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ON THE TRIANGULAR FORM OF 3 x 3-MATRIX OF SIMPLE STRUCTURE RELATIVE TO SEMISCALAR EQUIVALENCE

A special triangular form of polynomial 3×3-matrices of simple structure relative to semiscalar equivalence is established. The method of construction of matrices of such form is specified. Invariants and conditions for their semiscalar equivalence are established for matrices of this form. The method of construction of transforming matrices at transition from one matrix of a special triangular form to another is proposed.

Key words*: *matrix of simple structure, semiscalar equivalence of matrices, special triangular form of matrices, oriented by characteristic roots reduced matrix.

Introduction. This work completes the cycle of studies of the semiscalar equivalence of polynomial 3×3-matrices of simple structure, which began in the works of the author [14] and [13]. The concept of semiscalar equivalence of matrices was introduced by P. S. Kazimirsky and V. M. Petrychkovych in [5, 6]. By definition, polynomial matrices $F(x)$, $G(x)$ are called *semiscalar equivalent* (ssk.e.), if one of them can be obtained by multiplying the other on the left by the numerical non-singular matrix and on the right by the polynomial invertible matrix. In the simplest formulation, the problem consist in the establishment of the conditions for two matrices to be ssk.e. This task is multi-component and include, in particular the construction of a simpler form of the matrix using ssk.e transformations, determination of invariants of a matrix relative to such transformations, finding of transforming matrices etc. The importance of this problem also lies in its possible application to the known problem of classification of sets of numerical matrices accurate up to similarity, to the solution of matrix equations over a ring of polynomials and to other applied problems.

In the present paper, this problem is considered under certain (sometimes quite strong) constraints due to its complexity in the general case. In particular, it is assumed that the matrices under study have a simple structure. It is said that a polynomial *matrix has a simple structure* if all its elementary divisors are linear (see [5]). In other words, for a matrix of simple structure, its last invariant factor (as a polynomial) has no multiple roots. This notion of a polynomial matrix of simple structure correlates well with the notion of a numerical *matrix of simple structure* introduced in [1]. A matrix of a simple structure can have multiple characteristic roots. But the algebraic multiplicity of each characteristic root of a matrix of simple structure coincides with its geometric multiplicity. The set of matrices of simple structure contains a subset of matrices with all simple eigenvalues (of multiplicity 1).

As already mentioned, the problem investigated in this article is directly related to the well-known problem of the similarity of matrix pairs. The latter has attracted the attention of researchers for many decades, especially in the second half of the 20th century, and a satisfactory solution has obtained only in partial cases. Here, for example, we should cite the work [3], in which the above-mentioned problem is solved for a pair of nilpotent matrices that annul each other out. In [2], it was proved that the classification relative to the similarity of pairs of commutative matrices is in fact equivalent to the classification of pairs of arbitrary matrices. As it turned out later [4], the only exception to this rule is the case of classification of pairs of commutative

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matrices, considered in [3]. The authors of [12] applied a powerful apparatus of graph theory to the study of the similarity of matrix pairs, one of which has no multiple eigenvalues. A similar problem for a pair of matrices, one of which has a simple structure and all eigenvalues of multiplicity not higher than 2, is the subject of research in [7]. In the context of the study of the similarity of matrix pairs and related problems, it is also worth mentioning the works [8, 11]. The results obtained in the proposed work can be applied to solving matrix equations over a ring of polynomials. Some aspects of this topic can be found in [9, 10].

1. Preliminary assertions. We assume that matrix $F(x) \in M(3, \mathbb{C}[x])$ has a simple structure and full rank and its first invariant factor is equal to 1. Under such conditions, according to [5, 6] instead of $F(x)$ we can consider a matrix $A(x)$ that is ssk.e. to $F(x)$ and has the form

$$A(x) = \begin{Vmatrix} 1 & 0 & 0 \\ a_1(x) & \varphi_1(x) & 0 \\ a_3(x) & a_2(x) & \varphi_2(x) \end{Vmatrix}, \tag{1}$$

where 1, $\varphi_1(x)$ and $\varphi_2(x)$ are invariant factors of the matrix $A(x)$ and $\deg a_1(x) < \deg \varphi_1(x)$, $\deg a_2(x), \deg a_3(x) < \deg \varphi_2(x)$ (see Theorem [6]). Then, obviously, $\varphi_1(x)$ divides $a_2(x)$ and $\varphi_2(x)$. We assume that $\deg \varphi_1(x), \deg(\varphi_2(x)/\varphi_1(x)) > 1$. Otherwise, the problem is greatly simplified. Denote by $\varphi_{12}(x)$ and $a_2'(x)$ the fractions $\varphi_2(x)/\varphi_1(x)$ and $a_2(x)/\varphi_1(x)$, and by M_1 and M_2 the sets of roots of polynomials $\varphi_1(x)$ and $\varphi_{12}(x)$, respectively. Union of sets $M_1 \cup M_2$ obviously coincides with the set of roots of polynomial $\varphi_2(x)$ and is the set of characteristic roots of matrix $A(x)$ or the set of class ssk.e of $A(x)$ matrices. It is easy to see that matrix $A(x)$ can be chosen so that

$$\begin{Vmatrix} a_1(\alpha_0) & a_3(\alpha_0) \end{Vmatrix} = \begin{Vmatrix} 0 & 0 \end{Vmatrix} \tag{2}$$

for some root $\alpha_0 \in M_1$. In [14] it is proved that for a fixed root α_0 the greatest common divisor $(a_1(x), a_3(x), \varphi_1(x))$ does not depend on the choice $A(x)$ of the class of ssk.e. matrices (see Proposition 1 [14]). In the mentioned work the case $(a_1(x), a_3(x), \varphi_1(x)) = \varphi_1(x)$ is considered. In what follows, we will assume the opposite, i.e. $(a_1(x), a_3(x), \varphi_1(x)) \neq \varphi_1(x)$. Then there exists a root $\alpha_1 \in M_1$ such that $\begin{Vmatrix} a_1(\alpha_1) & a_3(\alpha_1) \end{Vmatrix} \neq \begin{Vmatrix} 0 & 0 \end{Vmatrix}$ and the matrix $A(x)$ can be chosen so that

$$\begin{Vmatrix} a_1(\alpha_1) & a_3(\alpha_1) \end{Vmatrix} = \begin{Vmatrix} 1 & 0 \end{Vmatrix} \tag{3}$$

(see Proposition 2 [13]).

Proposition 1. *Let* $A(x)$ *and* $B(x)$ *are two given matrices,* $A(x)$ *is of form* (1) *with conditions* (2), (3) *and*

$$B(x) = \begin{Vmatrix} 1 & 0 & 0 \\ b_1(x) & \varphi_1(x) & 0 \\ b_3(x) & b_2(x) & \varphi_2(x) \end{Vmatrix} \tag{4}$$

with invariant factors 1, $\varphi_1(x)$, $\varphi_2(x)$ *and conditions* $\deg b_1(x) < \deg \varphi_1(x)$, $\deg b_2(x), \deg b_3(x) < \deg \varphi_2(x)$, $\begin{Vmatrix} b_1(\alpha_0) & b_3(\alpha_0) \end{Vmatrix} = \begin{Vmatrix} 0 & 0 \end{Vmatrix}$, $\begin{Vmatrix} b_1(\alpha_1) & b_3(\alpha_1) \end{Vmatrix} = \begin{Vmatrix} 1 & 0 \end{Vmatrix}$. *Let also*

$$\delta_A(x) := \det\begin{Vmatrix} a_1(x) & 1 \\ a_3(x) & a_2'(x) \end{Vmatrix}, \qquad \delta_B(x) := \det\begin{Vmatrix} b_1(x) & 1 \\ b_3(x) & b_2'(x) \end{Vmatrix}, \tag{5}$$

where $a_2'(x) = a_2(x)/\varphi_1(x)$, $b_2'(x) = b_2(x)/\varphi_1(x)$. *Then matrices* $A(x)$ *and* $B(x)$ *are ssk.e. if and only if there are numbers* $s_{11} \neq 0$, $s_{22} \neq 0$, $s_{33} \neq 0$, s_{12}, s_{13}, s_{23} *such that the following congruences are fulfilled*

$$s_{22}a_1(x) + s_{23}a_3(x) \equiv b_1(x)(s_{11} + s_{12}a_1(x) + s_{13}a_3(x)) \pmod{\varphi_1(x)},$$

$$a_3(x)(s_{33} + s_{13}\delta_B(x) - s_{23}b_2'(x)) + \delta_B(x)(s_{11} + s_{12}a_1(x)) \equiv$$

$$\equiv s_{22}a_1(x)b_2'(x)\ (\mathrm{mod}\ \varphi_2(x),$$

$$a_2'(x)(s_{33} + s_{13}\delta_B(x) - s_{23}b_2'(x)) + s_{12}\delta_B(x) \equiv s_{22}b_2'(x) \pmod{\varphi_{12}(x)}. \tag{6}$$

P r o o f. *Necessity*. Let the given matrices $A(x)$ and $B(x)$ are ssk.e. Then there are matrices $\|s_{ij}\|_1^3 \in GL(3,\mathbb{C})$ and $\|r_{ij}(x)\|_1^3 \in GL(3,\mathbb{C}[x])$ such that the equality

$$\|s_{ij}\|_1^3\, A(x) = B(x)\, \|r_{ij}(x)\|_1^3 \tag{7}$$

holds, where according to Proposition 3 [13] the matrix $\|s_{ij}\|_1^3$ has an upper triangular form. Comparing the elements in positions $(2,1)$, $(3,1)$, $(3,2)$ in both sides of equality (7), we arrive at the congruences (6).

Sufficiency. From elements s_{ij}, $i, j = 1, 2, 3$, $i \leq j$, which satisfy the congruences (6), and from elements of matrices $A(x)$ and $B(x)$, we construct matrices

$$\|s_{ij}\|_1^3 = \begin{Vmatrix} s_{11} & s_{12} & s_{13} \\ 0 & s_{22} & s_{23} \\ 0 & 0 & s_{33} \end{Vmatrix}, \qquad \|r_{ij}(x)\|_1^3,$$

where

$$r_{11}(x) = s_{11} + s_{12}a_1(x) + s_{13}a_3(x), \qquad r_{12}(x) = s_{12}\varphi_1(x) + s_{13}a_2(x),$$

$$r_{13}(x) = s_{13}\varphi_2(x),$$

$$r_{21}(x) = \frac{s_{22}a_1(x) + s_{23}a_3(x) - b_1(x)r_{11}(x)}{\varphi_1(x)},$$

$$r_{22}(x) = s_{22} + s_{23}a_2'(x) - b_1(x)(s_{12} + s_{13}a_2'(x)),$$

$$r_{23}(x) = \varphi_{12}(x)(s_{23} - s_{13}b_1(x)),$$

$$r_{31}(x) = \frac{s_{33}a_3(x) - b_3(x)r_{11}(x) - b_2(x)r_{21}(x)}{\varphi_2(x)},$$

$$r_{32}(x) = \frac{s_{33}a_2'(x) - b_3(x)(s_{12} + s_{13}a_2'(x)) - b_2'(x)r_{22}(x)}{\varphi_{12}(x)},$$

$$r_{33}(x) = s_{33} - s_{13}b_3(x) - b_2'(x)(s_{23} - s_{13}b_1(x)).$$

In view of the congruences (6) it is clear that $r_{21}(x), r_{31}(x), r_{32}(x) \in \mathbb{C}[x]$. This means that $\|r_{ij}(x)\|_1^3$ is a polynomial matrix. By direct verification we make sure that its determinant is equal to $s_{11}s_{22}s_{33} \neq 0$ and, moreover, the matrices $\|r_{ij}(x)\|_1^3$, $\|s_{ij}\|_1^3$, $A(x)$ and $B(x)$ satisfy equality (7). So, the matrices $A(x)$ and $B(x)$ are ssk.e. ♦

Suppose that conditions (2) and (3) hold for matrix $A(x)$. According to Proposition 3 [13] at fixed α_0, α_1 the greatest common divisor $(a_3(x), \varphi_1(x))$ does not depend on the choice of matrix $A(x)$ from the class of matrices that are with it ssk.e. The case $(a_3(x), \varphi_1(x)) = \varphi_1(x)$ was studied in the work [13]. Note that $(a_3(x), \varphi_1(x)) \neq \varphi_1(x)$ means that there exists a root $\alpha_2 \in M_1$ such that $a_3(\alpha_2) \neq 0$.

2. Reduction of a matrix to a special triangular form.

Proposition 2. *Suppose that for matrix $A(x)$ satisfying conditions* (2), (3) *we have $(a_3(x), \varphi_1(x)) \neq \varphi_1(x)$. Then matrix $A(x)$ is ssk.e. to a matrix $B(x)$ of the form* (4) *that satisfies the conditions of Proposition* 1 *and is such that $b_1(\alpha_2) = b_3(\alpha_2) \neq 0$ for some root $\alpha_2 \in M_1$.*

P r o o f. As already noted, for $A(x)$ there is a root $\alpha_2 \in M_1$ such that $a_3(\alpha_2) \neq 0$. If $a_1(\alpha_2) \neq 0$, then the desired matrix $B(x)$ is obtained by multiplying the last row and column of the matrix $A(x)$ by the corresponding constants. If $a_1(\alpha_2) = 0$, then we first pass from $A(x)$ to ssk.e. matrix of the form (4) with non-zero values in positions $(2,1)$, $(3,1)$ at $x = \alpha_2$, and then to the desired matrix. To do this, we choose a number $s_{23} \neq 0$ such that $1 + s_{23}a_2'(\alpha) \neq 0$ for every $\alpha \in M_1 \cup M_2$. Next, from the congruences

$$a_1(x) + s_{23}a_3(\alpha) - b_{10}(x) \equiv 0 \pmod{\varphi_1(x)},$$

$$a_2(x) - b_{20}(x)(1 + s_{23}a_2'(\alpha)) \equiv 0 \pmod{\varphi_2(x)},$$

$$a_3(x) - b_{30}(x) - b_{20}(x)\frac{a_1(x) + s_{23}a_3(\alpha) - b_{10}(x)}{\varphi_1(x)} \equiv 0 \pmod{\varphi_2(x)},$$

we successively find polynomials $b_{10}(x)$, $b_{20}(x)$, $b_{30}(x)$, $\deg b_{10}(x) < \deg \varphi_1(x)$, $\deg b_{20}(x), \deg b_{30}(x) < \deg \varphi_2(x)$ and construct a matrix $B_0(x)$ of the form (4) with elements $b_{10}(x)$, $b_{20}(x)$, $b_{30}(x)$. It is clear that $b_{10}(\alpha_0) = b_{30}(\alpha_0) = 0$, $b_{10}(\alpha_1) = 1$, $b_{30}(\alpha_1) = 0$, $b_{10}(\alpha_2) \neq 0$, $b_{30}(\alpha_2) = a_3(\alpha_2) \neq 0$. The above number $s_{23} \neq 0$ together with the numbers $s_{11} = s_{22} = s_{33} = 1$, $s_{12} = s_{13} = 0$ and the elements of the matrices $A(x)$, $B_0(x)$, as can be seen from the definition of $b_{10}(x)$, $b_{20}(x)$, $b_{30}(x)$, satisfy the congruence (6). Therefore, according to Proposition 1 the matrices $A(x)$ and $B_0(x)$ are ssk.e. The transition from $B_0(x)$ to the desired matrix is indicated above. ♦

Next, for matrix $A(x)$ with conditions (2), (3) consider the following partitions of the sets M_1, M_2:

$$M_1 = M_{11} \cup M_{12}, \qquad M_2 = M_{21} \cup M_{22}, \tag{8}$$

where $M_{11} = \{\alpha_i : a_3(\alpha_i) = 0\}$, $M_{21} = \{\beta_j : \delta_A(\beta_j) = 0\}$ and $\delta_A(x)$ is defined in (5).

Proposition 3. *Partitions* (8) *of the sets M_1, M_2 with fixed $\alpha_0, \alpha_1 \in M_1$ do not depend on the choice of $A(x)$ from the class of ssk.e. matrices.*

P r o o f. The invariance of the partition of the set M_1 from (8) follows from Proposition 3 [13]. Let the matrices $A(x)$ and $B(x)$ with the conditions of Proposition 1 are ssk.e. Then by Proposition 1 the congruences (6) are fulfilled, from which after exclusion s_{12} we come to

$$\delta_A(x)(s_{33} + s_{13}(x)\delta_B(x) - s_{23}b_2'(x)) \equiv s_{11}\delta_B(x) \pmod{\varphi_{12}(x)}, \tag{9}$$

where $\delta_B(x)$ is defined in (5). From the last congruence it follows the invariance of the partition M_2 from (8). ♦

3. Oriented by characteristic roots reduced matrix and its invariants.

Definition 1. Matrix $A(x)$ with conditions (2), (3) and the condition $a_1(\alpha_2) = a_3(\alpha_2) \neq 0$ for some root $\alpha_2 \in M_1$ is called *oriented by characteristic roots* $\alpha_0, \alpha_1, \alpha_2$ *reduced matrix.*

Proposition 4. *Let* $A(x)$ (1), $B(x)$ (4) *are oriented by characteristic roots* $\alpha_0, \alpha_1, \alpha_2$ *reduced matrices. Let also* $\delta_A(x)$, $\delta_B(x)$, $a_2'(x)$ *and* $b_2'(x)$ *are defined in* (5). *If matrices* $A(x)$, $B(x)$ *are ssk.e., then the following conditions are met*:

(***i***) $a_1(\alpha_i) = 0 \Leftrightarrow b_1(\alpha_i) = 0, \quad a_1(\alpha_i) = 1 \Leftrightarrow b_1(\alpha_i) = 1,$

$a_1(\alpha_i) = a_1(\alpha_j) \Leftrightarrow b_1(\alpha_i) = b_1(\alpha_j)$ *for each pair* $\alpha_i, \alpha_j \in M_{11}$;

(***ii***) $a_1(\alpha_k) = a_3(\alpha_k) \Leftrightarrow b_1(\alpha_k) = b_3(\alpha_k),$

$$\frac{a_1(\alpha_k)}{a_3(\alpha_k)} = \frac{a_1(\alpha_\ell)}{a_3(\alpha_\ell)} \Leftrightarrow \frac{b_1(\alpha_k)}{b_3(\alpha_k)} = \frac{b_1(\alpha_\ell)}{b_3(\alpha_\ell)} \quad \text{for each pair} \quad \alpha_k, \alpha_\ell \in M_{12};$$

(***iii***) $a_2'(\beta_i) = 0 \Leftrightarrow b_2'(\beta_i) = 0, \quad a_2'(\beta_i) = 1 \Leftrightarrow b_2'(\beta_i) = 1,$

$a_2'(\beta_i) = a_2'(\beta_j) \Leftrightarrow b_2'(\beta_i) = b_2'(\beta_j)$ *for each pair* $\beta_i, \beta_j \in M_{21}$;

(***iv***) $a_2'(\beta_k) = \delta_A(\beta_k) \Leftrightarrow b_2'(\beta_k) = \delta_B(\beta_k),$

$$\frac{a_2'(\beta_k)}{\delta_A(\beta_k)} = \frac{a_2'(\beta_\ell)}{\delta_A(\beta_\ell)} \Leftrightarrow \frac{b_2'(\beta_k)}{\delta_B(\beta_k)} = \frac{b_2'(\beta_\ell)}{\delta_B(\beta_\ell)} \quad \text{for each pair} \quad \beta_k, \beta_\ell \in M_{22}.$$

P r o o f. (***i***). If matrices $A(x)$, $B(x)$ are ssk.e., then congruences (6) are satisfied. From the first of them for α_1 and $\alpha_i, \alpha_j \in M_{11}$ we have

$$\begin{aligned} &s_{22} - s_{11} - s_{12} = 0, \\ &s_{22}a_1(\alpha_i) - b_1(\alpha_i)(s_{11} + s_{12}a_1(\alpha_i)) = 0, \\ &s_{22}a_1(\alpha_j) - b_1(\alpha_j)(s_{11} + s_{12}a_1(\alpha_j)) = 0. \end{aligned} \tag{10}$$

Since $s_{11}, s_{22} \neq 0$, the first equivalence of condition (***i***) follows from the second equality of system (10). The second equivalence of (***i***) follows from the first and second equalities of the system (10). Finally, the third equivalence for non-zero and non-unit values $a_1(\alpha_i)$, $a_1(\alpha_j)$, $b_1(\alpha_i)$, $b_1(\alpha_j)$ follows from the second and third equalities (10).

(***ii***). From (7) we have

$$s_{33}a_3(x) \equiv b_3(x)(s_{11} + s_{12}a_1(x) + s_{13}a_3(x)) \pmod{\varphi_1(x)}. \tag{11}$$

If we exclude s_{11}, s_{12}, s_{13} from the first congruence of (6) and (11), we come to

$$s_{22}a_1(x)b_3(x) + s_{23}a_3(x)b_3(x) - s_{33}a_3(x)b_1(x) \equiv 0 \pmod{\varphi_1(x)}. \tag{12}$$

From (12) for α_2 and $\alpha_k, \alpha_\ell \in M_{12}$ we have

$$s_{22} + s_{23} - s_{33} = 0\,,$$

$$s_{22}\frac{a_1(\alpha_k)}{a_3(\alpha_k)} + s_{23} - s_{33}\frac{b_1(\alpha_k)}{b_3(\alpha_k)} = 0\,,$$

$$s_{22}\frac{a_1(\alpha_\ell)}{a_3(\alpha_\ell)} + s_{23} - s_{33}\frac{b_1(\alpha_\ell)}{b_3(\alpha_\ell)} = 0\,. \qquad (13)$$

Since $s_{22}, s_{33} \neq 0$, from the first and second equalities (13) it follows the first equivalence of condition (***ii***), and from the second and third it follows the second equivalence of (***ii***).

(***iii***). From the third congruence of system (6) for an arbitrary pair $\beta_i, \beta_j \in M_{21}$ we obtain

$$s_{33}a_2'(\beta_i) - b_2'(\beta_i)(s_{22} + s_{23}a_2'(\beta_i)) = 0\,,$$

$$s_{33}a_2'(\beta_j) - b_2'(\beta_j)(s_{22} + s_{23}a_2'(\beta_j)) = 0\,. \qquad (14)$$

The first equivalence of condition (***iii***) follows from the first equality (14). From the first equalities (13) and (14) it follows the second equivalence of (***iii***). The third equivalence of (***iii***) for non-zero and non-unit values $a_2'(\beta_i)$, $a_2'(\beta_j)$, $b_2'(\beta_i)$, $b_2'(\beta_j)$ follows from both equalities (14).

(***iv***). Excluding s_{13}, s_{23}, s_{33} from the second and third congruences of (6), we can obtain

$$s_{22}b_2'(x)\delta_A(x) - s_{11}a_2'(x)\delta_B(x) - s_{12}\delta_A(x)\delta_B(x) \equiv 0 \pmod{\varphi_{12}(x)}. \qquad (15)$$

Taking $x = \beta_k$, $x = \beta_\ell$, $\beta_k, \beta_\ell \in M_{22}$, in (15), we can write the result in the form

$$s_{22}\frac{b_2'(\beta_k)}{\delta_B(\beta_k)} - s_{11}\frac{a_2'(\beta_k)}{\delta_A(\beta_k)} - s_{12} = 0\,,$$

$$s_{22}\frac{b_2'(\beta_\ell)}{\delta_B(\beta_\ell)} - s_{11}\frac{a_2'(\beta_\ell)}{\delta_A(\beta_\ell)} - s_{12} = 0\,. \qquad (16)$$

From the first equalities (10), (16) it follows the first equivalence of condition (***iv***), and from both equalities (16) the second equivalence of (***iv***). ♦

In what follows we use the following notations for matrix $A(x)$ (1):

$$\pi_A(\gamma) = \begin{cases} 1/a_1(\gamma), & \gamma \in M_{11},\ a_1(\gamma) \neq 0, \\ a_2'(\gamma)/\delta_A(\gamma), & \gamma \in M_{22}, \end{cases} \qquad (17)$$

$$\tilde{\pi}_A(\lambda) = \begin{cases} 1/a_2'(\lambda), & \lambda \in M_{21},\ a_2'(\lambda) \neq 0, \\ a_1(\lambda)/a_3(\lambda), & \lambda \in M_{12}. \end{cases} \qquad (18)$$

Proposition 5. *Suppose that for oriented by characteristic roots reduced matrix* $A(x)$ (1) *there exists a root* $\gamma_0 \in M_{11} \cup M_{22}$ *such that* $\pi_A(\gamma_0) \neq 1$ (*see* (17)) *or there exists a root* $\lambda_0 \in M_{12} \cup M_{21}$ *such that* $\tilde{\pi}_A(\lambda_0) \neq 1$ (*see* (18)). *Then the matrix* $A(x)$ *is ssk.e. to oriented by the same characteristic roots reduced matrix* $B(x)$ (4) *with a predetermined value* $\pi_B(\gamma_0) \neq 1$ *or* $\tilde{\pi}_B(\lambda_0) \neq 1$.

P r o o f. If we have $\pi_A(\gamma_0) \neq 1$ for some $\gamma_0 \in M_{11} \cup M_{22}$, then in the first step we construct a matrix $B(x)$ of the form (4) with the given value of

$\pi_B(\gamma_0)\neq\pi_A(\gamma_0)$ that is ssk.e. to the matrix $A(x)$. To do this, for each $\gamma\in M_{22}$ and for each $\gamma\in M_{11}$ such that $a_1(\gamma)\neq 0$, denote $I(\gamma):=(\pi_A(\gamma)-1)/(\pi_A(\gamma_0)-1)$. Choose some non-zero value of $\pi_B(\gamma_0)\neq 1$, different from $(I(\gamma)-1)/I(\gamma)$ and $(\pi_A(\gamma_0)a_1(\alpha)-1)/(a_1(\alpha)-1)$ for all $\alpha\in M_{12}$. For each of these γ and α we find

$$\pi_B(\gamma)=I(\gamma)(\pi_B(\gamma_0)-1)+1\neq 0 \tag{19}$$

and

$$b_3(\alpha)=\frac{a_3(\alpha)(1-\pi_A(\gamma_0))}{a_1(\alpha)\big(\pi_B(\gamma_0)-\pi_A(\gamma_0)\big)-\pi_B(\gamma_0)+1}, \tag{20}$$

respectively. It is easy to see that $\pi_B(\gamma_0)$ is chosen so that the denominator in (20) is non-zero. Define a polynomial $b_1(x)$ of degree $\deg b_1(x)<\deg\varphi_1(x)$ by its values $b_1(\alpha)$ on the set M_1 in the following way:

$$b_1(\alpha)=\begin{cases}0, & \alpha\in M_{11},\ a_1(\alpha)=0,\\ 1/\pi_B(\gamma), & \alpha=\gamma\in M_{11},\ a_1(\alpha)\neq 0,\\ b_3(\alpha)a_1(\alpha)/a_3(\alpha), & \alpha\in M_{12},\end{cases} \tag{21}$$

where $\pi_B(\gamma)$ and $b_3(\alpha)$ are defined in (19) and (20). Since $\deg b_1(x)<\operatorname{card} M_1$, then $b_1(x)$ will be defined unambiguously. We find some non-zero solution of equation

$$\begin{Vmatrix}1 & -1 & 1\\ \pi_A(\gamma_0) & -\pi_B(\gamma_0) & 1\end{Vmatrix}\begin{Vmatrix}x\\ y\\ z\end{Vmatrix}=\begin{Vmatrix}0\\ 0\end{Vmatrix} \tag{22}$$

with unknown $\|x\quad y\quad z\|^{\top}$. Since $\pi_A(\gamma_0),\pi_B(\gamma_0)\neq 1$ and $\pi_A(\gamma_0)\neq\pi_B(\gamma_0)$, there is a solution $\|x\quad y\quad z\|^{\top}=\|s_{11}\quad s_{22}\quad s_{12}\|^{\top}$ of equation (22) with non-zero all its components. Since

$$(\pi_A(\gamma)-1)/(\pi_A(\gamma_0)-1)=(\pi_B(\gamma)-1)/(\pi_B(\gamma_0)-1)=I(\gamma)$$

(see (19) and definition $I(\gamma)$), the equality

$$s_{22}a_1(\gamma)-b_1(\gamma)\big(s_{11}+s_{12}a_1(\gamma)\big)=0$$

holds for each root $\gamma\in M_{11}$ (including γ_0 and a root γ such that $a_1(\gamma)=$ $=b_1(\gamma)=0$). Taking into account the definitions $b_1(\alpha)$ (21) and $b_3(\alpha)$ (20), it is easy to see that the last equality holds for any $\gamma=\alpha\in M_{12}$. Therefore, the congruence

$$s_{22}a_1(x)\equiv b_1(x)\big(s_{11}+s_{12}a_1(x)\big)\ (\operatorname{mod}\varphi_1(x)), \tag{23}$$

is true. Here $\big(s_{11}+s_{12}a_1(x),\varphi_1(x)\big)=1$. Construct a matrix $\|r_{ij}(x)\|_1^2$, where

$$r_{11}(x)=s_{11}+s_{12}a_1(x),\qquad r_{12}(x)=s_{12}\varphi_1(x),$$

$$r_{21}(x)=\frac{s_{22}a_1(x)-b_1(x)(s_{11}+s_{12}a_1(x))}{\varphi_1(x)}\in\mathbb{C}[x]\ \text{(see the congruence (23))},$$

$$r_{22}(x)=s_{22}-s_{12}b_1(x).$$

Since matrix $\|r_{ij}(x)\|_1^2$ is not singular (its determinant is $s_{11}s_{22}\neq 0$), then from the congruence

$$s_{22}\left\|a_3(x) \quad a_2(x)\right\| \equiv \left\|b_3(x) \quad b_2(x)\right\| \left\|r_{ij}(x)\right\|_1^2 \pmod{\varphi_2(x)} \tag{24}$$

we get polynomials $b_2(x)$, $b_3(x)$ of degrees $\deg b_2(x), \deg b_3(x) < \deg \varphi_2(x)$.

Elements $a_1(x)$, $a_2(x)$, $a_3(x)$ of matrix $A(x)$, polynomials $b_1(x)$, $b_2(x)$, $b_3(x)$ defined above and numbers $s_{11}, s_{22}, s_{33} = s_{22}$, $s_{12}, s_{13} = s_{23} = 0$ satisfy congruences (6) (see (23), (24)). Therefore, according to Proposition 1, the matrix $B(x)$ of the form (4) constructed on the elements $b_1(x)$, $b_2(x)$, $b_3(x)$ and the matrix $A(x)$ are ssk.e. In addition, $B(x)$ is an oriented by the characteristic roots reduced matrix (it is oriented by the same characteristic roots as the matrix $A(x)$). If $\gamma_0 \in M_{11}$, then from (21) we have $\pi_B(\gamma_0) = 1/b_1(\gamma_0)$. If $\gamma_0 \in M_{22}$, then from (15) we have $\pi_B(\gamma_0) = b_2'(\gamma_0)/\delta_B(\gamma_0)$. Therefore, in each case ($\gamma_0 \in M_{11}$ or $\gamma_0 \in M_{22}$), the value $\pi_B(\gamma_0)$ for $B(x)$ is predetermined.

If for the matrix $B(x)$, as well as for $A(x)$, for every $\lambda \in M_{12} \cup M_{21}$ we have $\tilde{\pi}_B(\lambda) = 1$ (respectively $\tilde{\pi}_A(\lambda) = 1$, see Proposition 4), then everything is already proved. Otherwise, in order not to introduce new notations, we will assume that the value $\pi_A(\gamma_0)$ for matrix $A(x)$ coincides with the pre-selected one. Suppose that for some $\lambda_0 \in M_{12} \cup M_{21}$ we have $\tilde{\pi}_A(\lambda_0) \neq 1$. Then in the second step we pass from matrix $A(x)$ to matrix $B(x)$ of the form (4), which is ssk.e. to $A(x)$, oriented by the same characteristic roots and reduced and for which the value $\tilde{\pi}_B(\lambda_0)$ is pre-selected, and the value $\pi_B(\gamma_0)$ coincides with the corresponding value $\pi_A(\gamma_0)$ for $A(x)$. We introduce the notation $J(\lambda) := (\tilde{\pi}_A(\lambda) - 1)/(\tilde{\pi}_A(\lambda_0) - 1)$ for every $\lambda \in M_{12}$ and for every $\lambda \in M_{21}$ such that $a_2'(\lambda) \neq 0$. We fix some non-zero value $\tilde{\pi}_B(\lambda_0) \neq 1$, different from $(J(\gamma) - 1)/J(\gamma)$ and $(\tilde{\pi}_A(\lambda_0)a_2'(\beta) - 1)/(a_2'(\beta) - 1)$ for each $\beta \in M_{22}$. For each of the above $\lambda \in M_{12}$ and $\beta \in M_{22}$ we find, respectively,

$$\tilde{\pi}_B(\lambda) = J(\lambda)\left(\tilde{\pi}_B(\lambda_0) - 1\right) + 1 \neq 0 \tag{25}$$

and

$$\delta_B(\beta) = \frac{\delta_A(\beta)\left(1 - \tilde{\pi}_A(\lambda_0)\right)}{a_2'(\beta)\left(\tilde{\pi}_B(\lambda_0) - \tilde{\pi}_A(\lambda_0)\right) - \tilde{\pi}_B(\lambda_0) + 1}. \tag{26}$$

It should be noted that $\tilde{\pi}_B(\lambda_0)$ is chosen so that the denominator in relation (26) is non-zero. Define a polynomial $b_2'(x)$ of degree less than $\deg \varphi_{12}(x)$ by values $b_2'(\beta)$ on the set M_2 such that

$$b_2'(\beta) = \begin{cases} 0, & \beta \in M_{21},\, a_2'(\beta) = 0, \\ 1/\tilde{\pi}_B(\lambda), & \beta = \lambda \in M_{21},\, a_2'(\beta) \neq 0, \\ \delta_B(\beta)a_2'(\beta)/\delta_A(\beta), & \beta \in M_{22}, \end{cases} \tag{27}$$

where $\tilde{\pi}_B(\lambda)$ and $\delta_B(\beta)$ are given by (25) and (26). Polynomial $b_2'(x)$ is uniquely defined as if $\deg b_2'(x) < \operatorname{card} M_2$. Let us find some non-zero solution of the equation

$$\begin{Vmatrix} 1 & -1 & 1 \\ \tilde{\pi}_A(\lambda_0) & -\tilde{\pi}_B(\lambda_0) & 1 \end{Vmatrix} \begin{Vmatrix} x \\ y \\ z \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \end{Vmatrix} \tag{28}$$

with respect to the unknown $\|x \quad y \quad z\|^{\top}$. Obviously, since $\tilde{\pi}_A(\lambda_0), \tilde{\pi}_B(\lambda_0) \neq 1$ and $\tilde{\pi}_A(\lambda_0) \neq \tilde{\pi}_B(\lambda_0)$, we can find solution $\|x \quad y \quad z\|^{\top} = \|s_{22} \quad s_{33} \quad s_{23}\|^{\top}$, where $s_{22}s_{33}s_{23} \neq 0$. By values $b_1(\alpha)$ for all $\alpha \in M_1 = M_{11} \cup M_{12}$ from congruence

$$s_{22}a_1(x) + s_{23}a_3(x) \equiv s_{22}b_1(x) \pmod{\varphi_1(x)}, \tag{29}$$

where s_{22}, s_{23} are taken from the solution of equation (28), we uniquely define a polynomial $b_1(x)$ such that $\deg b_1(x) < \deg \varphi_1(x)$. Also from the congruence

$$s_{33}a_3(x) - b_2'(x)\big(s_{22}(a_1(x) - b_1(x)) + s_{23}a_3(x)\big) \equiv s_{22}b_3(x) \pmod{\varphi_2(x)} \tag{30}$$

by polynomial $b_2'(x)$, defined above from (27), we find polynomial $b_3(x)$ such that $\deg b_3(x) < \deg \varphi_2(x)$.

Since

$$\det\begin{Vmatrix} 1 & -1 & 1 \\ \tilde{\pi}_A(\lambda_0) & -\tilde{\pi}_B(\lambda_0) & 1 \\ \tilde{\pi}_A(\lambda) & -\tilde{\pi}_B(\lambda) & 1 \end{Vmatrix} = 0, \quad \det\begin{Vmatrix} 1 & -1 & 1 \\ \tilde{\pi}_A(\lambda_0) & -\tilde{\pi}_B(\lambda_0) & 1 \\ 1/\delta_A(\beta) & -1/\delta_B(\beta) & a_2'(\beta)/\delta_A(\beta) \end{Vmatrix} = 0,$$

(see definitions $\tilde{\pi}_B(\lambda)$ (25) and $\delta_B(\beta)$ (26)), we have

$$s_{22}b_2'(\lambda) - s_{33}a_2'(\lambda) + s_{23}a_2'(\lambda)b_2'(\lambda) = 0, \qquad \lambda \in M_{21}, \tag{31}$$

and

$$s_{22}/\delta_A(\beta) - s_{33}/\delta_B(\beta) + s_{23}a_2'(\beta)/\delta_A(\beta) = 0$$

or

$$s_{22}\delta_B(\beta) - s_{33}\delta_A(\beta) + s_{23}a_2'(\beta)\delta_B(\beta) = 0.$$

Taking into account $a_2'(\beta)/\delta_A(\beta) = b_2'(\beta)/\delta_B(\beta)$ (see (27)), from the last equality we obtain

$$s_{22}b_2'(\beta) - s_{33}a_2'(\beta) + s_{23}a_2'(\beta)b_2'(\beta) = 0, \qquad \beta \in M_{22}. \tag{32}$$

From (31) and (32) it follows that

$$s_{22}b_2'(x) - s_{33}a_2'(x) + s_{23}a_2'(x)b_2'(x) \equiv 0 \pmod{\varphi_{12}(x)}. \tag{33}$$

From the elements $b_1(x)$, $b_2(x) = b_2'(x)\varphi_{12}(x)$ and $b_3(x)$ defined from (29), (27) and (30), we construct a matrix $B(x)$ of the form (4) and show that it is ssk.e to $A(x)$. To do this, based on (29), (30) and (33), it suffices to make sure that the elements of matrices $A(x)$ and $B(x)$ together with the numbers $s_{11} = s_{22}, s_{22}, s_{33}$, $s_{12} = s_{13} = 0$ and s_{23} satisfy the congruences (6) and then apply Proposition 1.

For $\gamma_0 \in M_{22}$, it follows from (27) that $b_2'(\gamma_0)/\delta_B(\gamma_0) = a_2'(\gamma_0)/\delta_A(\gamma_0)$. For $\gamma_0 \in M_{11}$, it follows from (29) that $a_1(\gamma_0) = b_1(\gamma_0)$. Therefore, in each case

$$\pi_B(\gamma_0) = \pi_A(\gamma_0).$$

If $\lambda_0 \in M_{12}$, then from congruences (29) and (30) we have

$$s_{22}\big(a_1(\lambda_0) - b_1(\lambda_0)\big) + s_{23}a_3(\lambda_0) = 0$$

and

$$s_{33}a_3(\lambda_0) - s_{22}b_3(\lambda_0) = 0,$$

respectively, whence it follows

$$s_{22}a_1(\lambda_0)/a_3(\lambda_0) - s_{33}b_1(\lambda_0)/b_3(\lambda_0) + s_{23} = 0 .$$

Since $\tilde{\pi}_A(\lambda_0) = a_1(\lambda_0)/a_3(\lambda_0)$, we have $b_1(\lambda_0)/b_3(\lambda_0) = \tilde{\pi}_B(\lambda_0)$. If $\lambda_0 \in M_{21}$, then from (27) we have $\tilde{\pi}_B(\lambda_0) = 1/b_2'(\lambda_0)$. This means that the value $\tilde{\pi}_B(\lambda_0)$ for $B(x)$ coincides with the selected one, regardless of whether $\lambda_0 \in M_{12}$ or $\lambda_0 \in M_{21}$. This proves the existence of the required matrix $B(x)$. ♦

5. Main results.

Theorem 1. *Suppose that for oriented by the same characteristic roots* α_0, α_1, α_2 *reduced matrices* $A(x)$ (1) *and* $B(x)$ (4), *we have* $\pi_A(\gamma_0) = \pi_B(\gamma_0) \neq 1$ *and* $\tilde{\pi}_A(\lambda_0) = \tilde{\pi}_B(\lambda_0) \neq 1$ *for some* $\gamma_0 \in M_{11} \cup M_{22}$ *and* $\lambda_0 \in M_{12} \cup M_{21}$ (*see* (17) *and* (18)). *Matrices* $A(x)$, $B(x)$ *are ssk.e. if and only if the following conditions are met*:

(***i***) $a_1(\alpha_i) = b_1(\alpha_i)$ *for each root* $\alpha_i \in M_{11}$;

(***ii***) $\dfrac{a_1(\alpha_j)}{a_3(\alpha_j)} = \dfrac{b_1(\alpha_j)}{b_3(\alpha_j)}$ *for each root* $\alpha_j \in M_{12}$;

(***iii***) $a_2'(\beta_k) = b_2'(\beta_k)$ *for each root* $\beta_k \in M_{21}$;

(***iv***) $\dfrac{a_2'(\beta_\ell)}{\delta_A(\beta_\ell)} = \dfrac{b_2'(\beta_\ell)}{\delta_B(\beta_\ell)}$ *for each root* $\beta_\ell \in M_{22}$;

(***v***) $\dfrac{1}{\delta_A(\beta_\ell)} + \dfrac{1}{a_1(\alpha_2)} = \dfrac{1}{\delta_B(\beta_\ell)} + \dfrac{1}{b_1(\alpha_2)}$ *for each root* $\beta_\ell \in M_{22}$;

(***vi***) $\dfrac{1}{a_3(\alpha_j)} - \dfrac{1}{a_1(\alpha_2)} = \dfrac{1}{b_3(\alpha_j)} - \dfrac{1}{b_1(\alpha_2)}$ *for each root* $\alpha_j \in M_{12}$.

P r o o f. *Necessity.* For ssk.e. matrices $A(x)$ and $B(x)$ equality (7) is true where according to Proposition 3 [13], the matrix $\|s_{ij}\|_1^3$ has an upper triangular form. Its elements together with the elements of matrices $A(x)$ and $B(x)$ satisfy the congruences (6). From the first of them for $x = \alpha_1$ and for $x = \gamma_0$ in case $\gamma_0 \in M_{11}$ we have

$$s_{22} - s_{11} - s_{12} = 0 \tag{34}$$

and

$$s_{22}\pi_B(\gamma_0) - s_{11}\pi_A(\gamma_0) - s_{12}\pi_A(\gamma_0)\pi_B(\gamma_0) = 0 , \tag{35}$$

respectively. The last equality is obtained from (15) for $\gamma_0 \in M_{22}$. Since $0 \neq \pi_A(\gamma_0) = \pi_B(\gamma_0) \neq 1$, then from (34) and (35) we have $s_{12} = 0$, $s_{11} = s_{22}$. From (12) for $x = \alpha_2$ and $x = \lambda_0$ in case $\lambda_0 \in M_{12}$ it follows that

$$s_{33} - s_{22} - s_{23} = 0 \tag{36}$$

and

$$s_{33}\tilde{\pi}_B(\lambda_0) - s_{22}\tilde{\pi}_A(\lambda_0) - s_{23}\tilde{\pi}_A(\lambda_0)\tilde{\pi}_B(\lambda_0) = 0 , \tag{37}$$

respectively. The last equality is a result of substitution $x = \lambda_0$ in the third congruence of (6), if $\lambda_0 \in M_{21}$. Since $0 \neq \tilde{\pi}_A(\lambda_0) = \tilde{\pi}_B(\lambda_0) \neq 1$, from (36) and (37) it follows that $s_{23} = 0$, $s_{22} = s_{33}$. If we remember that $s_{11} = s_{22} = s_{33}$ and $s_{12} = s_{23} = 0$, then the first congruence of (6) together with (12) will take the form

$$s_{11}\big(a_1(x)-b_1(x)\big)-s_{13}a_3(x)b_1(x)\equiv 0 \ (\mathrm{mod}\ \varphi_1(x)),$$

$$s_{11}\big(a_1(x)b_3(x)-a_3(x)b_1(x)\big)\equiv 0 \ (\mathrm{mod}\ \varphi_1(x)), \tag{38}$$

and the third congruence of (6) and (15) will take the form

$$s_{11}\big(a_2'(x)-b_2'(x)\big)+s_{13}a_2'(x)\delta_B(x)\equiv 0 \ (\mathrm{mod}\ \varphi_{12}(x)),$$

$$s_{11}\big(a_2'(x)\delta_B(x)-b_2'(x)\delta_A(x)\big)\equiv 0 \ (\mathrm{mod}\ \varphi_{12}(x)). \tag{39}$$

Also now the congruence (9) can be written as

$$s_{11}\big(\delta_A(x)-\delta_B(x)\big)+s_{13}\delta_A(x)\delta_B(x)\equiv 0 \ (\mathrm{mod}\ \varphi_{12}(x)). \tag{40}$$

From (38) and (39) we obtain conditions (***i***), (***ii***) and (***iii***), (***iv***), respectively. From the first congruence of (38) for α_2 and from (40) for any $\beta_\ell \in M_{22}$ we have

$$s_{11}\big(1/b_1(\alpha_2)-1/a_1(\alpha_2)\big)-s_{13}=0 \tag{41}$$

and

$$s_{11}\big(1/\delta_B(\beta_\ell)-1/\delta_A(\beta_\ell)\big)+s_{13}=0, \tag{42}$$

respectively. Since $s_{11}\neq 0$, condition (***v***) follows from (41) and (42). From (11) for any $\alpha_j \in M_{12}$ we have the equality

$$s_{11}\big(1/b_3(\alpha_j)-1/a_3(\alpha_j)\big)-s_{13}=0.$$

Comparing it with (41), we obtain the condition (***vi***).

Sufficiency. Suppose that conditions (***i***)−(***vi***) of Theorem hold for matrices $A(x)$, $B(x)$. Consider the equation

$$\begin{Vmatrix} 1/a_1(\alpha_2)-1/b_1(\alpha_2) & 1 \\ 1/\delta_A(\beta_\ell)-1/\delta_B(\beta_\ell) & 1 \\ 1/a_3(\alpha_j)-1/\delta_3(a_j) & 1 \end{Vmatrix} \begin{Vmatrix} x \\ y \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix} \tag{43}$$

with respect to the unknown $\|x \quad y\|^\top$, where α_j and β_ℓ are arbitrary elements of sets M_{12} and M_{22}, respectively. From conditions (***v***), (***vi***) it follows that the first row of the matrix of this equation is linearly dependent on each of its next two rows. Therefore, this equation has a solution $\|x \quad y\|^\top = \|s_{11} \quad s_{13}\|^\top$ with its non-zero first component s_{11}. The fact that $\|s_{11} \quad s_{13}\|^\top$ satisfies equation (43) yields the equality

$$s_{11}\big(a_3(\alpha_j)-b_3(\alpha_j)\big)-s_{13}a_3(\alpha_j)b_3(\alpha_j)=0. \tag{44}$$

Whence, taking into account the condition (***ii***), we will have

$$s_{11}\big(a_1(\alpha_j)-b_1(\alpha_j)\big)-s_{13}a_3(\alpha_j)b_1(\alpha_j)=0$$

for all $\alpha_j \in M_{12}$. Under condition (***i***), the obtained equality holds for any $\alpha_j \in M_{11}$. Therefore, the first congruence of the system (38) is true. Since $\|s_{11} \quad s_{13}\|^\top$ is the solution of equation (43), we have

$$s_{11}\big(\delta_A(\beta_\ell)-\delta_B(\beta_\ell)\big)+s_{13}\delta_A(\beta_\ell)\delta_B(\beta_\ell)=0 \tag{45}$$

for any $\beta_\ell \in M_{22}$. Taking into account condition (***iv***), the last equality we can write as

$$s_{11}\big(a_2'(\beta_\ell)-b_2'(\beta_\ell)\big)+s_{13}a_2'(\beta_\ell)\delta_B(\beta_\ell)=0.$$

Under condition (***iii***), the obtained equality holds for all $\beta_\ell \in M_{21}$. Therefore, the first congruence of the system (39) is true. Equality (45) confirms the truth of congruence (40). Subtracting the latter from the first congruence of (39) multiplied by $a_1(x)$ we obtain

$$s_{11}(a_3(x) - b_3(x)) + s_{11}b_2'(x)(b_1(x) - a_1(x)) + \\ + s_{13}a_3(x)\delta_B(x) \equiv 0 \pmod{\varphi_{12}(x)} \tag{46}$$

or

$$s_{11}(a_3(x) - b_3(x)) - s_{13}a_3(x)b_3(x) \equiv \\ \equiv b_2'(x)(s_{11}(a_1(x) - b_1(x)) - s_{13}a_3(x)b_1(x)) \pmod{\varphi_{12}(x)}$$

given that $\delta_B(x) = b_1(x)b_2'(x) - b_3(x)$. The left and right sides of the resulting congruence are divided by $\varphi_1(x)$. This follows from equality (44) and the first congruence of (38). This means that the left side of the congruence (46) is divisible by $\varphi_1(x)$. Therefore, this congruence is performed according by module $\varphi_2(x) = \varphi_1(x)\varphi_{12}(x)$, due to the mutual simplicity of $\varphi_1(x)$ and $\varphi_{12}(x)$. It is clear that the first congruence of (38), the congruence (46) modulo $\varphi_2(x)$ and the first congruence of (39) coincide with the first, second and third congruences of system (6) for the numbers $s_{11} = s_{22} = s_{33}$, $s_{12} = s_{23} = 0$, s_{13}, respectively. Here s_{11} and s_{13} are components of the solution of equation (43). Therefore, according to Proposition 1, the matrices $A(x)$ and $B(x)$ are ssk.e. ♦

Theorem 2. *Suppose that for oriented by the same characteristic roots* α_0, α_1, α_2 *reduced matrices* $A(x)$ (1) *and* $B(x)$ (4), *we have* $\pi_A(\gamma_0) = \pi_B(\gamma_0) \neq 1$ *for some root* $\gamma_0 \in M_{11} \cup M_{22}$ *and* $\tilde{\pi}_A(\lambda) = \tilde{\pi}_B(\lambda) = 1$ *for every root* $\lambda \in M_{12} \cup M_{21}$ *(see* (17) *and* (18)). *Matrices* $A(x)$ *and* $B(x)$ *are ssk.e. if and only if the following conditions are met*:

(***i***) $a_1(\alpha_i) = b_1(\alpha_i)$ *for each root* $\alpha_i \in M_{11}$;

(***ii***) $\dfrac{a_2'(\beta_\ell)}{\delta_A(\beta_\ell)} = \dfrac{b_2'(\beta_\ell)}{\delta_B(\beta_\ell)}$ *for each root* $\beta_\ell \in M_{22}$;

(***iii***) $a_3(\alpha_j) = a_3(\alpha_k) \Leftrightarrow b_3(\alpha_j) = b_3(\alpha_k)$ *for each pair of roots* $\alpha_j, \alpha_k \in M_{12}$;

(***iv***) $\delta_A(\beta_\ell) - a_3(\alpha_2)(a_2'(\beta_\ell) - 1) = 0 \Leftrightarrow \delta_B(\beta_\ell) - b_3(\alpha_2)(b_2'(\beta_\ell) - 1) = 0$ *for each root* $\beta_\ell \in M_{22}$;

(***v***) *if, for some root* $\alpha \in M_{12}$, *we have* $a_3(\alpha) \neq a_3(\alpha_2)$ *or for some root* $\beta \in M_{22}$, *we have* $\delta_A(\beta) - a_3(\alpha_2)(a_2'(\beta) - 1) \neq 0$ *then*

$$\frac{(a_3(\alpha_k) - a_3(\alpha_2))a_3(\alpha)}{(a_3(\alpha) - a_3(\alpha_2))a_3(\alpha_k)} = \frac{(b_3(\alpha_k) - b_3(\alpha_2))b_3(\alpha)}{(b_3(\alpha) - b_3(\alpha_2))b_3(\alpha_k)} \tag{47}$$

and

$$\frac{(\delta_A(\beta_\ell) - a_3(\alpha_2)(a_2'(\beta_\ell) - 1))a_3(\alpha)}{(a_3(\alpha) - a_3(\alpha_2))\delta_A(\beta_\ell)} = \\ = \frac{(\delta_B(\beta_\ell) - b_3(\alpha_2)(b_2'(\beta_\ell) - 1))b_3(\alpha)}{(b_3(\alpha) - b_3(\alpha_2))\delta_B(\beta_\ell)} \tag{48}$$

or

$$\frac{(a_3(\alpha_k)-a_3(\alpha_2))\delta_A(\beta)}{(\delta_A(\beta)-a_3(\alpha_2)(a_2'(\beta)-1))a_3(\alpha_k)}=$$

$$=\frac{(b_3(\alpha_k)-b_3(\alpha_2))\delta_B(\beta)}{(\delta_B(\beta)-b_3(\alpha_2)(b_2'(\beta)-1))b_3(\alpha_k)} \tag{49}$$

and

$$\frac{(\delta_A(\beta_\ell)-a_3(\alpha_2)(a_2'(\beta_\ell)-1))\delta_A(\beta)}{(\delta_A(\beta)-a_3(\alpha_2)(a_2'(\beta)-1))\delta_A(\beta_\ell)}=$$

$$=\frac{(\delta_B(\beta_\ell)-b_3(\alpha_2)(b_2'(\beta_\ell)-1))\delta_B(\beta)}{(\delta_B(\beta)-b_3(\alpha_2)(b_2'(\beta)-1))\delta_B(\beta_\ell)} \tag{50}$$

for each pair of roots $\alpha_k \in M_{12}$, $\beta_\ell \in M_{22}$.

P r o o f. *Necessity*. Assume that matrices $A(x)$ and $B(x)$ are ssk.e. Then relation (7) holds for them. By the same reasoning as in the proof of Theorem 1, we come to the conclusion that in this relation the matrix $\|s_{ij}\|_1^3$ has an upper triangular form, where $s_{11}=s_{22}$ and $s_{12}=0$. Then the first congruence of (6) and congruences (15), (11) will take the form

$$s_{22}a_1(x)+s_{23}a_3(x)\equiv b_1(x)(s_{11}+s_{13}a_3(x)) \pmod{\varphi_1(x)}$$

and

$$s_{22}b_2'(x)\delta_A(x)-s_{11}a_2'(x)\delta_B(x)\equiv 0 \pmod{\varphi_{12}(x)},$$

$$s_{33}a_3(x)\equiv b_3(x)(s_{11}+s_{13}a_3(x)) \pmod{\varphi_1(x)}, \tag{51}$$

respectively. The latter three congruences yields, respectively, conditions (***i***), (***ii***) and (***iii***). Substituting $x=\alpha_2$ in (14) and (51) and taking arbitrary $\beta_\ell \in$ $\in M_{22}$ for x in (9) we obtain three equalities, which can be written in matrix form as

$$\begin{Vmatrix} 1 & -1 & 0 & 1 \\ 1/a_3(\alpha_2) & -1/b_3(\alpha_2) & 1 & 0 \\ 1/\delta_A(\beta_\ell) & -1/\delta_B(\beta_\ell) & -1 & b_2'(\beta_\ell)/\delta_B(\beta_\ell) \end{Vmatrix} \begin{Vmatrix} s_{11} \\ s_{33} \\ s_{13} \\ s_{23} \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix}. \tag{52}$$

From the equality (52) we have relation

$$s_{11}\frac{\delta_A(\beta_\ell)-a_3(\alpha_2)(a_2'(\beta_\ell)-1)}{\delta_A(\beta_\ell)a_3(\alpha_2)}-s_{33}\frac{\delta_B(\beta_\ell)-b_3(\alpha_2)(b_2'(\beta_\ell)-1)}{\delta_B(\beta_\ell)b_3(\alpha_2)}=0$$

which yields condition (***iv***), since $s_{11}, s_{33}\neq 0$. If for some fixed $\alpha \in M_{12}$ we have $a_3(\alpha)\neq a_3(\alpha_2)$ ($b_3(\alpha)\neq b_3(\alpha_2)$ under condition (***iii***)), then substituting $x=\alpha$ and arbitrary $x=\alpha_k \in M_{12}$ into (51) we get two equalities, which in combination with (52) can be written in the form

$$\begin{Vmatrix} 1 & -1 & 0 & 1 \\ 1/a_3(\alpha_2) & -1/b_3(\alpha_2) & 1 & 0 \\ 1/a_3(\alpha) & -1/b_3(\alpha) & 1 & 0 \\ 1/a_3(\alpha_k) & -1/b_3(\alpha_k) & 1 & 0 \\ 1/\delta_A(\beta_\ell) & -1/\delta_B(\beta_\ell) & -1 & b_2'(\beta_\ell)/\delta_B(\beta_\ell) \end{Vmatrix} \begin{Vmatrix} s_{11} \\ s_{33} \\ s_{13} \\ s_{23} \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Vmatrix}. \tag{53}$$

Given $s_{11}, s_{33} \neq 0$, we have

$$\det \begin{Vmatrix} 1/a_3(\alpha_2) & 1/b_3(\alpha_2) & 1 \\ 1/a_3(\alpha) & 1/b_3(\alpha) & 1 \\ 1/a_3(\alpha_k) & 1/b_3(\alpha_k) & 1 \end{Vmatrix} = 0$$

and

$$\det \begin{Vmatrix} 1 & 1 & 0 & 1 \\ 1/a_3(\alpha_2) & 1/b_3(\alpha_2) & 1 & 0 \\ 1/a_3(\alpha) & 1/b_3(\alpha) & 1 & 0 \\ 1/\delta_A(\beta_\ell) & 1/\delta_B(\beta_\ell) & -1 & b_2'(\beta_\ell)/\delta_B(\beta_\ell) \end{Vmatrix} = 0,$$

from which (47) and (48) follows, respectively. If for some $\beta \in M_{22}$, we have $\delta_A(\beta) - a_3(\alpha_2)(a_2'(\beta) - 1) \neq 0$ (or $\delta_B(\beta) - b_3(\alpha_2)(b_2'(\beta) - 1) \neq 0$, see condition (***iv***)), then on the basis of (52) and the two equalities obtained from (9), respectively, at $x = \beta$ and at any $x = \alpha_k \in M_{12}$, we can write the following matrix equality

$$\begin{Vmatrix} 1 & -1 & 0 & 1 \\ 1/a_3(\alpha_2) & -1/b_3(\alpha_2) & 1 & 0 \\ 1/\delta_A(\beta) & -1/b_B(\beta) & -1 & b_2'(\beta)/\delta_B(\beta) \\ 1/a_3(\alpha_k) & -1/b_3(\alpha_k) & 1 & 0 \\ 1/\delta_A(\beta_\ell) & -1/\delta_B(\beta_\ell) & -1 & b_2'(\beta_\ell)/\delta_B(\beta_\ell) \end{Vmatrix} \begin{Vmatrix} s_{11} \\ s_{33} \\ s_{13} \\ s_{23} \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Vmatrix}, \qquad (54)$$

from which equality (49) and (50) follow given that $s_{11}, s_{33} \neq 0$.

Sufficiency. Suppose that conditions (***i***)–(***v***) hold for the matrices $A(x)$ and $B(x)$, that are oriented by the same characteristic roots and reduced. If there is a root $\alpha \in M_{12}$ such that $a_3(\alpha) \neq a_3(\alpha_2)$ (under condition (***iii***) $b_3(\alpha) \neq b_3(\alpha_2)$), then in the 5×4-matrix of equation (53) the first three rows are linearly independent. Under conditions (47), (48), the fourth and fifth rows of this matrix (for arbitrary $\alpha_k \in M_{12}$ and $\beta_\ell \in M_{22}$) depend linearly on the first three rows of it. Also in the case of the existence of $\beta \in M_{22}$ such that $\delta_A(\beta) - a_3(\alpha_2)(a_2'(\beta) - 1) \neq 0$ (under condition (***iv***) $\delta_B(\beta) - b_3(\alpha_2)(b_2'(\beta) - 1) \neq 0$), in the matrix of equality (54) the first three rows are linearly independent, and the fourth and fifth rows can be expressed as linear combinations of them. This conclusion can be made if we take into account (49) and (50). If we denote by N_1 the matrix from equation (53) or if we denote by N_2 the matrix from equation (54), then the equation

$$N_1 \|x \quad y \quad u \quad v\|^\top = \|0 \quad 0 \quad 0 \quad 0 \quad 0\|^\top \qquad (55)$$

or equation

$$N_2 \|x \quad y \quad u \quad v\|^\top = \|0 \quad 0 \quad 0 \quad 0 \quad 0\|^\top \qquad (56)$$

has a solution with non-zero first two components. An solution with a similar property has an equation

$$\begin{Vmatrix} 1 & -1 & 0 & 1 \\ 1/a_3(\alpha_2) & -1/b_3(\alpha_2) & 1 & 0 \end{Vmatrix} \|x \quad y \quad u \quad v\|^\top = \|0 \quad 0\|^\top, \qquad (57)$$

if for every $\alpha_k \in M_{12}$ and every $\beta_\ell \in M_{22}$ we have $a_3(\alpha) = a_3(\alpha_2)$ and $\delta_A(\beta) - a_3(\alpha_2)(a_2'(\beta) - 1) = 0$, respectively. If $\|x \quad y \quad u \quad v\|^\top = \|s_{11} \quad s_{33} \quad s_{13} \quad s_{23}\|^\top$, where $s_{11}, s_{33} \neq 0$, is the solution of one of equations (55), (56) or (57), then

$$s_{11} - s_{33} + s_{23} = 0$$

and

$$s_{11}b_3(\alpha_k) - s_{33}a_3(\alpha_k) + s_{23}a_3(\alpha_k)b_3(\alpha_k) = 0 \qquad (58)$$

for every $\alpha_k \in M_{12}$. Excluding s_{33} from the last two equations, we obtain

$$s_{11}\big(a_3(\alpha_k) - s_{33}b_3(\alpha_k)\big) + s_{23}a_3(\alpha_k) - s_{13}a_3(\alpha_k)b_3(\alpha_k) = 0$$

or

$$s_{11}\big(a_1(\alpha_k) - s_{33}b_1(\alpha_k)\big) + s_{23}a_3(\alpha_k) - s_{13}a_3(\alpha_k)b_1(\alpha_k) = 0\,,$$

since we have for each root $\alpha_k \in M_{12}$ $a_3(\alpha_k) = a_1(\alpha_k)$ and $b_3(\alpha_k) = b_1(\alpha_k)$. Under condition (***i***), all roots $\alpha_k \in M_{11}$ also satisfy the last equality. Therefore, the following congruence

$$s_{11}\big(a_1(x) - b_1(x)\big) + s_{23}a_3(x) - s_{13}a_3(x)b_1(x) \equiv 0 \ (\mathrm{mod}\ \varphi_1(x)). \qquad (59)$$

holds. If M_{22} is not an empty set and $\|s_{11} \quad s_{33} \quad s_{13} \quad s_{23}\|^{\top}$ is the solution of equation (55) or (56), then we have the equality

$$s_{11}/\delta_A(\beta_\ell) - s_{33}/\delta_B(\beta_\ell) - s_{13} + s_{23}b_2'(\beta_\ell)/\delta_B(\beta_\ell) = 0 \qquad (60)$$

for all roots $\beta_\ell \in M_{22}$. This equality is also satisfied if $\delta_A(\beta_\ell) - a_3(\alpha_2)(a_2'(\beta_\ell) - 1) =$ $= 0$ for every $\beta_\ell \in M_{22}$, because in this case a row

$$\|1/\delta_A(\beta_\ell) \quad -1/\delta_B(\beta_\ell) \quad -1 \quad b_2'(\beta_\ell)/\delta_B(\beta_\ell)\|$$

depends linearly on the rows of the matrix of equation (57). Under condition (***ii***) from (60) we can proceed to equality

$$s_{11}b_2'(\beta_\ell) - s_{33}a_2'(\beta_\ell) - s_{13}a_2'(\beta_\ell)\delta_B(\beta_\ell) + s_{23}a_2'(\beta_\ell)b_2'(\beta_\ell) = 0\,.$$

The last equality is obviously performed for any $\beta_\ell \in M_{21}$, because in this case $a_2'(\beta_\ell)$ i and $b_2'(\beta_\ell)$ are simultaneously equal to zero or one and $\delta_B(\beta_\ell) = 0$. Therefore, we can write the congruence

$$s_{11}b_2'(x) - s_{33}a_2'(x) - s_{13}a_2'(x)\delta_B(x) + s_{23}a_2'(x)b_2'(x) \equiv 0 \ (\mathrm{mod}\ \varphi_{12}(x)). \qquad (61)$$

This congruence is also true when the set M_{22} is empty. Equality (60) is the basis for such a congruence

$$s_{11}\delta_B(x) - s_{33}\delta_A(x) - s_{13}\delta_A(x)\delta_B(x) + s_{23}b_2'(x)\delta_A(x) \equiv 0 \ (\mathrm{mod}\ \varphi_{12}(x)).$$

Subtracting the latter from the congruence (61) multiplied by $a_1(x)$ we obtain

$$s_{33}a_3(x) - s_{11}b_3(x) - s_{13}a_3(x)b_3(x) \equiv b_2'(x)\big(s_{11}(a_1(x) - b_1(x)) +$$

$$+ s_{23}a_3(x) - s_{13}a_3(x)b_1(x)\big) \equiv 0 \ (\mathrm{mod}\ \varphi_{12}(x)).$$

The left and right sides of the resulting congruence are divided by $\varphi_1(x)$ on the basis of (58) and (59), respectively. Since $(\varphi_1(x), \varphi_{12}(x)) = 1$ is true, the congruence

$$s_{33}a_3(x) + s_{11}\delta_B(x) + s_{13}a_3(x)\delta_B(x) \equiv$$

$$\equiv b_2'(x)\big(s_{11}a_1(x) + s_{23}a_3(x)\big) \ (\mathrm{mod}\ \varphi_2(x)). \qquad (62)$$

holds. Due to the fact that the congruences (59), (62) and (61) coincide with the congruences of system (6), according to Proposition 1 the matrices $A(x)$ and $B(x)$ are ssk.e. ♦

Theorem 3. *Suppose that for oriented by the same characteristic roots* α_0, α_1, α_2 *reduced matrices* $A(x)$ (1) *and* $B(x)$ (4), *we have* $\tilde{\pi}_A(\lambda_0) = \tilde{\pi}_B(\lambda_0) \neq 1$ *for some root* $\lambda_0 \in M_{12} \bigcup M_{21}$ *and* $\pi_A(\gamma) = \pi_B(\gamma) = 1$ *for every root* $\gamma \in M_{11} \bigcup M_{22}$ *(see* (17) *and* (18)*). Matrices* $A(x)$, $B(x)$ *are ssk.e. if and only if the following conditions are met*:

(***i***) $a_2'(\beta_i) = b_2'(\beta_i)$ *for each root* $\beta_i \in M_{21}$;

(***ii***) $\dfrac{a_1(\alpha_k)}{a_3(\alpha_k)} = \dfrac{b_1(\alpha_k)}{b_3(\alpha_k)}$ *for each root* $\alpha_k \in M_{12}$;

(***iii***) $\delta_A(\beta_\ell) - a_3(\alpha_2)(\delta_A(\beta_\ell) - 1) = 0 \ \Leftrightarrow\ \delta_B(\beta_\ell) - b_3(\alpha_2)(\delta_B(\beta_\ell) - 1) = 0$ *for each root* $\beta_l \in M_{22}$;

(***iv***) $a_3(\alpha_k) - a_3(\alpha_2)(a_3(\alpha_k) - a_1(\alpha_k) + 1) = 0 \Leftrightarrow$
$\Leftrightarrow b_3(\alpha_k) - b_3(\alpha_2)(b_3(\alpha_k) - b_1(\alpha_k) + 1) = 0$ *for each root* $\alpha_k \in M_{12}$;

(***v***) *if for some root* $\alpha \in M_{12}$, *we have* $a_3(\alpha) - a_3(\alpha_2)(a_3(\alpha) - a_1(\alpha) + 1) \neq 0$ *or for some root* $\beta \in M_{22}$, *we have* $\delta_A(\beta) - a_3(\alpha_2)(\delta_A(\beta) - 1) \neq 0$, *then*

$$\frac{(a_3(\alpha_k) - a_3(\alpha_2)(a_3(\alpha_k) - a_1(\alpha_k) + 1))a_3(\alpha)}{(a_3(\alpha) - a_3(\alpha_2)(a_3(\alpha) - a_1(\alpha) + 1))a_3(\alpha_k)} = \frac{(b_3(\alpha_k) - b_3(\alpha_2)(b_3(\alpha_k) - b_1(\alpha_k) + 1))b_3(\alpha)}{(b_3(\alpha) - b_3(\alpha_2)(b_3(\alpha) - b_1(\alpha) + 1))b_3(\alpha_k)} \tag{63}$$

and

$$\frac{(\delta_A(\beta_\ell) - a_3(\alpha_2)(\delta_A(\beta_\ell) - 1))a_3(\alpha)}{(a_3(\alpha) - a_3(\alpha_2)(a_3(\alpha) - a_1(\alpha) + 1))\delta_A(\beta_\ell)} = \frac{(\delta_B(\beta_\ell) - b_3(\alpha_2)(\delta_B(\beta_\ell) - 1))b_3(\alpha)}{(b_3(\alpha) - b_3(\alpha_2)(b_3(\alpha) - b_1(\alpha) + 1))\delta_B(\beta_\ell)} \tag{64}$$

or

$$\frac{(a_3(\alpha_k) - a_3(\alpha_2)(a_3(\alpha_k) - a_1(\alpha_k) + 1))\delta_A(\beta)}{(\delta_A(\beta) - a_3(\alpha_2)(\delta_A(\beta) - 1))a_3(\alpha_k)} = \frac{(b_3(\alpha_k) - b_3(\alpha_2)(b_3(\alpha_k) - b_1(\alpha_k) + 1))\delta_B(\beta)}{(\delta_B(\beta) - b_3(\alpha_2)(\delta_B(\beta) - 1))b_3(\alpha_k)} \tag{65}$$

and

$$\frac{\delta_A(\beta_\ell) - a_3(\alpha_2)(\delta_A(\beta_\ell) - 1)\delta_A(\beta)}{\delta_A(\beta) - a_3(\alpha_2)(\delta_A(\beta) - 1)\delta_A(\beta_\ell)} = \frac{\delta_B(\beta_\ell) - b_3(\alpha_2)(\delta_B(\beta_\ell) - 1)\delta_B(\beta)}{\delta_B(\beta) - b_3(\alpha_2)(\delta_B(\beta) - 1)\delta_B(\beta_\ell)} \tag{66}$$

for each pair of roots $\alpha_k \in M_{12}$, $\beta_\ell \in M_{22}$.

P r o o f. *Necessity.* Let the matrices $A(x)$ and $B(x)$ are ssk.e. Considerations similar to the proof of Theorem 1 lead to the conclusion that in relation (7) the left transforming matrix $\left\| s_{ij} \right\|_1^3$ has an upper triangular form and, moreover, $s_{22} = s_{33}$, $s_{23} = 0$. Then the third congruence of (6) yields condition (*i*) and (12) yields condition (***ii***). If there is a root $\alpha \in M_{12}$ such that $a_3(\alpha) - - a_3(\alpha)(a_3(\alpha) - a_1(\alpha) + 1) \neq 0$, then by substituting $x = \alpha_1$ into the first congruence of (6), $x = \alpha_2$, $x = \alpha$, $x = \alpha_k \in M_{12}$ into (11) and $x = \beta_\ell \in M_{22}$ into (9), we obtain five equalities, which can be written in the form

$$\left\| \begin{matrix} 1 & -1 & 0 & 1 \\ 1/a_3(\alpha_2) & -1/b_3(\alpha_2) & 1 & 1 \\ 1/a_3(\alpha) & -1/b_3(\alpha) & a_1(\alpha)/a_3(\alpha) & 1 \\ 1/a_3(\alpha_k) & -1/b_3(\alpha_k) & a_1(\alpha_k)/a_3(\alpha_k) & 1 \\ 1/\delta_A(\beta_\ell) & -1/\delta_B(\beta_\ell) & 0 & -1 \end{matrix} \right\| \left\| \begin{matrix} s_{11} \\ s_{33} \\ s_{12} \\ s_{13} \end{matrix} \right\| = \left\| \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} \right\|. \tag{67}$$

Recall that α_k and β_ℓ are arbitrary elements of M_{12} and M_{22}, respectively, and α is a fixed root of M_{12}. By scalarly equivalent transformations of rows of the matrix from (67) we pass to the equality from which we obtain

$$\left\| \begin{matrix} \dfrac{a_3(\alpha) - a_3(\alpha_2)\big(a_3(\alpha) - a_1(\alpha) + 1\big)}{a_3(\alpha)a_3(\alpha_2)} & -\dfrac{b_3(\alpha) - b_3(\alpha_2)\big(b_3(\alpha) - b_1(\alpha) + 1\big)}{b_3(\alpha)b_3(\alpha_2)} \\ \dfrac{a_3(\alpha_k) - a_3(\alpha_2)\big(a_3(\alpha_k) - a_1(\alpha_k) + 1\big)}{a_3(\alpha_k)a_3(\alpha_2)} & -\dfrac{b_3(\alpha_k) - b_3(\alpha_2)\big(b_3(\alpha_k) - b_1(\alpha_k) + 1\big)}{b_3(\alpha_k)b_3(\alpha_2)} \\ \dfrac{\delta_A(\beta_\ell) - a_3(\alpha_2)\big(\delta_A(\beta_\ell) - 1\big)}{\delta_A(\beta_\ell)a_3(\alpha_2)} & -\dfrac{\delta_B(\beta_\ell) - b_3(\alpha_2)\big(\delta_B(\beta_\ell) - 1\big)}{\delta_B(\beta_\ell)b_3(\alpha_2)} \end{matrix} \right\| \times$$

$$\times \left\| \begin{matrix} s_{11} \\ s_{33} \end{matrix} \right\| = \left\| \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \right\|.$$

From this, since $s_{11}, s_{33} \neq 0$, we obtain condition (***iv***), as well as relations (63), (64).

If there is a root $\beta \in M_{22}$ such that $\delta_A(\beta) - a_3(\alpha_2)(\delta_A(\beta) - 1) \neq 0$, then substituting $x = \alpha_1$ into the first congruence of (6), $x = \alpha_2$, $x = \alpha_k \in M_{12}$ into (11) and $x = \beta$ $x = \beta_\ell \in M_{22}$ into (9) we come to five equalities, which can be written in matrix form as

$$\left\| \begin{matrix} 1 & -1 & 1 & 0 \\ 1/a_3(\alpha_2) & -1/b_3(\alpha_2) & 1 & 1 \\ 1/\delta_A(\beta) & -1/b_B(\beta) & 0 & -1 \\ 1/a_3(\alpha_k) & -1/b_3(\alpha_k) & a_1(\alpha_k)/a_3(\alpha_k) & 1 \\ 1/\delta_A(\beta_\ell) & -1/\delta_B(\beta_\ell) & 0 & -1 \end{matrix} \right\| \left\| \begin{matrix} s_{11} \\ s_{33} \\ s_{12} \\ s_{13} \end{matrix} \right\| = \left\| \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} \right\|. \tag{68}$$

Here, as above α_k and β_ℓ, are arbitrary elements of M_{12} and M_{22}, respectively, and β is a fixed root of M_{22}. From equality (68) with the help of left equivalent transformations we come to the equality

$$\left\| \begin{matrix} \dfrac{\delta_A(\beta) - a_3(\alpha_2)\big(\delta_A(\beta) - 1\big)}{\delta_A(\beta)a_3(\alpha_2)} & -\dfrac{\delta_B(\beta) - b_3(\alpha_2)\big(\delta_B(\beta) - 1\big)}{\delta_B(\beta)b_3(\alpha_2)} \\ \dfrac{a_3(\alpha_k) - a_3(\alpha_2)\big(a_3(\alpha_k) - a_1(\alpha_k) + 1\big)}{a_3(\alpha_k)a_3(\alpha_2)} & -\dfrac{b_3(\alpha_k) - b_3(\alpha_2)\big(b_3(\alpha_k) - b_1(\alpha_k) + 1\big)}{b_3(\alpha_k)b_3(\alpha_2)} \\ \dfrac{\delta_A(\beta_\ell) - a_3(\alpha_2)\big(\delta_A(\beta_\ell) - 1\big)}{\delta_A(\beta_\ell)a_3(\alpha_2)} & -\dfrac{\delta_B(\beta_\ell) - b_3(\alpha_2)\big(\delta_B(\beta_\ell) - 1\big)}{\delta_B(\beta_\ell)b_3(\alpha_2)} \end{matrix} \right\| \times$$

$$\times \left\| \begin{matrix} s_{11} \\ s_{33} \end{matrix} \right\| = \left\| \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \right\|.$$

Whence it is easy to obtain condition (***iii***) and relations (65), (66).

Sufficiency. Suppose that the conditions of Theorem hold for matrices $A(x)$ and $B(x)$. Suppose that there exists the root $\alpha \in M_{12}$ specified in condition (***v***). Then, under condition (***ii***), the first three rows of the matrix from equation (67) are linearly independent, and as can be seen from (63), (64), its other two rows can be expressed as linear combinations of them. Similarly, in the case of the existence of the root $\beta \in M_{22}$ specified in condition (***v***), it is easy to find out that the matrix from equation (68) has rank three with its linearly independent first three rows. The linear dependence of each of the last two rows of this matrix on its first three rows follows from (65) and (66). Denote 5×4-matrices from equations (67) and (68) by K_1 and K_2, respectively. Then the equation

$$K_1 \|x \quad y \quad u \quad v\|^{\top} = \|0 \quad 0 \quad 0 \quad 0 \quad 0\|^{\top}$$

or equation

$$K_2 \|x \quad y \quad u \quad v\|^{\top} = \|0 \quad 0 \quad 0 \quad 0 \quad 0\|^{\top}$$

has solution $\|x \quad y \quad u \quad v\|^{\top} = \|s_{11} \quad s_{33} \quad s_{12} \quad s_{13}\|^{\top}$, where $s_{11}, s_{33} \neq 0$. If the roots α and β specified in condition (***v***) do not exist, then a solution of the equation

$$\left\| \begin{matrix} 1 & -1 & 1 & 0 \\ 1/a_3(\alpha_2) & -1/b_3(\alpha_2) & 1 & 1 \end{matrix} \right\| \|x \quad y \quad u \quad v\|^{\top} = \|0 \quad 0\|^{\top},$$

has the same property (i.e. its first two components are non-zero). Therefore, in each case for the components of solution $\|s_{11} \quad s_{33} \quad s_{12} \quad s_{13}\|^{\top}$ we have

$$s_{11} - s_{33} + s_{12} = 0,$$

$$s_{11}b_3(\alpha_k) - s_{33}a_3(\alpha_k) + b_3(\alpha_k)(s_{12}a_1(\alpha_k) + s_{13}a_3(\alpha_k)) = 0, \tag{69}$$

$$s_{11}\delta_B(\beta_\ell) - s_{33}\delta_A(\beta_\ell) - s_{13}\delta_A(\beta_\ell)\delta_B(\beta_\ell) = 0, \tag{70}$$

where α_k and β_ℓ are arbitrary elements of M_{12} and M_{22}, respectively. From (69), given condition (***ii***), we can proceed to equality

$$s_{11}b_1(\alpha_k) - s_{33}a_1(\alpha_k) + b_1(\alpha_k)(s_{12}a_1(\alpha_k) + s_{13}a_3(\alpha_k)) = 0,$$

and then to congruence

$$s_{11}b_1(x) - s_{33}a_1(x) + b_1(x)(s_{12}a_1(x) + s_{13}a_3(x)) \equiv 0 \pmod{\varphi_1(x)}. \tag{71}$$

Here we use the fact that for every $\alpha_i \in M_{11}$ polynomials $a_1(x)$, $b_1(x)$ simultaneously acquire the value 0 or 1 and, in addition, $a_3(\alpha_i) = 0$. Recall that for every $\beta_i \in M_{21}$ and for every $\beta_\ell \in M_{22}$ we have $a_2'(\beta_i) = b_2'(\beta_i)$ (see condition (***i***)), $\delta_A(\beta_i) = \delta_B(\beta_i) = 0$ and $\delta_A(\beta_\ell) = a_2'(\beta_i)$, $\delta_B(\beta_\ell) = b_2'(\beta_i)$, respectively. Moreover, if we take into account that $s_{11} = s_{33} - s_{12}$, then from (70) we get the following congruences:

$$s_{11}\delta_B(x) - s_{33}\delta_A(x) - s_{13}\delta_A(x)\delta_B(x) \equiv 0 \pmod{\varphi_{12}(x)},$$

$$s_{33}b_2'(x) - s_{12}\delta_B(x) - s_{23}a_2'(x) - s_{13}a_2'(x)\delta_B(x) \equiv 0 \pmod{\varphi_{12}(x)}. \tag{72}$$

If we add the first congruence of (72) multiplied by $-a_1(x)$ to the second one, then we get

$$s_{33}a_3(x) + s_{11}\delta_B(x) + s_{12}a_1(x)\delta_B(x) + s_{13}a_3(x)\delta_B(x) \equiv$$
$$\equiv s_{33}a_1(x)b_2'(x) \pmod{\varphi_{12}(x)}$$

or otherwise

$$s_{33}a_3(x) - s_{11}b_3(x) - b_3(x)(s_{12}a_1(x) + s_{13}a_3(x)) \equiv b_2'(x)(s_{33}a_1(x) -$$
$$- s_{11}b_1(x) - b_1(x)(s_{12}a_1(x) + s_{13}a_3(x))) \pmod{\varphi_{12}(x)}.$$

Equality (69) and congruence (71) show that the left and right sides of the last congruence are divisible by $\varphi_1(x)$. Therefore, due to the mutual simplicity of polynomials $\varphi_1(x)$ and $\varphi_{12}(x)$, the congruence

$$s_{33}a_3(x) + s_{11}\delta_B(x) + s_{12}a_1(x)\delta_B(x) + s_{13}a_3(x)\delta_B(x) \equiv$$
$$\equiv s_{33}a_1(x)b_2'(x) \pmod{\varphi_2(x)} \tag{73}$$

is true. Congruences (71)−(73) are evidence of the fulfillment of congruences (6) for the numbers $s_{11}, s_{22} = s_{33}$, $s_{12}, s_{13}, s_{23} = 0$. According to Proposition 1 this means that the matrices $A(x)$ and $B(x)$ are ssk.e. ♦

Theorem 4. *Suppose that for oriented by the same characteristic roots α_0, α_1, α_2 reduced matrices $A(x)$ (1) and $B(x)$ (4), we have $\pi_A(\gamma) = \pi_B(\gamma) = 1$ for every root $\gamma \in M_{11} \cup M_{22}$ and $\tilde{\pi}_A(\lambda) = \tilde{\pi}_B(\lambda) = 1$ for every root $\lambda \in M_{12} \cup M_{21}$ (see (17) and (18)). Matrices $A(x)$, $B(x)$ are ssk.e. if and only if the following conditions are met*:

(***i***) $a_3(\alpha_i) = a_3(\alpha_k) \Leftrightarrow b_3(\alpha_i) = b_3(\alpha_k)$ *for each pair* $\alpha_i, \alpha_k \in M_{12}$;

(***ii***) $\delta_A(\beta_\ell) - a_3(\alpha_2)(\delta_A(\beta_\ell) - 1) = 0 \Leftrightarrow \delta_B(\beta_\ell) - b_3(\alpha_2)(\delta_B(\beta_\ell) - 1) = 0$ *for each root* $\beta_\ell \in M_{22}$;

(***iii***) *if for some root $\alpha \in M_{12}$ we have $a_3(\alpha) \neq a_3(\alpha_2)$ or for some root $\beta \in M_{22}$ we have $\delta_A(\beta) - a_3(\alpha_2)(\delta_A(\beta) - 1) \neq 0$, then*

$$\frac{(a_3(\alpha_k) - a_3(\alpha_2))a_3(\alpha)}{(a_3(\alpha) - a_3(\alpha_2))a_3(\alpha_k)} = \frac{(b_3(\alpha_k) - b_3(\alpha_2))b_3(\alpha)}{(b_3(\alpha) - b_3(\alpha_2))b_3(\alpha_k)} \tag{74}$$

and

$$\frac{(\delta_A(\beta_\ell) - a_3(\alpha_2)(\delta_A(\beta_\ell) - 1))a_3(\alpha)}{(a_3(\alpha) - a_3(\alpha_2))\delta_A(\beta_\ell)} =$$
$$= \frac{(\delta_B(\beta_\ell) - b_3(\alpha_2)(\delta_B(\beta_\ell) - 1))b_3(\alpha)}{(b_3(\alpha) - b_3(\alpha_2))\delta_B(\beta_\ell)} \tag{75}$$

or

$$\frac{(a_3(\alpha_k) - a_3(\alpha_2))\delta_A(\beta)}{(\delta_A(\beta) - a_3(\alpha_2)(\delta_A(\beta) - 1))a_3(\alpha_k)} =$$
$$= \frac{(b_3(\alpha_k) - b_3(\alpha_2))\delta_B(\beta)}{(\delta_B(\beta) - b_3(\alpha_2)(\delta_B(\beta) - 1))b_3(\alpha_k)} \tag{76}$$

and

$$\frac{(\delta_A(\beta_\ell)-a_3(\alpha_2)(\delta_A(\beta_\ell)-1))\delta_A(\beta)}{(\delta_A(\beta)-a_3(\alpha_2)(\delta_A(\beta)-1))\delta_A(\beta_\ell)}=$$

$$=\frac{(\delta_B(\beta_\ell)-b_3(\alpha_2)(\delta_B(\beta_\ell)-1))\delta_B(\beta)}{(\delta_B(\beta)-b_3(\alpha_2)(\delta_B(\beta)-1))\delta_B(\beta_\ell)} \tag{77}$$

for each pair $\alpha_k \in M_{12}$, $\beta_\ell \in M_{22}$.

P r o o f. *Necessity.* For the elements of matrices $A(x)$ and $B(x)$, which are ssk.e., the congruence (11) holds. Substituting $x=\alpha_i$ and $x=\alpha_k$, where α_i and α_k are an arbitrary pair of elements from the set M_{12}, into (11) and excluding s_{12}, s_{13}, s_{23} we obtain

$$s_{11}(1/a_3(\alpha_i)-1/a_3(\alpha_k))-s_{33}(1/b_3(\alpha_i)-1/b_3(\alpha_k))=0.$$

From the last relation, since $s_{11}, s_{33}\neq 0$, it follows condition (***i***). Recall that for any $\alpha_k\in M_{12}$ and $\beta_\ell\in M_{22}$ we have $a_1(\alpha_k)=a_3(\alpha_k)$, $b_1(\alpha_k)=b_3(\alpha_k)$ and $a_2'(\beta_\ell)=\delta_A(\beta_\ell)$, $b_2'(\beta_\ell)=\delta_B(\beta_\ell)$, respectively. Therefore, substituting $x=\alpha_k$, $x=\beta_\ell$ into (12), (15), respectively, we get

$$s_{22}-s_{33}+s_{23}=0, \tag{78}$$

$$s_{11}-s_{22}+s_{12}=0. \tag{79}$$

Also, after substitutions $x=\alpha_2$ into (11) and arbitrary $x=\beta_\ell\in M_{22}$ into (9) we get, respectively,

$$s_{11}/a_3(\alpha_2)-s_{33}/b_3(\alpha_2)+s_{12}+s_{13}=0 \tag{80}$$

and

$$s_{11}/\delta_A(\beta_\ell)-s_{33}/\delta_B(\beta_\ell)-s_{13}+s_{23}=0. \tag{81}$$

Excluding from (78)–(81) terms containing s_{12}, s_{13}, s_{23} we arrive at equality

$$s_{11}\frac{\delta_A(\beta_\ell)-a_3(\alpha_2)(\delta_A(\beta_\ell)-1)}{\delta_A(\beta_\ell)a_3(\alpha_2)}-s_{33}\frac{\delta_B(\beta_\ell)-b_3(\alpha_2)(\delta_B(\beta_\ell)-1)}{\delta_B(\beta_\ell)b_3(\alpha_2)}=0, \tag{82}$$

which implies condition (***ii***).

Let α_k be an arbitrary root of the set M_{12} and $\alpha\in M_{12}$ be a fixed root for which we have $a_3(\alpha)\neq a_3(\alpha_2)$. Substitutions $x=\alpha$, $x=\alpha_k$ into (11) give

$$s_{11}/a_3(\alpha)-s_{33}/b_3(\alpha)+s_{12}+s_{13}=0,$$

$$s_{11}/a_3(\alpha_k)-s_{33}/b_3(\alpha_k)+s_{12}+s_{13}=0,$$

whence together with (80) we get

$$s_{11}\frac{a_3(\alpha_k)-a_3(\alpha_2)}{a_3(\alpha_k)a_3(\alpha_2)}-s_{33}\frac{b_3(\alpha_k)-b_3(\alpha_2)}{b_3(\alpha_k)b_3(\alpha_2)}=0, \tag{83}$$

$$s_{11}\frac{a_3(\alpha)-a_3(\alpha_2)}{a_3(\alpha)a_3(\alpha_2)}-s_{33}\frac{b_3(\alpha)-b_3(\alpha_2)}{b_3(\alpha)b_3(\alpha_2)}=0. \tag{84}$$

Since $a_3(\alpha)\neq a_3(\alpha_2)$ (and according to the proved $b_3(\alpha)\neq b_3(\alpha_2)$) and $s_{11}, s_{33}\neq 0$, equalities (82)–(84) yield (74), (75).

Now let β be the fixed root of M_{22}, for which we have $\delta_A(\beta)-a_3(\alpha_2)\times$ $\times(\delta_A(\beta)-1)\neq 0$ (and $\delta_B(\beta)-b_3(\alpha_2)(\delta_B(\beta)-1)\neq 0$ by the proved). We write

(82) for $\beta_\ell = \beta$:

$$s_{11}\frac{\delta_A(\beta) - a_3(\alpha_2)(\delta_A(\beta) - 1)}{\delta_A(\beta) a_3(\alpha_2)} - s_{33}\frac{\delta_B(\beta) - b_3(\alpha_2)(\delta_B(\beta) - 1)}{\delta_B(\beta) b_3(\alpha_2)} = 0. \qquad (85)$$

From (83) and (85) it follows (76), and from (82) and (85) it follows (77).

Sufficiency. Suppose that for matrix $A(x)$ and for some $\alpha \in M_{12}$, inequality $a_3(\alpha) \neq a_3(\alpha_2)$ holds. Then under condition (***i***), for matrix $B(x)$ we have $b_3(\alpha) \neq b_3(\alpha_2)$. In this case we consider the equation

$$\begin{Vmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 1/a_3(\alpha_2) & 0 & -1/b_3(\alpha_2) & 1 & 1 & 0 \\ 1/a_3(\alpha) & 0 & -1/b_3(\alpha) & 1 & 1 & 0 \\ 1/a_3(\alpha_k) & 0 & -1/b_3(\alpha_k) & 1 & 1 & 0 \\ 1/\delta_A(\beta_\ell) & 0 & -1/\delta_B(\beta_\ell) & 0 & -1 & 1 \end{Vmatrix} \begin{Vmatrix} x \\ y \\ z \\ u \\ v \\ w \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Vmatrix}. \qquad (86)$$

Also, in the case of the existence of $\beta \in M_{22}$ such that $\delta_A(\beta) - a_3(\alpha_2) \times$ $\times (\delta_A(\beta) - 1) \neq 0$ (or $\delta_B(\beta) - b_3(\alpha_2)(\delta_B(\beta) - 1) \neq 0$ by condition (***ii***)), consider the equation

$$\begin{Vmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 1/a_3(\alpha_2) & 0 & -1/b_3(\alpha_2) & 1 & 1 & 0 \\ 1/\delta_A(\beta) & 0 & -1/b_B(\beta) & 0 & -1 & 1 \\ 1/a_3(\alpha_k) & 0 & -1/b_3(\alpha_k) & 1 & 1 & 0 \\ 1/\delta_A(\beta_\ell) & 0 & -1/\delta_B(\beta_\ell) & 0 & -1 & 1 \end{Vmatrix} \begin{Vmatrix} x \\ y \\ z \\ u \\ v \\ w \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Vmatrix}. \qquad (87)$$

Recall that in these equations α_k and β_ℓ are arbitrary roots of M_{12} and M_{22}, respectively. It is easy to see that the first four rows of the 6×6-matrix of equation (86) are linearly independent. As follows from (74), (75), the last two rows of the specified matrix are expressed as linear combinations of the first four rows. On the basis of (76), (77) we get a similar conclusion about the linear independence of the first four rows of the matrix of equation (87). Each of the last two rows of the specified matrix are linearly dependent on these rows. All this gives grounds to assert the existence of a non-zero solution of each of equations (86), (87). Moreover, it is easy to see that these equations have solution $\|x \quad y \quad z \quad u \quad v \quad w\|^\top = \|s_{11} \quad s_{22} \quad s_{33} \quad s_{12} \quad s_{13} \quad s_{23}\|^\top$ with non-zero first three components. Equation

$$\begin{Vmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 1/a_3(\alpha_2) & 0 & -1/b_3(\alpha_2) & 1 & 1 & 0 \end{Vmatrix} \begin{Vmatrix} x \\ y \\ z \\ u \\ v \\ w \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix} \qquad (88)$$

has the same solution in the absence of the roots $\alpha \in M_{12}$ and $\beta \in M_{22}$ specified in condition (***iii***). If we take into account the fifth row of the matrix from (86) (or from (87)), then for the components of the solution

$$\|s_{11} \quad s_{22} \quad s_{33} \quad s_{12} \quad s_{13} \quad s_{23}\|^\top$$

we have

$$s_{33}a_3(\alpha_k) - b_3(\alpha_k)(s_{11} + s_{12}a_1(\alpha_k) + s_{13}a_3(\alpha_k)) = 0, \qquad (89)$$

because there are $a_3(\alpha_k) = a_1(\alpha_k)$. Since $s_{33} = s_{22} + s_{23}$ and $b_3(\alpha_k) = b_1(\alpha_k)$, equality (89) can be written as

$$s_{22}a_1(\alpha_k)+s_{23}a_3(\alpha_k)-b_1(\alpha_k)\big(s_{11}+s_{12}a_1(\alpha_k)+s_{13}a_3(\alpha_k)\big)=0.$$

This equality is obviously fulfilled for each root $\alpha_k \in M_{11}$, because $s_{22}-s_{11}-s_{12}=0$ and for such α_k both values $a_1(\alpha_k)$, $b_1(\alpha_k)$ are simultaneously equal to 0 or 1 and $a_3(\alpha_k)=b_3(\alpha_k)=0$. All this gives grounds to assert that for the component $s_{11}, s_{22}, s_{33}, s_{12}, s_{13}, s_{23}$, $s_{11}s_{22}s_{33}\neq 0$, of equation (86) ((87) or (88)) and the elements of matrices $A(x)$ and $B(x)$, the first congruence of (6) holds. Also, if we take into account the sixth row of the matrix of equation (86) (or (87)), then for the components of the above solution of this equation we can write

$$\delta_A(\beta_\ell)\big(s_{33}+s_{13}\delta_B(\beta_\ell)-s_{23}b_2'(\beta_\ell)\big)-s_{11}\delta_B(\beta_\ell)=0, \tag{90}$$

since $\delta_B(\beta_\ell)=b_2'(\beta_\ell)$. Moreover, taking into account that $\delta_A(\beta_\ell)=a_2'(\beta_\ell)$ and $s_{11}=s_{22}-s_{12}$, we get

$$a_2'(\beta_\ell)\big(s_{33}+s_{13}\delta_B(\beta_\ell)-s_{23}b_2'(\beta_\ell)\big)+s_{12}\delta_B(\beta_\ell)-s_{22}b_2'(\beta_\ell)=0 \tag{91}$$

for each root $\beta_\ell \in M_{22}$. Since $s_{33}=s_{22}+s_{23}$ and polynomials $\delta_A(x)$, $\delta_B(x)$ on the set M_{21} take only zero values, and polynomials $a_2'(x)$, $b_2'(x)$ at each point of this set simultaneously acquire zero or unit values, equality (91), as well as (90), also holds for every $\beta_\ell \in M_{21}$. Thus, on the basis of these equalities we can write the following congruences

$$\delta_A(x)\big(s_{33}+s_{13}\delta_B(x)-s_{23}b_2'(x)\big)-s_{11}\delta_B(x)\equiv 0 \pmod{\varphi_{12}(x)},$$

$$a_2'(x)\big(s_{33}+s_{13}\delta_B(x)-s_{23}b_2'(x)\big)+s_{12}\delta_B(x)-s_{22}b_2'(x)\equiv 0 \pmod{\varphi_{12}(x)}. \tag{92}$$

Subtracting the previous one from the congruence (92) multiplied by $a_1(x)$, we get

$$s_{33}a_3(x)+s_{11}\delta_B(x)+s_{12}a_1(x)\delta_B(x)+s_{13}a_3(x)\delta_B(x)-$$
$$-s_{22}a_1(x)b_2'(x)-s_{23}a_3(x)b_2'(x)\equiv 0 \pmod{\varphi_{12}(x)} \tag{93}$$

or

$$s_{33}a_3(x)-b_3(x)\big(s_{11}+s_{12}a_1(x)+s_{13}a_3(x)\big)\equiv b_2'(x)\big(s_{22}a_1(x)+s_{23}a_3(x)-$$
$$-b_1(x)\big(s_{11}+s_{12}a_1(x)+s_{13}a_3(x)\big)\big) \pmod{\varphi_{12}(x)}$$

in another form. The left and right sides of the last congruence are divided into $\varphi_1(x)$ (see (89) and the first congruence of (6)). Since $(\varphi_1(x), \varphi_{12}(x))=1$, the congruence (93) is performed modulo $\varphi_2(x)=\varphi_1(x)\varphi_{12}(x)$, i.e. the second congruence of (6) is true. The third congruence of (6) coincides with (92). Therefore, according to Proposition 1 matrices $A(x)$ and $B(x)$ are ssk.e. ♦

Corollary 1. *If for matrices* $A(x)$ *and* $B(x)$ *the conditions of one of Theorems* 1 – 4 *are satisfied, then the left transforming matrix* $\|s_{ij}\|_1^3$ *in relation* (7) *has an upper triangular form. In addition, by conditions of Theorem* 1, *all diagonal elements are equal and there are zero elements in positions* $(1,2)$, $(2,3)$. *By conditions of Theorem* 2, *first two diagonal elements of the matrix* $\|s_{ij}\|_1^3$ *are equal and there is a zero element at position* $(1,2)$. *By conditions of Theorem* 3, *the last two diagonal elements of the matrix are equal and there is a zero element at position* $(2,3)$.

Conclusions. This work completes the study of the problem of classification of polynomial 3x3-matrices of simple structure up to semiscalar equivalence. Proposition 1 is technical in nature. The use of Proposition 1 made it possible to significantly shorten the proof of other statements and theorems. Propositions 2, 5 are statements of existence. They establish the reducibility of the matrix from the selected class by transforming the semiscalar equivalence to the oriented by characteristic roots reduced matrix. The method of constructing the latter is provided on the basis of Propositions 2, 5. Propositions 3 and 4 specify the invariants of the oriented by characteristic roots reduced matrix. Theorems 1 – 4 give the necessary and sufficient conditions for the semiscalar equivalence of oriented by the same characteristic roots reduced matrices. Corollary 1 indicates the form of the left transforming matrix during the transition from one reduced matrix to another semiscalarly equivalent reduced matrix. The method of constructing this matrix can be taken from the proofs of Theorems 1 – 4. In particular, by conditions of Theorem 1, the nonzero elements of the left transforming matrix can be found by the solution of equation (43). The obtained results are applicable to the classification problem accurate up to the similarity of pairs of numerical matrices and to determination of the invertible solutions of matrix equations of type $XA(x) - B(x)Y(x) = 0$ over a ring of polynomials with respect to unknown X, $Y(x)$.

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ПРО ТРИКУТНУ ФОРМУ 3 × 3-МАТРИЦІ ПРОСТОЇ СТРУКТУРИ СТОСОВНО НАПІВСКАЛЯРНОЇ ЕКВІВАЛЕНТНОСТІ

Для поліноміальних 3×3 *-матриць простої структури відносно напівскалярної еквівалентності встановлено спеціальну трикутну форму. Вказано метод побудови матриць такої форми. Для матриць такої форми знайдено інваріанти та встановлено необхідні та достатні умови їхньої напівскалярної еквівалентності. Запропоновано метод побудови перетворювальних матриць при переході від однієї матриці спеціальної трикутної форми до іншої.*

Ключові слова: *матриця простої структури, напівскалярна еквівалентність матриць, спеціальна трикутна форма матриць, зведена орієнтована за характеристичними коренями матриця.*

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