UDC 515.12

K. M. Koporkh[1], M. M. Zarichnyi[2]✉

ON THE SPACE OF OPEN MAPS OF THE CANTOR SET

The space $\Psi(C)$ of the equivalence classes of continuous open maps defined on the Cantor set C endowed with the Vietoris topology is considered. It is shown that $\Psi(C)$ is homeomorphic to the space of irrational numbers.

Key words: *Cantor set, space of irrational numbers, open map, hyperspace, quotient object.*

Introduction. The hyperspaces, i.e., spaces whose elements are subsets of a topological space are important objects of topology. They are considered in numerous publications during decades (see, e.g., [8, 11] and the bibliography therein). They have numerous applications not only in mathematics but also in other disciplines. In particular, fixed point theorems for multivalued maps (i.e., maps with values in hyperspaces) are widely used in the general equilibrium theory and related areas.

From categorical point of view, the hyperspaces are nothing but the spaces of subobjects. The dual notion, i.e., the space of quotient objects of a topological space is considerably less known. Note that E. Shchepin in [5] (see also [6]) considered the sets of quotient objects in his theory of inverse spectra (inverse systems) but did not use this terminology. Actually, he introduced the set $\Psi(X)$ of equivalence classes of open maps of a compact Hausdorff space X.

In the compact Hausdorff setting, there is a natural way to topologize $\Psi(X)$ by using the fact that in this case the fibers of any open map of X form a continuous decomposition of X (see the details below).

The spaces $\Psi(X)$ were investigated in some publications of the first-named author. In some cases, it was possible to describe topology of the spaces $\Psi(X)$. In [1] the space of open quotient objects of a convergent sequence is considered and it is proved that this space is a perfect zero-dimensional noncompact metric space which can be decomposed into the union of two sets, one of which is dense and the other is homeomorphic to the set of irrational numbers.

Some general results in this direction are obtained in [2]. In [10] the author considered the space of open maps of the segment and obtained a description of topology of its connectedness components.

In [9] the topology of the connectedness components of the space $\Psi(S^1)$ is described. In both cases it turns out that the space of quotient objects is not the one-point compactification of its connectedness components.

The aim of the present note is to investigate the space of equivalence classes of open maps, i.e., the quotient objects of the Cantor set C. The main result is the following: the space $\Psi(C)$ is homeomorphic to the space of irrational numbers.

1. Preliminaries. We suppose that all maps are continuous, unless their continuity is to be proved. By $\bar{A}$ we denote the closure of A. If X, Y are topological spaces, $X \cong Y$ means "X is homeomorphic to Y".

A topological space is called zero-dimensional if there is a base of its topology consisting of sets which are simultaneously open and closed.

A metrizable topological space is called topologically complete if there is a complete metric on it that induces its topology.

✉ zarichnyi@yahoo.com

The hyperspace $\exp X$ is the set of all nonempty closed subsets on X, endowed with the Vietoris topology.

In the case of a compact metric space (X, d), which is sufficient for our future considerations, the Vietoris topology can be generated by the Hausdorff metric d_H,

$$d_H(E,F) = \inf\{\varepsilon > 0 \mid E \subseteq O_\varepsilon(F) \text{ and } F \subseteq O_\varepsilon(E)\}$$

(hereafter, $O_\varepsilon(A)$ stands for the ε-neighborhood of A).

Recall that a map $f : X \to Y$ of topological spaces is open if the image of every open set is open. We will use the following result [4, 7]:

Theorem 1. *Each onto map $f : X \to Y$ of compact metric spaces is open if and only if the inverse map $f^{-1} : Y \to \exp X$ is continuous.*

By $\exp^2 X$ we denote the space $\exp(\exp X)$. If d is a metric on X that induces its topology, then the metric on $\exp^2 X$ is $(d_H)_H = d_{HH}$.

Let $u_X : \exp^2 X \to \exp X$ denote the union map. It is well-known that u_X is continuous. The following result is a characterization of the space $\mathbb{P}$ of irrational numbers (see, e.g., [3, 13]).

Theorem 2 (characterization theorem for the set of irrational numbers). *The space $\mathbb{P}$ is topologically the unique nonempty, topologically complete, nowhere locally compact and zero-dimensional space.*

2. The space of quotient maps. In the following we deal with compact metric spaces.

Following [6] we say that two continuous onto maps $f_i : X \to Y_i$, $i = 1, 2$, are equivalent, $f_1 \sim f_2$, if there exists a homeomorphism $h : Y_1 \to Y_2$ such that $f_2 = hf_1$.

Then «$\sim$» is an equivalence relation on the class of all continuous open onto maps. By $\langle f \rangle$ we denote the class of all open onto maps (the open quotient object) equivalent to $f : X \to Y$. Let

$$\Psi(X) = \{\langle f \rangle \mid f : X \to Y \text{ is an open onto map}\}.$$

Thus, every quotient object $\langle f \rangle$ uniquely determines a disjoint family $\mathcal{F}$ of closed subsets of space X.

On the contrary, let $\mathcal{A} \in \exp^2 X$ and $\mathcal{A}$ be a disjoint family. Let $g : X \to X / \mathcal{A}$ be the quotient map. Then $\langle g \rangle \in \Psi(X)$.

We therefore come to the following result:

Proposition 1. *For every compact space X there is a one-to-one correspondence between the quotient objects of X and the disjoint elements of $\exp^2 X$.*

In the sequel, we use this correspondence and endow the set $\Psi(X)$ with the topology induced from $\exp^2 X$.

3. Topological type of the space $\Psi(C)$. Let C be the middle-third Cantor set. Then $\exp^2 C$ is a zero-dimensional separable compact metric space.

Proposition 2. *Let X be a metric compact space. Then $\Psi(X)$ is a G_δ-subset in the space $\exp^2 X$.*

P r o o f. Note that $u_X^{-1}(X)$ is a closed subset of $\exp^2 X$ and

$\Psi(X) \subset u_X^{-1}(X)$. We are going to show that $u_X^{-1}(X) \setminus \Psi(X)$ is an F_σ-set. To this end, show that $u_X^{-1}(X) \setminus \Psi(X) = \bigcup_{n=1}^{\infty} \mathcal{F}_n$, where $\mathcal{F}_n$ is a closed subset in the space $u_X^{-1}(X)$. Let

$\mathcal{F}_n = \{\mathcal{A} \in u_X^{-1}(X) \mid \exists A, B \in \mathcal{A},\ A \cap B \neq \varnothing$

and one of the conditions holds:

(*1*) there exists $a \in A$ such that $d(a, B) \geq 1/n$;

(*2*) there exists $b \in B$ such that $d(b, A) \geq 1/n\}$.

Then for every $n \in \mathbb{N}$ and $\mathcal{A} \in \mathcal{F}_n$ we see that $\mathcal{A}$ is not disjoint, therefore $\mathcal{F}_n \cap \Psi(X) = \varnothing$. On the other hand, if $\mathcal{A} \in \Psi(X) \subset u_X^{-1}(X)$, then $\mathcal{A}$ is not disjoint, therefore there exist $A, B \in \mathcal{A}$ such that $A \neq B$ and $A \cap B \neq \varnothing$. Without loss of generality one may assume that there exists $x \in A \setminus B$. Consequently, there exists $n \in \mathbb{N}$ such that $d(x, B) \geq 1/n$. Thus, $\mathcal{A} \in \mathcal{F}_n$.

We are going to show that the set $\mathcal{F}_n$ is closed for every n. Let $\{\mathcal{A}_i\}_{i=1}^{\infty}$ be a sequence of elements in $\mathcal{F}_n$. Since $\exp^2 X$ is compact, passing, if necessary, to a subsequence one may assume that $\{\mathcal{A}_i\}_{i=1}^{\infty}$ is convergent. Let $\mathcal{A} = \lim_{i \to \infty} \mathcal{A}_i$.

Without loss of generality one may assume that for every $i \in \mathbb{N}$ there exist $A_i \in \mathcal{A}_i$ and $B_i \in \mathcal{A}_i$ such that

(*1*) $A_i \cap B_i \neq \varnothing$;

(*2*) $\exists a_i \in A_i$ such that $d(a_i, B_i) \geq 1/n$.

Consider the sequences $\{a_i\}_{i=1}^{\infty}$, $\{A_i\}_{i=1}^{\infty}$ and $\{B_i\}_{i=1}^{\infty}$. Since X and $\exp X$ are compact, passing, if necessary, to subsequences one may assume that these sequences are convergent.

Suppose that $\{a_i\}_{i=1}^{\infty} \to a$, $\{A_i\}_{i=1}^{\infty} \to A$ and $\{B_i\}_{i=1}^{\infty} \to B$ as $i \to \infty$.

For all $i \in \mathbb{N}$ there exists $x_i \in A_i \cap B_i \neq \varnothing$. Again, one may assume that the sequence $\{x_i\}_{i=1}^{\infty} \subset X$ is convergent. Suppose that $\{x_i\}_{i=1}^{\infty} \to x$.

Then clearly $x \in A \cap B$ and therefore $A \cap B \neq \varnothing$.

Since the distance function $d : X \times \exp X \to \mathbb{R}$ is continuous, we conclude that

$$d(a, B) = \lim_{i \to \infty} d(a_i, B_i) \geq 1/n.$$

Finally, $\mathcal{A} \in \mathcal{F}_n$ and therefore the set $\mathcal{F}_n$ is closed. ♦

Being a G_δ-subset of a compact metric space $\exp^2 X$ the set $\Psi(X)$ is topologically complete.

Lemma 1. *Let A and B be subsets in a metric space (X, d) such that $A \cong C$, $B \cong C$ and $d_H(A, B) < r$ (where $r > 0$). Then there exists a homeomorphism $h : A \to B$ such that $d(x, h(x)) < 2r$ for every $x \in A$.*

P r o o f. One can decompose $A = A_1 \cup A_2 \cup \ldots \cup A_n$ and $B = B_1 \cup B_2 \cup \ldots \cup B_m$, where all A_i, B_j are open and closed subsets in A and B respectively such that:

(*1*) the family $\{A_1,\dots,A_n\}$ is disjoint;

(*2*) the family $\{B_1,\dots,B_m\}$ is disjoint;

(*3*) $\operatorname{diam}(A_i)<r$ for every i;

(*4*) $\operatorname{diam}(B_j)<r$ for every j;

(*5*) for every i there exists j such that $O_{2r}(A_i)\supset B_j$;

(*6*) for every j there exists i such that $O_{2r}(B_j)\supset A_i$.

By decomposing, if necessary, each A_i and B_j into disjoint finite sum of nonempty open and closed parts and changing the numeration one may suppose that $m=n$ and $d_H(A_i,B_i)<2r$, for every $i=1,\dots,n$.

For every i let $g_i : A_i\to B_{h(i)}$ be a homeomorphism. Finally, define $g : A\to B$ by the condition $g(x)=g_i(x)$ whenever $x\in A_i$, $i=1,\dots,n$.

Then the distance between g and the identity map $1_A : A\to A$ is

$$d(g,1_A)< 2r. \qquad \blacklozenge$$

Proposition 3. *The set* $\Psi(C)$ *is a dense subset in the space*

$$\{\mathcal{A}\in\exp^2 C \mid \bigcup A=C\}.$$

P r o o f. Suppose that $\mathcal{A}\in\exp^2 C$ and $\bigcup\mathcal{A}=C$. Let $\varepsilon>0$ and $r<\varepsilon/2$.

By known properties of hyperspaces, there exists a finite family $\mathcal{U}\subset\exp C$ such that $d_{HH}(\mathcal{A},\mathcal{U})<\varepsilon/2$. Since the union map is nonexpanding, $d_H(\bigcup\mathcal{U},C)<\varepsilon/2$. Passing, if necessary, to closed neighborhoods of the elements of $\mathcal{U}$ one may assume that

(*1*) $\bigcup\mathcal{U}=C$;

(*2*) $d_{HH}(\mathcal{A},\mathcal{U})<\varepsilon$;

(*3*) every element of $\mathcal{U}$ is homeomorphic to C.

Suppose that $\mathcal{U}=\{U_1,\dots,U_n\}$.

Consider the set $\{1,2,\dots,n\}$ as a finite metric space, whose metric ρ is defined by the condition $\rho(i,j)<r$ for every $1\neq j$.

We identify the space C with the subset

$$C_1=C\times\{1\}\subset C\times\{1,2,\dots,n\}$$

by the embedding $x\mapsto(x,1)$, $x\in C$. Let

$$C_2=\bigcup_{i=1}^{n}U_i\times\{\mathrm{i}\}\subset C\times\{1,2,\dots,n\}.$$

Clearly, C_1 and C_2 are homeomorphic to C.

We endow the product $C\times\{1,2,\dots,n\}$ with the max-metric, which we denote by d.

Then it is easy to check that

$$d_{HH}(\mathcal{A},\{U_i\times\{i\}\mid\{1,2,\dots,n\}\})<r.$$

One can apply Lemma 1 in order to obtain a homeomorphism $h : C_2\to C_1$ such that $d(h,1_{C\times\{1,2,\dots,n\}})<2r$. Then the finite family $\mathcal{U}'=\{h(U_i)\mid i=1,\dots,n\}$ is an element of $\Psi(C_1)=\Psi(C)$ such that $d_{HH}(\mathcal{U}',\mathcal{A})<2r<\varepsilon$. $\blacklozenge$

Remark 1. Actually, it is proved that the set of all finite disjoint families $\mathcal{A}$ with $\bigcup\mathcal{A}=C$ is a dense subset in $\Psi(C)$.

Lemma 2. *There exist sequences* (A_i), (B_i) *in the space* $\exp C$ *satisfying*:

(*1*) $\lim_{i\to\infty} A_i = \lim_{i\to\infty} B_i = C$;

(*2*) $A_i \cup B_i = C$;

(*3*) $A_i \cap B_i = \varnothing$.

P r o o f. Represent C as $\prod_{i=1}^{\infty}\{0,1\}_i$, where $\{0,1\}_i = \{0,1\}$ for every i. We consider the discrete topology on $\{0,1\}$. Let

$$A_i = \{(x_j) \in C \mid x_i = 0\}, \quad B_i = \{(x_j) \in C \mid x_i = 1\}.$$

It is easy to verify that (A_i), (B_i) are as required. ♦

Proposition 4. *The set* $\Psi(C)$ *is a nowhere locally compact subset in the space* $\exp^2 C$.

P r o o f. It suffices to show that there exists a dense subset A in $\Psi(C)$ such that no point in A has a neighborhood with compact closure. By Remark 1, it is enough to consider the case of finite disjoint family $\mathcal{A} \in \exp^2 C$ with $\bigcup\mathcal{A} = C$

Suppose that $D \in \mathcal{A}$ and $r > 0$.

Let $D = C_1 \cup C_2$ where C_1, C_2 are homeomorphic to C and $C_1 \cap C_2 = \varnothing$.

By Lemma 2 there exist sequences $\{A_i\}_{i=1}^{\infty}$ and $\{B_i\}_{i=1}^{\infty}$ in $\exp C$ satisfying

(*1*) $\lim_{i\to\infty} A_i = C_1$;

(*2*) $\lim_{i\to\infty} B_i = C_1$;

(*3*) $A_i \cup B_i = C_1$;

(*4*) $A_i \cap B_i = \varnothing$;

(*5*) $d_H(A_i, C_1) < r$ for every i;

(*6*) $d_H(B_i, C_1) < r$ for every i.

For every i we define $\mathcal{A}_i = \mathcal{A} \setminus \{D\} \cup \{C_2 \cup A_i\} \cup \{B_i\}$. Then all $\mathcal{A}_i$ belong to the closed r-neighborhood of $\mathcal{A}$ in $\Psi(C)$. However,

$$\lim_{i\to\infty} \mathcal{A}_i = \mathcal{A} \cup \{C_1\} \notin \Psi(C).$$

This shows that the space $\Psi(C)$ is nowhere locally compact. ♦

Theorem 3. *The space of open quotient objects of the Cantor set is homeomorphic to the set of irrational numbers.*

The p r o o f of the theorem is based on Propositions 2, 3, 4 and Theorem 2 (characterization theorem for the set of irrational numbers). ♦

4. Questions. Recall that a supersequence is a one-point compactification of a discrete topological space.

In connection with the main result of [1] we formulate the following problem: describe the topology of the space $\Psi(S)$ for supersequences S of arbitrary cardinality. Also, since the ordinal number $\omega+1$ is nothing but a convergent sequence, the results of [1] lead to the problem of description of topology of the spaces $\Psi(\tau)$ for arbitrary ordinal number τ.

There is an uncountable counterpart of the Cantor set, namely, the space D^τ, where D is a discrete two-point space and τ is an uncountable cardinal number. The natural problem of description of topology of the space $\Psi(D^\tau)$ remains open.

We now turn to the convex setting. It is proved in [12] that the hyperspace of compact convex subsets of the cube $[0,1]^n$, $2 \le n \le \omega$, is homeomorphic to the Hilbert cube. A similar result can be proved for another compact convex sets. It seems natural to consider the space of the affine open quotient objects of compact convex sets.

1. *Копорх К. М.* Простір відкритих факторвідображень збіжної послідовності // Карпат. мат. публікації. – 2012. – **4**, № 1. – С. 58–66.
2. *Копорх К. М.* Топологія Вієторіса на просторі відкритих фактороб'єктів компактного гаусдорфового простору // Праці міжнар. геометр. центру. – 2010. – **3**, № 4. – С. 35–42.
3. *Куратовский К.* Топология. – Т. 1. – Москва: Мир, 1966. – 594 с.
4. *Федорчук В. В.*, *Филиппов В. В.* Общая топология. Основные конструкции: Учеб. пособие. – Москва: МГУ, 1988. – 252 с.
5. *Щепин Е. В.* Топология предельных пространств несчетных обратных спектров // Успехи мат. наук. – 1976. – **31**, № 5(191). – Р. 191–226.
   Shchepin E. V. Topology of limit spaces of uncountable inverse spectra // Rus. Math. Surv. – 1976. – **31**, No. 5. – P. 155–191.
6. *Щепин Е. В.* Функторы и несчетные степени компактов // Успехи мат. наук. – 1981. – **36**, № 3(219). – Р. 3–62.
   Shchepin E. V. Functors and uncountable powers of compacta // Rus. Math. Surv. – 1981. – **36**, No. 3. – P. 1–71.
7. *Энгелькинг Р.* Общая топология. – Москва: Мир, 1986. – 752 с.
   Engelking R. General topology. – Berlin: Heldermann, 1989. – 539 p.
8. *Beer G.* Topologies on closed and closed convex sets. – Dordrecht: Kluwer Academic Publishers, 1993. – xii+340 p.
9. *Koporkh K.* On space of open maps of the circle // Вісн. Львів. ун-ту. Сер. мех.-мат. – 2011. – Вип. 75. – С. 151–157.
10. *Koporkh K.* On space of open maps of the unit segment // Вісн. Львів. ун-ту. Сер. мех.-мат. – 2009. – Вип. 71. – С. 135–141.
11. *Nadler S. B. Jr.* Hyperspaces of sets: A text with research questions. – New York etc.: Marcel Dekker, Inc., 1978. – 707 p. – Ser. Monographs and textbooks in pure and applied mathematics. – Vol. 49.
12. *Nadler S. B., Jr.*, *Quinn J. E.*, *Stavrokas N. M.* Hyperspaces of compact convex sets // Pacif. J. Math. – 1979. – **83**, No. 2. – P. 441–462.
   – https://doi.org/10.2140/pjm.1979.83.441.
13. *Van Mill J.* The infinite-dimensional topology of function spaces. – Amsterdam: Elsevier, 2001. – 642 p.

ПРО ПРОСТІР ВІДКРИТИХ ВІДОБРАЖЕНЬ КАНТОРОВОЇ МНОЖИНИ

Розглядаєтья простір $\Psi(C)$ класів еквівалентності неперервних відкритих відображень, означених на канторовій множині C, наділений топологією Вієторіса. Доведено, що цей простір гомеоморфний простору ірраціональних чисел.

Ключові слова: *канторова множина, простір ірраціональних чисел, відкрите відображення, гіперпростір, фактороб'єкт.*

[1] Vasyl' Stefanyk National Pre-Carpathian University, Ivano-Frankivs'k,

[2] Ivan Franko National University of Lviv, Lviv

Received 10.01.21