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MATHEMATICAL MODELING OF THE GAS-FILTRATION IN THE BOTTOMHOLE ZONE OF UNDERGROUND GAS-STORAGE WELLS USING FRACTIONAL DERIVATIVES

The finite element method is applied for the numerical simulation of the gas filtration in a porous inhomogeneous media. The model is based on the fractional derivatives with respect to time in terms of the Grünvald – Letnikov operator. The numerical analysis results are verified by comparing with the real-life empirical data on the physical and geometrical parameters to reveal their excellent agreement.

Key words: *mathematical model, nonstationary gas motion, fractional derivatives, linearization, finite element method.*

Introduction. Modeling the mass transfer in complex natural porous media is concerned with enormous indetermination of the parameters characterizing both the media and transferred matter. Practical evaluations are usually based on the averaged (homogenized) empirical parameters for making a model adequate to the processes of interest. This results in narrowing the time-spatial frameworks of the model and inducing errors of different kinds, which are to be encountered for specific processes under investigation.

Recently, the adequateness criteria became much tighter and thus the computational models are to fit more strict requirements with regard to the accuracy of computational results. Hence, the advancement in the existing models along with the development of the new ones is a topical problem. An efficient method for answering this challenge is the application of fractional derivatives with respect to time for modeling the mass transfer in complex porous media. The advantage of this approach is an opportunity to encounter the memory of a process that is critical for the computation of the hydrocarbons filtration parameters.

There exist a number of fractional-differential models of diffusion-like transfer processes, see, e. g. [2, 3, 5]. Such models are efficiently used, in particular, for covering the processes of impurity transfer in geological formations with a complex and inhomogeneous internal structure [6]. This approach has not been widely used, however, for filtration processes in complex inhomogeneous porous media (which is engaged with strong nonlinear effects) for most of the currently proposed fractional-differential models are linear. Hence, the adaptation of the fractional-differential approach for modeling of filtration processes in porous media and construction of corresponding nonlinear mathematical models of filtration presents a challenge. The construction of such models, in particular, will make it possible to adequately predict the volume of gas production in formations with natural fractures.

It is worth noting that the application of fractional derivatives for modeling various processes is concerned with mathematical issues of both theoretical and practical nature. One of those is the selection of a proper order of the fractional derivative. As shown by numerical evaluations, such models appear to be very sensitive to the variation of the order of a fractional derivative.

This work is motivated by the development and analysis of a model for gas filtration in porous inhomogeneous media by using the fractional derivatives with respect to time. It is also aimed to clarify the efficient application of the fractional derivatives within the framework of mathematical models of gas filtration in underground storages.

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1. Mathematical model. Consider the mass-transfer process in a porous medium related to the coordinate system (x, y_1, y_2). In the case of gas and fluid filtration, the process is described by the following equation formulated with the fractional derivative in time [1, 2]:

$$\frac{\partial}{\partial x}\left(\frac{kh}{\mu z}\frac{\partial p^{\ell}}{\partial x}\right)+\frac{\partial}{\partial y_1}\left(\frac{kh}{\mu z}\frac{\partial p^{\ell}}{\partial y_1}\right)+\frac{\partial}{\partial y_2}\left(\frac{kh}{\mu z}\frac{\partial p^{\ell}}{\partial y_2}\right)=$$

$$=2mh\left(\frac{\partial^{\alpha}}{\partial t^{\alpha}}\left(\frac{p}{z}\right)+2qp_{\text{at}}\right). \qquad (1)$$

Here, $\ell = 1$ is the power in the case of an incompressible fluid and $\ell = 2$ stands for a gas, $\alpha \in (0,2]$ is the order of fractional derivative that describes the nature of the non-stationary process, $k = k(x, y_1, y_2, t)$ is the coefficients of permeability, $m = m(x, y_1, y_2)$ is the porosity coefficient, $h = h(x, y_1, y_2)$ is the thickness of the medium, μ is the dynamic viscosity of substance, p_{at} denotes the atmospheric pressure, q is the extraction density, z is the coefficient of compressibility of gas, for calculation of which a significant number of empirical formulas based on experimental data is used, including $z = 1/(1 + f p_{[\text{AT}]})$, where $f = (24 - 0.21\times T[°\text{C}])\times 10^{-4}$, and $p_{[\text{AT}]}(x, y_1, y_2, t)$ is the operational gas pressure measured in the atmospheres [1]. Gas is extracted from a porous medium through I wells located at points $(x_i^0, y_{1,i}^0, y_{2,i}^0)$ and operated during certain periods of time $t \in [t_{1,i}, t_{2,i}]$, $i = 1, 2, \ldots, I$. The density of extraction is determined by the following formula:

$$q = \frac{1}{V}\sum_{i=1}^{I} q_i(x, y_1, y_2, t)\big(\eta(t - t_{1,i}) - \eta(t - t_{2,i})\big)\times$$

$$\times\delta(x - x_i^0)\delta(y_1 - y_{1,i}^0)\delta(y_2 - y_{2,i}^0).$$

Here, q_i is the gas extraction from the ith well at the moment t, $\delta(x)$ is the Dirac delta-function, $\eta(t - t_{ji})$ is the Heaviside function, and V is the storage volume.

The problem implies finding a solution $p(x, y_1, y_2, t)$ of equation (1) for given values of gas extractions in wells of the medium and an impermeability condition on an medium circuit under the necessary condition of the gas mass conservation in the medium:

$$M = \int_V \rho\, dv,$$

where M is the mass of gas in storage, ρ is the density of gas connected to pressure by an equation of state $p = \rho z R \Theta$, R is the gas constant, Θ is the absolute temperature of gas.

Let $\Omega \subset \mathbb{R}^3$ be a three-dimensional domain which occupies the porous medium (Fig. 1) and limited with a boundary surface $\partial\Omega$. Within domain Ω, a set of wells is located at points with coordinates $(x_i, y_{1,i}, y_{2,i})$, $i = 1, 2, \ldots, I$, and pressures $p(x_i, y_{1,i}, y_{2,i}, t_0)$ in these points is given at time t_0.

Let domain Ω to consist of a number of layers, the limiting surfaces of which are close to horizontal planes. In the case of layers being gas bearing, the gas differential pressure between the top and bottom interfaces of a layer is negligible. Hence, the processes of gas filtration within such layers is irrespective of the vertical coordinate y_2.

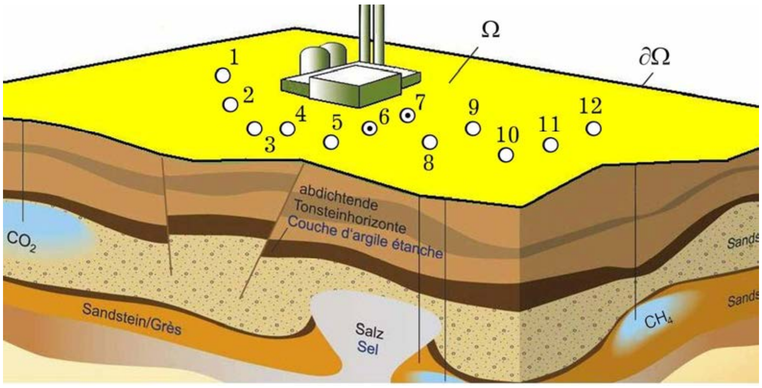

Fig. 1. Scheme of the considered gas storage.

Then equation (1) takes the following form:

$$\frac{\partial}{\partial x}\left(\frac{kh}{\mu z}\frac{\partial p^2}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{kh}{\mu z}\frac{\partial p^2}{\partial y}\right)=2mh\left(\frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{p}{z}\right)+2qp_{\text{at}}\right), \quad (2)$$

where $y = y_1$.

Assuming the absence of gas supplement into the volume of domain Ω through its boundary $\partial\Omega$, the pressure gradient along the normal to the boundary is dismissed and $\partial p / \partial n = 0$ can be regarded as an appropriate boundary condition. This condition implies $(\partial p / \partial x)|_{\partial\Omega} = (\partial p / \partial y)|_{\partial\Omega} = 0$.

By implying the end of the neutral period (which is the period when $q = 0$) to be a reference point, the initial pressure distribution can be assumed constant and equal to a measured value p_0.

2. Numerical solution scheme. Let us develop an iterative scheme for the calculation gas pressure in a layer. Equation (2) can thereby be represented in the following form:

$$\frac{\partial}{\partial x}\left(\frac{kh}{\mu z}p\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{kh}{\mu z}p\frac{\partial p}{\partial y}\right)=mh\left(\frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{p}{z}\right)+2qp_{\text{at}}\right). \quad (3)$$

The linearized version [1, 2] of equation (1) reads

$$\tilde{p}\frac{kh}{\mu z}\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial x}\right)+\tilde{p}\frac{kh}{\mu z}\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial y}\right)=$$

$$=mh\left(\frac{1}{z}\frac{\partial^\alpha p}{\partial t^\alpha}+2qp_{\text{at}}\right)+F(\tilde{p},k,h,\mu,z), \quad (4)$$

where

$$F(\tilde{p},k,h,\mu,z)=-\frac{\partial}{\partial x}\left(\frac{kh}{\mu z}\tilde{p}\right)\frac{\partial\tilde{p}}{\partial x}-\frac{\partial}{\partial y}\left(\frac{kh}{\mu z}\tilde{p}\right)\frac{\partial\tilde{p}}{\partial y} \quad (5)$$

and $\tilde{p}$ is an iterative approximate value of solution p on the previous step of iteration.

The operator of the fractional derivative can be presented via the Caputo formulation, as follows [4, 7]

$${}^{\text{C}}\text{D}_\tau^\alpha=\frac{{}^{\text{C}}\partial^\alpha}{\partial\tau^\alpha}\varphi(\tau)=\frac{1}{\Gamma(m+1-\alpha)}\int_0^\tau\frac{1}{(\tau-\xi)^{\alpha-m}}\frac{\partial^{m+1}\varphi(\xi)}{\partial\xi^{m+1}}d\xi,$$

where $m=[\alpha]$, $[\cdot]$ is an integer part of the real number, Γ is the gamma function.

This operator can also be presented via the Riemann – Liouville formulation, which has the following form:

$$ {}^{\mathrm{RL}}\mathrm{D}_{\tau}^{\alpha} = \frac{{}^{\mathrm{RL}}\partial^{\alpha}}{\partial\tau^{\alpha}}\varphi(\tau) = \frac{1}{\Gamma(m+1-\alpha)}\frac{\partial^{m+1}}{\partial\xi^{m+1}}\int_0^{\tau}\frac{\varphi(\xi)}{(\tau-\xi)^{\alpha-m}}d\xi . $$

These two formulations are connected via the following relationship [7]:

$$ {}^{\mathrm{C}}\mathrm{D}_{\tau}^{\alpha}\varphi(\tau) = {}^{\mathrm{RL}}\mathrm{D}_{\tau}^{\alpha}\varphi(\tau) - \sum_{k=0}^{m}\frac{\tau^{k-\alpha}}{\Gamma(k-\alpha+1)}\frac{\partial^{k}\varphi(0)}{\partial\tau^{k}} . $$

In order to construct a numerical model for non-stationary problems of gas filtration in a porous medium, we employ iteratively the finite element method combined with the difference scheme of time discretization in Grünwald – Letnikov scheme [1, 7]:

$$ {}^{\mathrm{GL}}\mathrm{D}_{\tau}^{\alpha}p = \lim_{\Delta t\to 0}(\Delta t)^{-\alpha}\sum_{j=0}^{[\tau/\Delta t]}(-1)^{j}\binom{\alpha}{j}p(\tau-j\Delta t) , $$

where $\binom{\alpha}{j}$ is the binomial coefficient (the Pochhammer symbol), τ is a current time moment, and Δt denotes the discrete time period. During the calculations, the linearized version of equation (4) is solved iteratively at each time interval. The Grünwald–Letnikov operator is approximated on an interval $[0, \tau]$ with the subinterval step Δt as

$$ {}^{\mathrm{GL}}\mathrm{D}_{\tau}^{\alpha}p(\tau) \approx \sum_{j=0}^{[\tau/\Delta t]} c_j^{(\alpha)}p(\tau-j\Delta t) , \tag{6} $$

where

$$ c_j^{(\alpha)} = (\Delta t)^{-\alpha}(-1)^{j}\binom{\alpha}{j} \tag{7} $$

are the Grünwald – Letnikov coefficients. We can calculate coefficients $c_j^{(\alpha)}$ by using the following recursive formulas [5]:

$$ c_j^{(\alpha)} = (\Delta t)^{-\alpha}, \quad jc_j^{(\alpha)} = (j-1-\alpha)c_{j-1}^{(\alpha)}, \quad c_1^{(\alpha)} = -\alpha(\Delta t)^{-\alpha} . \tag{8} $$

Having used linearization (4) and discretization scheme for the fractional derivative (6) – (8), we transform equation (3) to the form:

$$ \tilde{p}\frac{kh}{\mu z}\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial x}\right) + \tilde{p}\frac{kh}{\mu z}\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial y}\right) = $$

$$ = mh\left(\frac{1}{z}\frac{\partial^{\alpha}p}{\partial t^{\alpha}} + 2qp_{\mathrm{at}}\right) + F(\tilde{p}, k, h, \mu, z) , $$

where $F(\tilde{p}, k, h, \mu, z)$ is given through formula (5).

Consider approximation on some extended time interval $[0, T]$ that is divided into N subintervals where the subinterval length equals $\Delta t = T / N$ and the time-nodes are $t_i = i\Delta t$. Let the fractional derivative be represented via the Grünwald – Letnikov scheme:

$$ \frac{\partial^{\alpha}p(t)}{\partial t^{\alpha}} = \sum_{j=0}^{i}c_j^{(\alpha)}p(t_{i-j}) - \sum_{k=0}^{m}\frac{t_i^{k-\alpha}}{\Gamma(k-\alpha+1)}p(t_k) . $$

Finding a generalized solution of problem consists in the minimization of the functional

$$F(p) = \int_\Omega \left(a_1 \left(\frac{\partial p}{\partial x} \right)^2 + a_2 \left(\frac{\partial p}{\partial y} \right)^2 \right) d\Omega + \int_\Omega \mathrm{d} p^2 dxdy - 2\int_\Omega fpdxdy \,, \tag{9}$$

where

$$\mathrm{d} = \frac{c_0}{\tilde{p}} \frac{mh}{z} \,,$$

$$f = \frac{1}{\tilde{p}} \Bigg(-2mhp_{\mathrm{at}} q + F(\tilde{p}, k, h, \mu, z) + + \frac{mh}{z} \left(\sum_{j=1}^{i} c_j p(t_j - j\Delta t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} p(t_0) \right) \Bigg),$$

and a_1, a_2 are the finite element method coefficients [1].

For the representation of the linearized equation in the form (9), the scheme of time discretization is used considering that $\tilde{p}$ is an iterative approximate value of the solution p on the previous step of iteration.

An important step in this algorithm is to ensure the gas balance during the entire non-stationary process. During the sub-diffusion, the distribution of the permeability coefficient value is a particular issue. Simply following the traditional approach and assuming it to be a constant implies the gas imbalance. To avoid such a conflict at each iteration in the Grünwald – Letnikov scheme, the permeability coefficients in the geometric zones of the largest differences in gas pressure are corrected as

$$k_*(x, y) = \mu \frac{\Delta\ell \Delta Q}{\Delta S \Delta p}$$

in order to fulfill the condition

$$\lim_{k \to k_*} \left(M(t_0) - M(t_i) - \sum_{j=1}^{i} M_{\mathrm{ex}}(t_j) \right) = 0$$

for each time point t_i. Here, M is the mass of gas in the storage, M_{ex} is the mass of extracted gas, $\Delta\ell$ and ΔS are the length and cross-sectional area of the formation element through which the gas passes, ΔQ and Δp are the difference of gas consumption and gas pressure, respectively.

3. Numerical experiment. The method for adaptation of the gas permeability coefficient is verified with the following numerical experiment. A layer of an underground gas storage is considered with the seasonal withdrawal of gas from the storage containing a variable number of working wells. The gas drowning coefficient is selected according to the described algorithm in the areas of the included wells $k \in (0.8 \times 10^{-13} [\mathrm{m}^2], 8.8 \times 10^{-12} [\mathrm{m}^2])$. The coefficient of the fractional derivative of the gas pressure over time was chosen within the range $\alpha \in [0.94, 1]$.

The input information was set by the pressures in control, metering, and production wells in the neutral period and the values of volumetric gas extraction during gas extraction from the storage.

The numerical results demonstrate the feature of the gas distribution behavior in the layer cross-section near the wells (Fig. 2 – Fig. 5), as well as in the deposit of underground gas storage. The use of the fractional derivative allows for the registration of the adequate qualitative behavior of the pressure function in the well bottoming (Fig. 5 and Fig. 6). In contrast, the classical approach would involve some additional analysis for the proper selection of adaptive parameters.

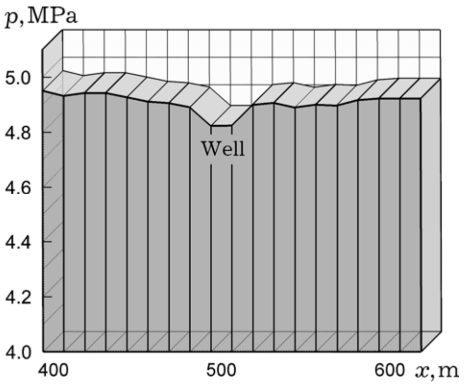

Fig. 2. The distribution of gas pressure for $\alpha = 1$, $t = 50$.

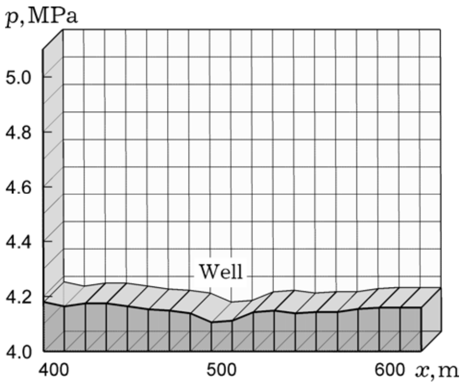

Fig 3. The distribution of gas pressure for $\alpha = 1$, $t = 150$.

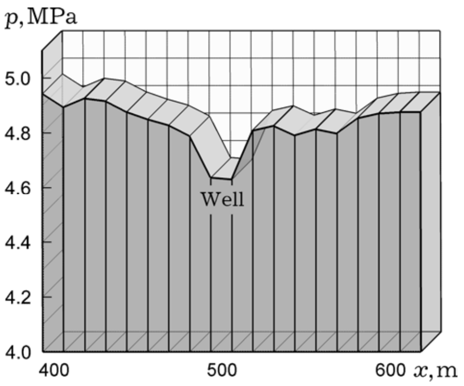

Fig. 4. The distribution of gas pressure for $\alpha = 0.97$, $t = 50$.

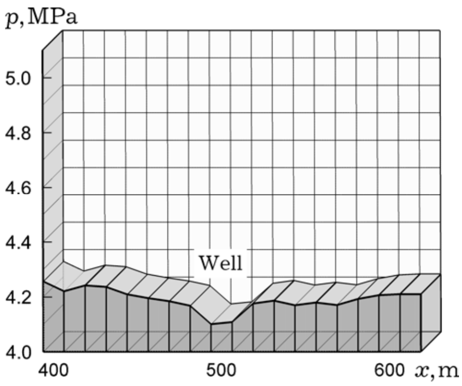

Fig 5. The distribution of gas pressure for $\alpha = 0.97$, $t = 150$.

It is worth noting that the criterion for choosing a proper value for the order of the fractional derivative depending on the parameters of the process remains disputable. In order to achieve a gas balance, the gas permeability coefficients are selected separately in different areas of the environment.

Figures 6 – 8 demonstrate the behavior of gas distribution in the zones around the wells and the relationship between permeability coefficients and the order of the fractional derivative in the process of gas withdrawal from the storage.

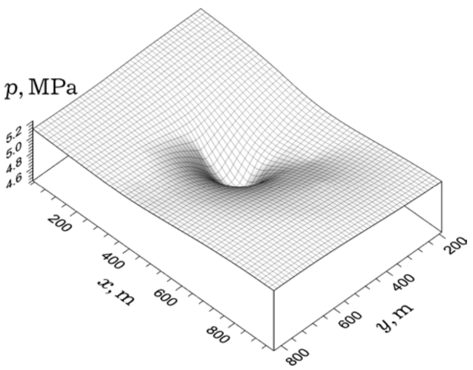

Fig. 6. The distribution of gas pressure for $\alpha = 0.94$, $\tilde{k} = 0.8 \times 10^{-13} [\mathrm{m}^2]$.

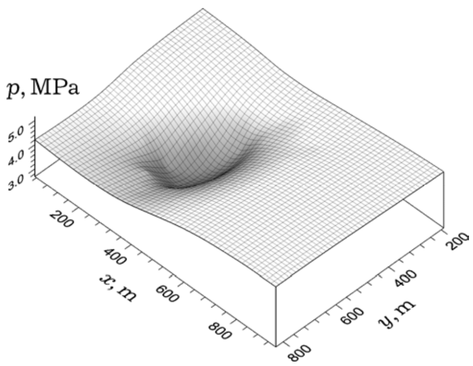

Fig 7. The distribution of gas pressure for $\alpha = 0.96$, $\tilde{k} = 0.8 \times 10^{-13} [\mathrm{m}^2]$.

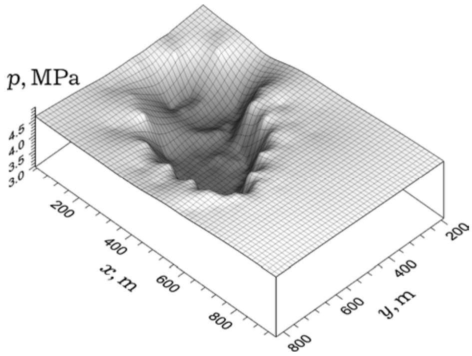

Fig. 8. The distribution of gas pressure for $\alpha = 0.94$,
$\tilde{k} = 1.2 \times 10^{-13} [\mathrm{m}^2]$.

The analysis of the figures shows that a decrease in the fractional derivative parameter leads to an increase in the area of reduced pressure in the well vicinity. It is obvious that in practice, the area of reduced pressure is closely related to the depression of pressure (or well flow). In the classic version, the so-called "pressure hole" in the well area was modeled by changing certain parameters of the environment. The use of derivatives of the fractional order ensures the presence of a "pit" without selecting the parameters of the environment. Thus, a qualitative picture of pressure behavior is presented. Conducting numerical experiments based on a real object and real measured data showed the effectiveness of the approaches proposed in this paper for the numerical modeling of mass transfer in porous media of complex structures. The study of the qualitative properties of the obtained equations, as well as the construction of their numerical solutions, are quite non-trivial tasks that require independent research in each specific case.

Conclusions. The approach used in the work to fractional-differential filtration models is phenomenological, so the possibility of their application in each case should be justified using experimental data confirming the validity of the corresponding fractional-differential generalizations. All the filtration models obtained in this work belong to the class of anomalous diffusion equations. The analysis of results of computing experiment shows that depending on a choice of parameter of the derivative order with respect to time α, the behavior of the average reservoir pressures doesn't change. It occurs because throughout all system operating time, the balance of gas in storage is ensured. In the borehole and in boundary area of the layer, the behavior of pressures depends on the derivative parameter α. The less the parameter α is then the more the difference between values of the pressure in the borehole and boundary area of the layer is. Results of the experiment confirm the behavior of gas pressure in the porous media at the presence of atypical filtration. If the parameter $\alpha = 1$, the values of calculated average reservoir pressures equal to the experimental data.

The problem of determining the relationship between the parameters of the well bottoming, the parameters of the gas filtration process, and the order of the fractional derivative remains unsolved. One of these practical criteria is the maintenance of balance ratios that are amenable to experimental measurements.

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МАТЕМАТИЧНЕ МОДЕЛЮВАННЯ ФІЛЬТРАЦІЇ ГАЗУ У ВИБІЙНИХ ЗОНАХ СВЕРДЛОВИН ПІДЗЕМНИХ СХОВИЩ ГАЗУ З ВИКОРИСТАННЯМ ДРОБОВИХ ПОХІДНИХ

Запропоновано числову модель фільтрації газу в пористих неоднорідних середовищах з використанням методу скінченних елементів та застосуванням дробових похідних за часом, обчислених на основі оператора Грюнвальда – Лєтнікова. Результати числового аналізу верифіковано з використанням отриманих з реального досліду емпіричних фізико-геометричних параметрів та виявлено їх якісне узгодження.

Ключові слова: *математична модель, нестаціонарний рух газу, дробові похідні, лінеаризація, метод скінченних елементів.*

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