete subsemigroup of* $(\mathcal{C}^0_{\mathbb{Z}},\tau)$ *for any integer* n.

P r o o f. First we observe that by Theorem 1 from [18] all non-zero elements of the semigroup $\mathcal{C}^0_{\mathbb{Z}}$ are isolated points in $(\mathcal{C}^0_{\mathbb{Z}},\tau)$.

Suppose to the contrary that there exist a non-discrete Hausdorff shift-continuous topology τ on $\mathcal{C}^0_{\mathbb{Z}}$ and an integer n such that $\mathcal{C}^0_{\mathbb{Z}}[n]$ is a discrete subsemigroup of $(\mathcal{C}^0_{\mathbb{Z}},\tau)$. Fix an arbitrary open neighbourhood $U(0)$ of zero 0 in $(\mathcal{C}^0_{\mathbb{Z}},\tau)$ such that $U(0)\cap\mathcal{C}^0_{\mathbb{Z}}[n]=\{0\}$. Then the separate continuity of the semigroup operation in $(\mathcal{C}^0_{\mathbb{Z}},\tau)$ implies that there exists an open neighbourhood $V(0)\subseteq U(0)$ of zero 0 in $(\mathcal{C}^0_{\mathbb{Z}},\tau)$ such that $(n,n)\cdot V(0)\cdot(n,n)\subseteq U(0)$. Our assumption implies that every open neighbourhood $W(0)\subseteq U(0)$ of zero 0 in $(\mathcal{C}^0_{\mathbb{Z}},\tau)$ contains infinitely many points (x,y) such that $x\le n$ or $y\le n$. Then for any non-zero $(x,y)\in V(0)$ by formula (1) we have that

$$(n,n)\cdot(x,y)\cdot(n,n)=(n,n-x+y)\cdot(n,n)=\begin{cases}(n+x-y,n), & y\le x,\\ (n,n-x+y), & y\ge x,\end{cases}$$

and hence $(n,n)\cdot V(0)\cdot(n,n)\cap\mathcal{C}_{\mathbb{Z}}[n]\neq\varnothing$ which contradicts the assumption $U(0)\cap\mathcal{C}^0_{\mathbb{Z}}[n]=\{0\}$. The obtained contradiction implies the statement of the lemma. ♦

For an arbitrary non-zero element $(a,b)\in\mathcal{C}^0_{\mathbb{Z}}$ we denote

$$\uparrow_{\preceq}(a,b)=\{(x,y)\in\mathcal{C}_{\mathbb{Z}}:(a,b)\preceq(x,y)\}$$

where $\preceq$ is the natural partial order on $\mathcal{C}^0_{\mathbb{Z}}$. It is obvious that

$$\uparrow_{\preceq}(a,b)=\{(x,y)\in\mathcal{C}_{\mathbb{Z}}: a-b=x-y,\ x\le a \quad \text{in } (\mathbb{Z},\le)\}.$$

Lemma 2. *Let* $(a,b),(c,d),(e,f)\in\mathcal{C}_{\mathbb{Z}}$ *be such that* $(a,b)\cdot(c,d)=(e,f)$. *Then the following statements hold*:

(***i***) *if* $b\le c$ *then* $(x,y)\cdot(c,d)=(e,f)$ *for any* $(x,y)\in\uparrow_{\preceq}(a,b)$, *and moreover, there exists a minimal element* $(\hat{a},\hat{b})\preceq(a,b)$ *in* $\mathcal{C}^0_{\mathbb{Z}}$ *such that* $(\hat{a},\hat{b})\cdot(c,d)=(e,f)$. *Also, there exist no other elements* $(x,y)\in\mathcal{C}_{\mathbb{Z}}$ *with the property* $(x,y)\cdot(c,d)=(e,f)$;

(***ii***) *if* $b \geq c$ *then* $(a,b)\cdot(x,y)=(e,f)$ *for any* $(x,y)\in\uparrow_{\preceq}(c,d)$, *and moreover, there exists a minimal element* $(\hat{c},\hat{d})\preceq(c,d)$ *in* $\mathcal{C}_{\mathbb{Z}}$ *such that* $(a,b)\cdot(\hat{c},\hat{d})=(e,f)$. *Also, there exist no other elements* $(x,y)\in\mathcal{C}_{\mathbb{Z}}$ *with the property* $(a,b)\cdot(x,y)=(e,f)$.

P r o o f. (***i***). Since $b\leq c$, the semigroup operation of $\mathcal{C}_{\mathbb{Z}}$ implies that $(b,b)\cdot(c,d)=(c,d)$. Also, if $(a,b)\preceq(x,y)$, then Lemma 1.4.6(5) from [28] implies that

$$(x,y)\cdot(b,b)=(x,y)\cdot(a,b)^{-1}\cdot(a,b)=(a,b),$$

and hence we have that

$$(x,y)\cdot(c,d)=(x,y)\cdot((b,b)\cdot(c,d))=$$

$$=((x,y)\cdot(b,b))\cdot(c,d)=(a,b)\cdot(c,d)=(e,f).$$

We put $(\hat{a},\hat{b})=(a-b+c,c)$. Then $(\hat{a},\hat{b})\preceq(a,b)$ and formula (1) implies that the element $(\hat{a},\hat{b})$ is required.

The last statement follows from Proposition 2.1 of [18] and formula (1).

The proof of statement (***ii***) is similar. ♦

Lemma 3. *Let* τ *be a non-discrete Hausdorff shift-continuous topology on* $\mathcal{C}_{\mathbb{Z}}^0$. *Then the natural partial order* $\preceq$ *is closed on* $(\mathcal{C}_{\mathbb{Z}}^0,\tau)$ *and* $\uparrow_{\preceq}(a,b)$ *is an open-and-closed subset of* $(\mathcal{C}_{\mathbb{Z}}^0,\tau)$ *for any non-zero element* (a,b) *of* $\mathcal{C}_{\mathbb{Z}}^0$.

P r o o f. By Theorem 1 of [18] all non-zero elements of the semigroup $\mathcal{C}_{\mathbb{Z}}^0$ are isolated points in $(\mathcal{C}_{\mathbb{Z}}^0,\tau)$. Since $0\preceq(a,b)$ for any $(a,b)\in\mathcal{C}_{\mathbb{Z}}^0$, the above implies the first statement of the lemma.

The definition of the natural partial order $\preceq$ on $\mathcal{C}_{\mathbb{Z}}^0$ and the separate continuity of the semigroup operation on $(\mathcal{C}_{\mathbb{Z}}^0,\tau)$ imply the second statement, because

$$\uparrow_{\preceq}(a,b)=\{(x,y)\in\mathcal{C}_{\mathbb{Z}}^0:(a,a)\cdot(x,y)=(a,b)\}.$$ ♦

Proposition 2. *Let the semigroup* $\mathcal{C}_{\mathbb{Z}}^0$ *admits a non-discrete Hausdorff locally compact shift-continuous topology* τ. *Then the following statements hold:*

(***i***) *for any open neighbourhood* $U(0)$ *of zero there exists a compact-and-open neighbourhood* $V(0)\subseteq U(0)$ *of* 0 *in* $(\mathcal{C}_{\mathbb{Z}}^0,\tau)$;

(***ii***) *the set* $\uparrow_{\preceq}(a,b)\cap U(0)$ *is finite for any compact-and-open neighbourhood* $V(0)\subseteq U(0)$ *of the zero* 0 *in* $(\mathcal{C}_{\mathbb{Z}}^0,\tau)$ *and any non-zero element* (a,b) *of* $\mathcal{C}_{\mathbb{Z}}^0$;

(***iii***) *for any open neighbourhood* $U(0)$ *of zero in* $(\mathcal{C}_{\mathbb{Z}}^0,\tau)$ *and any integer* n *the set* $U(0)\setminus\mathcal{C}_{\mathbb{Z}}^0[n]$ *is finite.*

P r o o f. Statement (***i***) follows from Theorem 1 of [18] and the local compactness of the space $(\mathcal{C}_{\mathbb{Z}}^0,\tau)$.

Statement (***ii***) follows from Lemma 3 and Theorem 1 of [18].

(***iii***). It is obvious that $\mathcal{C}_{\mathbb{Z}}^0[n]=(n,n)\cdot\mathcal{C}_{\mathbb{Z}}^0\cdot(n,n)$ for any integer n. This implies that $\mathcal{C}_{\mathbb{Z}}^0[n]$ is a closed subset of $(\mathcal{C}_{\mathbb{Z}}^0,\tau)$ because $\mathcal{C}_{\mathbb{Z}}^0[n]$ is a retract of the space $(\mathcal{C}_{\mathbb{Z}}^0,\tau)$, and hence by Corollary 3.3.10 from [17] it is locally compact. Since the topology τ is non-discrete, Lemma 1 and Theorem 1 from [20] implies that $\mathcal{C}_{\mathbb{Z}}^0[n]$ is a compact subspace of $(\mathcal{C}_{\mathbb{Z}}^0,\tau)$. Finally, we apply Theorem 1 from [18]. ♦

Next we shall construct an example of a non-discrete Hausdorff locally compact shift-continuous topology on the semigroup $\mathcal{C}_{\mathbb{Z}}^0$ that is neither compact nor discrete.

Example 1. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be two increasing sequences of positive integers with the following properties: $x_1, y_1 > 1$ and

$$x_n + 1 < x_{n+1}, \quad 2 < y_n + 1 < y_{n+1}$$

for any $n \in \mathbb{N}$.

We denote

$$A_0 = \uparrow_{\preceq}(0,0) \cup \bigcup_{i=1}^{x_1-1} \uparrow_{\preceq}(0,-i) \cup \bigcup_{i=1}^{y_1-1} \uparrow_{\preceq}(-j,0)$$

and

$$A_n^d = \bigcup_{i=x_n}^{x_{n+1}-1} \uparrow_{\preceq}(-x_n,-i), \quad A_n^{\ell} = \bigcup_{j=y_n}^{y_{n+1}-1} \uparrow_{\preceq}(-j,-y_n)$$

for any positive integer n.

Next, we put $D = A_0 \cup \bigcup_{i\in\mathbb{N}} (A_i^d \cup A_i^{\ell})$. For finitely many $(a_1,b_1),\ldots,(a_k,b_k) \in \mathcal{C}_{\mathbb{Z}}$ we denote

$$U_{(a_1,b_1),\ldots,(a_k,b_k)} = \mathcal{C}_{\mathbb{Z}}^0 \setminus \left(D \cup \uparrow_{\preceq}(a_1,b_1) \cup \ldots \cup \uparrow_{\preceq}(a_k,b_k)\right).$$

We define a topology $\tau_{\{x_n\}}^{\{y_n\}}$ on the semigroup $\mathcal{C}_{\mathbb{Z}}^0$ in the following way:

1°) all non-zero elements of $\mathcal{C}_{\mathbb{Z}}^0$ are isolated points;

2°) the family $\mathscr{B}^0_{\tau_{\{x_n\}}^{\{y_n\}}} = \left\{U_{(a_1,b_1),\ldots,(a_k,b_k)} : (a_1,b_1),\ldots,(a_k,b_k) \in \mathcal{C}_{\mathbb{Z}},\ k \in \mathbb{N}\right\}$ is the base of the topology $\tau_{\{x_n\}}^{\{y_n\}}$ at zero 0.

Proposition 3.

(***i***) *the set* $\uparrow_{\preceq}(a,b) \setminus D$ *is finite for any* $(a,b) \in \mathcal{C}_{\mathbb{Z}}$.

(***ii***) D *is a compact subset of the space* $(\mathcal{C}_{\mathbb{Z}}^0, \tau_{\{x_n\}}^{\{y_n\}})$.

(***iii***) *the space* $(\mathcal{C}_{\mathbb{Z}}^0, \tau_{\{x_n\}}^{\{y_n\}})$ *is locally compact and Hausdorff.*

P r o o f. (***i***) The statement is trivial for $(a,b) \in D$. Assume that $(a,b) \notin D$ and consider the following cases.

(***a***) If $a = b$, then $\uparrow_{\preceq}(a,b) \setminus D = \{(1,1),\ldots,(a,a)\}$.

(***b***) Suppose that $a < b$. Then either there exists a positive integer $i \ge 1$ such that $y_i \le b - a < y_{i+1}$ or $b - a < y_1$. In the first case we have that

$$\uparrow_{\preceq}(a,b) \setminus D = \{(-i+1-b+a,-i+1),\ldots,(a,b)\} =$$

$$= \bigcup \{(k-b+a,k) : k = -i+1,\ldots,b\}.$$

In the second case we have that $b > 0$ and hence

$$\uparrow_{\preceq}(a,b)\setminus D=\{(1-b+a,1),\ldots,(a,b)\}=$$

$$=\bigcup\{(k-b+a,k):k=1,\ldots,b\}.$$

(*c*) Suppose that $a>b$. Then either there exists a positive integer $j\geq 1$ such that $x_j\leq a-b<x_{j+1}$ or $a-b<x_1$. In the first case we have that

$$\uparrow_{\preceq}(a,b)\setminus D=\{(-j+1,-j+1-a+b),\ldots,(a,b)\}=$$

$$=\bigcup\{(k-a+b,k):k=-j+1,\ldots,a\}.$$

In the second case we have that $a>0$ and hence

$$\uparrow_{\preceq}(a,b)\setminus D=\{(1,1-a+b),\ldots,(a,b)\}=$$

$$=\bigcup\{(k,k-a+b):k=1,\ldots,a\}.$$

Statement (***i***) is proved. Statement (***ii***) follows from (***i***).

Since all non-zero elements of $\mathcal{C}_{\mathbb{Z}}^0$ are isolated points in $(\mathcal{C}_{\mathbb{Z}}^0,\tau_{\{x_n\}}^{\{y_n\}})$, statement (***iii***) follows from (***ii***). ♦

For any non-zero element (a,b) of $\mathcal{C}_{\mathbb{Z}}^0$ we denote

$$S^{\underline{b}\uparrow}=\{(x,y)\in\mathcal{C}_{\mathbb{Z}}:y\geq b\}\cup\{0\},$$

$$S^{\overrightarrow{|a}}=\{(x,y)\in\mathcal{C}_{\mathbb{Z}}:x\geq a\}\cup\{0\}.$$

It is obvious that $(a,b)\mathcal{C}_{\mathbb{Z}}^0=S^{\overrightarrow{|a}}$ and $\mathcal{C}_{\mathbb{Z}}^0(a,b)=S^{\underline{b}\uparrow}$ for any non-zero $(a,b)\in\mathcal{C}_{\mathbb{Z}}^0$.

Theorem 1. $(\mathcal{C}_{\mathbb{Z}}^0,\tau_{\{x_n\}}^{\{y_n\}})$ *is a semitopological semigroup.*

P r o o f. By the definition of the topology $\tau_{\{x_n\}}^{\{y_n\}}$ it is sufficient to prove that the left and right shifts of $\mathcal{C}_{\mathbb{Z}}^0$ are continuous at zero 0.

Fix any non-zero element $(a,b)\in\mathcal{C}_{\mathbb{Z}}^0$ and any basic open neighbourhood $U_{(a_1,b_1),\ldots,(a_k,b_k)}$ of zero 0 in $(\mathcal{C}_{\mathbb{Z}}^0,\tau_{\{x_n\}}^{\{y_n\}})$.

The definition of the topology $\tau_{\{x_n\}}^{\{y_n\}}$ implies that there exist finitely many non-zero elements $(e_1,f_1),\ldots,(e_m,f_m)$ of the semigroup $\mathcal{C}_{\mathbb{Z}}^0$ with $e_1,\ldots,e_m\geq a$ such that

$$U_{(a_1,b_1),\ldots,(a_k,b_k)}\cap S^{\overrightarrow{|a}}=S^{\overrightarrow{|a}}\setminus(\uparrow_{\preceq}(e_1,f_1)\cup\ldots\cup\uparrow_{\preceq}(e_m,f_m)).$$

Since $(a,b)\mathcal{C}_{\mathbb{Z}}^0=S^{\overrightarrow{|a}}$, by Lemma 2 (***ii***) there exist minimal elements $(\hat{c}_1,\hat{d}_1),\ldots,(\hat{c}_m,\hat{d}_m)$ in $\mathcal{C}_{\mathbb{Z}}$ such that

$$(a,b)\cdot(\hat{c}_1,\hat{d}_1)=(e_1,f_1),\quad\ldots,\quad(a,b)\cdot(\hat{c}_m,\hat{d}_m)=(e_m,f_m).$$

Then the last equalities imply that

$$(a,b)\cdot U_{(\hat{c}_1,\hat{d}_1),\ldots,(\hat{c}_m,\hat{d}_m)}\subseteq U_{(a_1,b_1),\ldots,(a_k,b_k)}.$$

Similarly, there exist finitely many non-zero elements $(e_1,f_1),\ldots,(e_p,f_p)$

of the semigroup $\mathcal{C}_{\mathbb{Z}}^0$ with $f_1,\ldots,f_p \geq b$ such that

$$U_{(a_1,b_1),\ldots,(a_k,b_k)} \cap S^{\underline{b}\uparrow} = S^{\underline{b}\uparrow} \setminus (\uparrow_{\preceq}(e_1,f_1) \cup \ldots \cup \uparrow_{\preceq}(e_p,f_p)).$$

Since $\mathcal{C}_{\mathbb{Z}}^0(a,b) = S^{\underline{b}\uparrow}$, by Lemma 2 (***i***) there exist minimal elements $(\hat{c}_1,\hat{d}_1),\ldots,(\hat{c}_p,\hat{d}_p)$ in $\mathcal{C}_{\mathbb{Z}}$ such that

$$(\hat{c}_1,\hat{d}_1)\cdot(a,b) = (e_1,f_1), \quad \ldots, \quad (\hat{c}_p,\hat{d}_p)\cdot(a,b) = (e_p,f_p).$$

Then the last equalities imply that $U_{(\hat{c}_1,\hat{d}_1),\ldots,(\hat{c}_p,\hat{d}_p)}(a,b) \subseteq U_{(a_1,b_1),\ldots,(a_k,b_k)}$, which completes the proof of the separate continuity of the semigroup operation in $(\mathcal{C}_{\mathbb{Z}}^0, \tau_{\{x_n\}}^{\{y_n\}})$. ♦

If in Example 1 we put $x_i = y_i$ for any $i \in \mathbb{N}$ and denote $\tau_{\{x_n\}} = \tau_{\{x_n\}}^{\{y_n\}}$, then $(U_{(a_1,b_1),\ldots,(a_k,b_k)})^{-1} = U_{(b_1,a_1),\ldots,(b_k,a_k)}$ for any $a_1,b_1,\ldots,a_k,b_k \in \mathbb{Z}$. This and Theorem 1 imply the following corollary:

Corollary 2. $(\mathcal{C}_{\mathbb{Z}}^0, \tau_{\{x_n\}}^{\{y_n\}})$ *is a Hausdorff locally compact semitopological semigroup with continuous inversion.*

Theorem 1 implies that on the semigroup $\mathcal{C}_{\mathbb{Z}}^0$ there exist $\mathfrak{c}$ many Hausdorff locally compact shift-continuous topologies. But Lemma 1 implies the following counterpart of Corollary 1 from [20]:

Corollary 3. *Every Hausdorff locally compact semigroup topology on the semigroup* $\mathcal{C}_{\mathbb{Z}}^0$ *is discrete.*

2. Minimal shift-continuous and inverse semigroup topologies on $\mathcal{C}_{\mathbb{Z}}^0$. The concept of a minimal topological group was introduced independently in the early 1970's by Doîtchinov [15] and Stephenson [32]. Both authors were motivated by the theory of minimal topological spaces, which was well understood at that time (cf. [11]). More than 20 years earlier Nachbin [29] had studied minimality in the context of division rings, and Banaschewski [6] investigated minimality in the more general setting of topological algebras. The concept of a minimal topological semigroup was introduced in [23].

Definition 1 **[**23**]**. A Hausdorff semitopological (respectively, topological, topological inverse) semigroup (S,τ) is said to be *minimal* if no Hausdorff shift-continuous (respectively, semigroup, semigroup inverse) topology on S is strictly contained in τ. If (S,τ) is minimal semitopological (respectively, topological, topological inverse) semigroup, then τ is called *minimal shift-continuous* (respectively, *semigroup, semigroup inverse*) topology.

It is obvious that every Hausdorff compact shift-continuous (respectively, semigroup, semigroup inverse) topology on a semigroup S is a minimal shift-continuous (respectively, semigroup, semigroup inverse) topology on S. But an infinite semigroup of matrix units admits a unique compact shift-continuous topology and non-compact minimal semigroup and inverse semigroup topologies [23]. Similar results were obtained in [9] for the bicyclic monoid with adjoined zero $\mathcal{C}^0$.

Example 2. For finitely many $(a_1,b_1),\ldots,(a_k,b_k) \in \mathcal{C}_{\mathbb{Z}}$ we denote

$$U^{\uparrow}_{(a_1,b_1),\ldots,(a_k,b_k)} = \mathcal{C}_{\mathbb{Z}}^0 \setminus (\uparrow_{\preceq}(a_1,b_1) \cup \ldots \cup \uparrow_{\preceq}(a_k,b_k)).$$

We define a topology $\tau_{\min}^{\mathrm{sh}}$ on the semigroup $\mathcal{C}_{\mathbb{Z}}^0$ in the following way:

1°) all non-zero elements of $\mathcal{C}_{\mathbb{Z}}^0$ are isolated points;

2°) the family $\mathscr{B}^0_{\tau^{\mathrm{sh}}_{\min}} = \{U^{\uparrow}_{(a_1,b_1),\ldots,(a_k,b_k)} : (a_1,b_1),\ldots,(a_k,b_k) \in \mathcal{C}_{\mathbb{Z}},\ k \in \mathbb{N}\}$ is the base of the topology $\tau^{\mathrm{sh}}_{\min}$ at zero 0.

We observe that by Lemma 3 the space $(\mathcal{C}^0_{\mathbb{Z}}, \tau^{\mathrm{sh}}_{\min})$ is Hausdorff, 0-dimensional and scattered, and hence it is regular. Since the base $\mathscr{B}^0_{\tau^{\mathrm{sh}}_{\min}}$ is countable, by the Urysohn Metrization Theorem (see [26, p. 123, Theorem 16]) the space $(\mathcal{C}^0_{\mathbb{Z}}, \tau^{\mathrm{sh}}_{\min})$ is metrizable and hence by Corollary 4.1.13 from [17] it is perfectly normal.

Proposition 4. $(\mathcal{C}^0_{\mathbb{Z}}, \tau^{\mathrm{sh}}_{\min})$ *is a minimal semitopological semigroup with continuous inversion.*

P r o o f. The definition of the topology $\tau^{\mathrm{sh}}_{\min}$ implies that it is sufficient to prove that the left and right shifts of $\mathcal{C}^0_{\mathbb{Z}}$ are continuous at zero 0.

Fix any non-zero element $(a,b) \in \mathcal{C}^0_{\mathbb{Z}}$ and any basic open neighbourhood $U^{\uparrow}_{(a_1,b_1),\ldots,(a_k,b_k)}$ of zero 0 in $(\mathcal{C}^0_{\mathbb{Z}}, \tau^{\mathrm{sh}}_{\min})$.

The definition of the topology $\tau^{\mathrm{sh}}_{\min}$ implies that there exist finitely many non-zero elements $(e_1,f_1),\ldots,(e_m,f_m)$ of the semigroup $\mathcal{C}^0_{\mathbb{Z}}$ with $e_1,\ldots,e_m \ge a$ such that

$$U^{\uparrow}_{(a_1,b_1),\ldots,(a_k,b_k)} \cap S^{\overrightarrow{|a}} = S^{\overrightarrow{|a}} \setminus (\uparrow_{\preceq} (e_1,f_1) \cup \ldots \cup \uparrow_{\preceq} (e_m,f_m)).$$

Since $(a,b)\mathcal{C}^0_{\mathbb{Z}} = S^{\overrightarrow{|a}}$, by Lemma 2 (***ii***) there exist minimal elements $(\hat{c}_1,\hat{d}_1),\ldots,(\hat{c}_m,\hat{d}_m)$ in $\mathcal{C}_{\mathbb{Z}}$ such that

$$(a,b)\cdot(\hat{c}_1,\hat{d}_1) = (e_1,f_1), \quad \ldots, \quad (a,b)\cdot(\hat{c}_m,\hat{d}_m) = (e_m,f_m).$$

Then the last equalities imply that

$$(a,b)\cdot U^{\uparrow}_{(\hat{c}_1,\hat{d}_1),\ldots,(\hat{c}_m,\hat{d}_m)} \subseteq U_{(a_1,b_1),\ldots,(a_k,b_k)}.$$

Again, by similar way there exists finitely many non-zero elements $(e_1,f_1),\ldots,(e_p,f_p)$ of the semigroup $\mathcal{C}^0_{\mathbb{Z}}$ with $f_1,\ldots,f_p \ge b$ such that

$$U^{\uparrow}_{(a_1,b_1),\ldots,(a_k,b_k)} \cap S^{\underline{b}\uparrow} = S^{\underline{b}\uparrow} \setminus (\uparrow_{\preceq} (e_1,f_1) \cup \ldots \cup \uparrow_{\preceq} (e_p,f_p)).$$

Since $\mathcal{C}^0_{\mathbb{Z}}(a,b) = S^{\underline{b}\uparrow}$, Lemma 2 (***i***) implies that there exist minimal elements $(\hat{c}_1,\hat{d}_1),\ldots,(\hat{c}_p,\hat{d}_p)$ in $\mathcal{C}_{\mathbb{Z}}$ such that

$$(\hat{c}_1,\hat{d}_1)\cdot(a,b) = (e_1,f_1), \quad \ldots, \quad (\hat{c}_m,\hat{d}_m)\cdot(a,b) = (e_p,f_p).$$

Then the last equalities imply that $U^{\uparrow}_{(\hat{c}_1,\hat{d}_1),\ldots,(\hat{c}_p,\hat{d}_p)}\cdot(a,b) \subseteq U_{(a_1,b_1),\ldots,(a_k,b_k)}$, which completes the proof of the separate continuity of the semigroup operation in $(\mathcal{C}^0_{\mathbb{Z}}, \tau^{\mathrm{sh}}_{\min})$.

Also, since $(U^{\uparrow}_{(a_1,b_1),\ldots,(a_k,b_k)})^{-1} = U^{\uparrow}_{(b_1,a_1),\ldots,(b_k,a_k)}$ for any $(a_1,b_1),\ldots,(a_k,b_k) \in$ $\in \mathcal{C}_{\mathbb{Z}}$, the inversion is continuous in $(\mathcal{C}^0_{\mathbb{Z}}, \tau^{\mathrm{sh}}_{\min})$ as well.

Lemma 3 implies that $\tau_{\min}^{\mathrm{sh}}$ is the coarsest Hausdorff shift-continuous topology on $\mathcal{C}_{\mathbb{Z}}^{0}$ and hence $(\mathcal{C}_{\mathbb{Z}}^{0}, \tau_{\min}^{\mathrm{sh}})$ is a minimal semitopological semigroup.

Example 3. We define a topology $\tau_{\min}^{\mathrm{i}}$ on the semigroup $\mathcal{C}_{\mathbb{Z}}^{0}$ in the following way:

1°) all non-zero elements of $\mathcal{C}_{\mathbb{Z}}^{0}$ are isolated points in the topological space $(\mathcal{C}_{\mathbb{Z}}^{0}, \tau_{\min}^{\mathrm{i}})$;

2°) the family $\mathscr{B}_{\tau_{\min}^{i}}^{0} = \{S^{\overrightarrow{|a}} \cap S^{\underline{b}\uparrow} : a, b \in \mathbb{Z}\}$ is the base of the topology $\tau_{\min}^{\mathrm{i}}$ at zero 0.

It is obvious that the space $(\mathcal{C}_{\mathbb{Z}}^{0}, \tau_{\min}^{\mathrm{sh}})$ is Hausdorff, 0-dimensional and scattered and hence it is regular. Since the base $\mathscr{B}_{\tau_{\min}^{\mathrm{i}}}^{0}$ is countable, similarly as in Example 2 we get that the space $(\mathcal{C}_{\mathbb{Z}}^{0}, \tau_{\min}^{\mathrm{i}})$ is metrizable.

Proposition 5. $(\mathcal{C}_{\mathbb{Z}}^{0}, \tau_{\min}^{\mathrm{i}})$ *is a minimal topological inverse semigroup.*

P r o o f. We have that for any $a, b \in \mathbb{Z}$ and any non-zero element $(x, y) \in \mathcal{C}_{\mathbb{Z}}^{0}$ there exists an integer n such that $(x, y) \in \mathcal{C}_{\mathbb{Z}}^{0}[n]$ and $S^{\overrightarrow{|a}} \cap S^{\underline{b}\uparrow} \subseteq \mathcal{C}_{\mathbb{Z}}^{0}[n]$. By Corollary 1 the semigroup $\mathcal{C}_{\mathbb{Z}}^{0}[n]$ is isomorphic to the bicyclic monoid with adjoined zero $\mathcal{C}^{0}$. Also, it is obvious that the topology $\tau_{\min}^{\mathrm{sh}}$ induces the topology τ on $\mathcal{C}_{\mathbb{Z}}^{0}[n]$ such that τ generates by the map $h : \mathcal{C}_{\mathbb{Z}}^{0}[n] \to \mathcal{C}^{0}$, $(a, b) \to q^{a-n}p^{b-n}$ and $0 \to 0$, the topology $\tau_{\min}$ on $\mathcal{C}^{0}$ [9]. Then the proof of Lemma 2 from [1] implies that $(\mathcal{C}^{0}, \tau_{\min})$ is a Hausdorff topological semigroup. This and the above arguments imply that $(\mathcal{C}_{\mathbb{Z}}^{0}, \tau_{\min}^{\mathrm{i}})$ is a topological inverse semigroup. The minimality of $(\mathcal{C}_{\mathbb{Z}}^{0}, \tau_{\min}^{\mathrm{i}})$ as topological inverse semigroup follows from Lemma 3, because

$$\mathcal{C}_{\mathbb{Z}}^{0} \setminus (S^{\overrightarrow{|a}} \cap S^{\underline{b}\uparrow}) = \{(x, y) : (x, y) \cdot (x, y)^{-1} \in \uparrow_{\preceq} (a-1, a-1)\} \cup$$
$$\cup \{(x, y) : (x, y)^{-1} \cdot (x, y) \in \uparrow_{\preceq} (b-1, b-1)\}.$$

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НАПІВТОПОЛОГІЧНІ РОЗШИРЕННЯ БІЦИКЛІЧНОЇ НАПІВГРУПИ З ПРИЄДНАНИМ НУЛЕМ

Доведено, що кожна гаусдорфова локально компактна напівгрупова топологія на розширеній біциклічній напівгрупі з приєднаним нулем $\mathcal{C}_{\mathbb{Z}}^0$ є дискретною, але на $\mathcal{C}_{\mathbb{Z}}^0$ існує $\mathfrak{c}$ різних гаусдорфових локально компактних трансляційно-неперервних топологій. Також на $\mathcal{C}_{\mathbb{Z}}^0$ побудовано єдину мінімальну трансляційно-неперервну топологію та єдину мінімальну інверсну напівгрупову топологію.

Ключові слова: *розширена біциклічна напівгрупа, локально компактний, напівтопологічна напівгрупа, топологічна напівгрупа, мінімальна топологічна напівгрупа, дискретний.*

ПОЛУТОПОЛОГИЧЕСКИЕ РАСШИРЕНИЯ БИЦИКЛИЧЕСКОЙ ПОЛУГРУППЫ С ПРИСОЕДИНЕННЫМ НУЛЕМ

Доказано, что каждая хаусдорфова локально компактная полугрупповая топология на расширенной бициклической полугруппе с присоединенным нулем $\mathcal{C}_{\mathbb{Z}}^0$ является дискретной, но на $\mathcal{C}_{\mathbb{Z}}^0$ существует $\mathfrak{c}$ разных хаусдорфовых локально компактных трансляционно-непрерывных топологий. Также на $\mathcal{C}_{\mathbb{Z}}^0$ построена единственная минимальная трансляционно-непрерывная топология и единственная минимальная инверсная полугрупповая топология.

Ключевые слова: *расширенная бициклическая полугруппа, локально компактный, полутопологическая полугруппа, топологическая полугруппа, минимальная топологическая полугруппа, дискретный.*

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