UDC 517.547.2

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BELONGING OF LAPLACE – STIELTJES-TYPE INTEGRALS TO CONVERGENCE Φ-CLASS

For a non-negative nondecreasing unbounded right-continuous function F on $[0,+\infty)$, an entire transcendental function $f(z)=\sum_{k=0}^{\infty} f_k z^k$ with $f_k \ge 0$ for all $k \ge 0$ and a non-negative on $[0,+\infty)$ function $a(x)$ the integral $I(r)=\int_0^{\infty} a(x) f(xr)\,dF(x)$ is called a Laplace − Stieltjes-type integral. Suppose that for a positive unbounded on $(-\infty,+\infty)$ function Φ the derivative Φ' is positive, continuously differentiable and increasing to $+\infty$. The conditions under which $\int_{r_0}^{\infty} \frac{\Phi'(r)\ln I(r)}{\Phi^2(r)}\,dr<+\infty$ have been found.

Key words: *Laplace − Stieltjes-type integral, convergence Φ-class.*

Introduction. Let V be a class of non-negative nondecreasing unbounded right-continuous functions F on $[0,+\infty)$ and $f(z)=\sum_{k=0}^{\infty} f_k z^k$ be an entire transcendental function with $f_k \ge 0$ for all $k \ge 0$. Assume that a function $a(x)\ge 0$ on $[0,+\infty)$ is such that the Lebesgue − Stieltjes integral $\int_0^{K} a(x) f(xr)\,dF(x)$ exists for every $r \ge 0$ and every $K \in [0,+\infty)$. The integral

$$I(r)=\int_0^{\infty} a(x) f(xr)\,dF(x), \qquad r \ge 0, \tag{1}$$

is called Laplace − Stieltjes-type integral and is direct generalization of the Laplace − Stieltjes integral

$$I^*(r)=\int_0^{\infty} a(x) e^{xr}\,dF(x). \tag{2}$$

Many authors studied the asymptotic properties of the Laplace − Stieltjes integrals I^* (see, for example, [2, 3, 4, 12–14]). The geometric properties of integrals (2) were studied in [7], and in [10] the conditions for belonging of I^* to the convergence Φ-class were found.

Denote by Ω a class of positive unbounded on $(-\infty,+\infty)$ functions Φ such that the derivatives Φ' are positive, continuously differentiable and increasing to $+\infty$ on $(-\infty,+\infty)$. Let φ denotes the inverse to Φ' function.

Let $\Psi(x)=x-\Phi(x)/\Phi'(x)$ be the function associated with Φ in the sense of Newton. Then Ψ is [6, p. 75] continuously differentiable and increasing to $+\infty$ on $(-\infty,+\infty)$.

Let $\Phi \in \Omega$. We say that integral (1) belongs to the convergence Φ-class if

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$$\int_{r_0}^{\infty} \frac{\Phi'(r)\ln I(r)}{\Phi^2(r)} dr < \infty . \qquad (3)$$

As in [6, p. 21] we say that a positive on $[0,+\infty)$ function $a(x)$ has regular variation with respect to $F \in V$ if there exist $b \geq 0$, $c \geq 0$ and $h > 0$ such that $\int_{x-b}^{x+c} a(t)\,dF(t) \geq ha(x)$ for all $x \geq b$.

For $I(r) = I^*(r)$, in [10] the following theorem is proved.

Theorem 1. *Let* $\Phi \in \Omega$, *the function* $\Phi'(r)/\Phi(r)$ *is nondecreasing on* $[r_0,+\infty)$ *and*

$$0 < h \leq \frac{\Phi''(r)\Phi(r)}{(\Phi'(r))^2} \leq H < +\infty . \qquad (4)$$

Suppose that $F \in V$, $a(x)$ *has regular variation with respect to* F *and*

$$\int_{x_0}^{\infty} \frac{\ln F(x)}{x\Phi'(\Psi(\varphi(x)))} dx < \infty . \qquad (5)$$

In order that integral (2) *belongs to the convergence* Φ *-class it is necessary and in the case where* $v(x) := -(\ln a(x))'$ *is continuous and increasing to* $+\infty$ *on* $[x_0,+\infty)$ *it is sufficient that*

$$\int_{x_0}^{\infty} \frac{dx}{\Phi'\left(\frac{1}{x}\ln\frac{1}{a(x)}\right)} < +\infty . \qquad (6)$$

The asymptotic properties of Laplace – Stieltjes-type integrals were studied in [8, 9, 11]. Here we will continue these studies and find the conditions under which integral (1) belongs to the convergence Φ -class.

1. Main result. For $r \geq 0$, let $\mu(r) = \mu(r, I) = \sup\{a(x)f(xr) : x \geq 0\}$ be the maximum of the integrand in (1) and as in [9], let $\nu(r) = \nu(r, I)$ be the central point of the maximum of the integrand. The following lemmas are proved in [9] and [10].

Lemma 1. *Let* $F \in V$, *the function* a *has regular variation with respect to* $F \in V$ *and* $\Gamma_f(r) := \frac{d\ln f(r)}{d\ln r} = O(r)$ *as* $r \to +\infty$. *Then*

$$\ln \mu(r) \leq (1 + o(1))\ln I(r) + O(r) \qquad as \quad r \to \infty .$$

Lemma 2. *Let* $F \in V$, *the function* f'/f *is nondecreasing and* $\tau := \overline{\lim}_{x\to+\infty} \frac{\ln F(x)}{\Gamma_f(x)} < +\infty$. *Then* $I(r) \leq K(\varepsilon)\mu(r + \tau + \varepsilon)$ *for every* $\varepsilon > 0$, *where* $K(\varepsilon) = \text{const} > 0$.

Lemma 3. *The central point* $\nu(r) \nearrow +\infty$ *as* $r \to \infty$ *and*

$$\ln \mu(r) - \ln \mu(r_0) = \int_{r_0}^{r} \frac{\Gamma_f(x\nu(x))}{x} dx .$$

Moreover, if the function $a(x)$ *is upper semi-continuous, then* $\nu(r) = \max\{x \geq 0 : a(x)f(xr) = \mu(r)\}$ *and* $\mu(r) = a(\nu(r))f(r\nu(r))$ *for each* $r \in [0,+\infty)$.

Using these lemmas, at first we prove the following statements.

Proposition 1. *Let* $F \in V$, *the function* a *has regular variation with respect to* $F \in V$, $\tau < +\infty$ *and* $f'(x)/f(x) \nearrow K < +\infty$ *as* $x \to +\infty$. *If* $\Phi(x + O(1)) = = O(\Phi(x))$ *as* $x \to +\infty$ *and* $\int_{r_0}^{\infty} \frac{r\Phi'(r)}{\Phi^2(r)} dr < +\infty$, *then in order that integral* (1) *belongs to the convergence* Φ *-class, it is necessary and sufficient that*

$$\int_{r_0}^{\infty} \frac{\Phi'(r)\ln\mu(r)}{\Phi^2(r)} dr < +\infty. \tag{7}$$

P r o o f. At first we remark that the condition $f'(x)/f(x) \nearrow K < +\infty$ as $x \to +\infty$ implies the condition $\Gamma_f(r) = O(r)$ as $r \to +\infty$. Therefore, by Lemma 1 for some $K_j > 0$, $j = 1, 2$, we have

$$\int_{r_0}^{\infty} \frac{\Phi'(r)\ln\mu(r)}{\Phi^2(r)} dr \le K_1 \int_{r_0}^{\infty} \frac{\Phi'(r)\ln I(r)}{\Phi^2(r)} dr + K_2 \int_{r_0}^{\infty} \frac{r\Phi'(r)}{\Phi^2(r)} dr,$$

i.e. (3) implies (7).

On the other hand, by Lemma 2 in view of the condition $\Phi(x + O(1)) = = O(\Phi(x))$ as $x \to +\infty$ we have

$$\int_{r_0}^{\infty} \frac{\Phi'(r)\ln I(r)}{\Phi^2(r)} dr \le K_3 + \int_{r_0}^{\infty} \frac{\Phi'(r)\ln\mu(r + \tau + \varepsilon)}{\Phi^2(r)} dr =$$

$$= K_3 + \int_{r_0^*}^{\infty} \frac{\Phi'(r - \tau - \varepsilon)\ln\mu(r)}{\Phi^2(r - \tau - \varepsilon)} dr \le$$

$$\le K_3 + \int_{r_0^*}^{\infty} \frac{\Phi'(r)\ln\mu(r)}{\Phi^2(r)} \left(\frac{\Phi(r)}{\Phi(r - \tau - \varepsilon)}\right)^2 dr \le$$

$$\le K_3 + K_4 \int_{r_0^*}^{\infty} \frac{\Phi'(r)\ln\mu(r)}{\Phi^2(r)} dr,$$

i.e. (7) implies (3). ♦

Proposition 2. *If* $\Gamma_f(r) \asymp r$ *as* $r \to +\infty$, *then* (7) *holds if and only if*

$$\int_{r_0}^{\infty} \Phi_1(r)\, d\nu(r) < +\infty, \qquad \Phi_1(r) = \int_{r}^{\infty} \frac{dx}{\Phi(x)}. \tag{8}$$

P r o o f. By Lemma 3

$$\int_{r_0}^{\infty} \frac{\Phi'(r)\ln\mu(r)}{\Phi^2(r)} dr = -\int_{r_0}^{\infty} \ln\mu(r) d\left(\frac{1}{\Phi(r)}\right) =$$

$$= -\frac{\ln\mu(r)}{\Phi(r)}\Bigg|_{r_0}^{\infty} + \int_{r_0}^{\infty} \frac{d\ln\mu(r)}{\Phi(r)} = -\frac{\ln\mu(r)}{\Phi(r)}\Bigg|_{r_0}^{\infty} + \int_{r_0}^{\infty} \frac{\Gamma_f(r\nu(r))}{r\Phi(r)} dr.$$

If (7) holds, then

$$0 < \frac{\ln\mu(r)}{\Phi(r)} \le \int_{r}^{\infty} \frac{\Phi'(t)\ln\mu(t)}{\Phi^2(t)} dt \to 0, \qquad r \to +\infty.$$

Thus, (7) holds if and only if

$$\int_{r_0}^{\infty} \frac{\Gamma_f(r\nu(r))}{r\Phi(r)} dr < +\infty. \tag{9}$$

Since $\Gamma_f(r) \asymp r$, i.e. $0 < c_1 \le \Gamma_f(r)/r \le c_2 < +\infty$, we have that (9) holds if and only if

$$\int_{r_0}^{\infty} \frac{\nu(r)}{\Phi(r)} dr < +\infty. \tag{10}$$

On the other hand,

$$\int_{r_0}^{\infty} \frac{\nu(r)}{\Phi(r)} dr = -\int_{r_0}^{\infty} \nu(r)\, d\Phi_1(r) = -\nu(r)\Phi_1(r)\Big|_{r_0}^{\infty} + \int_{r_0}^{\infty} \Phi_1(r)\, d\nu(r).$$

From (10) it follows that

$$\nu(r)\Phi_1(r) = \nu(r)\int_{r}^{\infty} \frac{dt}{\Phi(t)} \le \int_{r}^{\infty} \frac{\nu(t)}{\Phi(t)} dt \to 0, \qquad r \to +\infty.$$

Therefore, (8) holds if and only if (10) holds. ♦

Now we suppose that the function

$$w(x) = \frac{1}{x}\Gamma_f^{-1}\left(\frac{d\ln(1/a(x))}{d\ln x}\right)$$

is continuous and increasing to $+\infty$ on $[x_0,+\infty)$. Then $\nu(r)$ is a unique point of the maximum of the function $\ln a(x) + \ln f(rx)$ and the function $\nu(r)$ is continuous and increasing to $+\infty$ on $[r_0,+\infty)$.

To obtain an analog of Theorem 1, we also need the following lemma (see [1] and [5, p. 161]).

Lemma 4. *If $c(x)$ and $b(x)$ are continuous on $(0,+\infty)$ functions, $-\infty \le C < c(x) < B \le +\infty$, $b(x) \searrow b \ge 0$ as $x \to +\infty$ and for a positive function φ on (C,B) the function $\varphi^{1/p}$, $p > 1$, is convex on (C,B), then*

$$\int_0^y b(x)\varphi\left(\frac{1}{x}\int_0^x c(t)dt\right) dx \le \left(\frac{p}{p-1}\right)^p \int_0^y b(x)\varphi(c(x))\, dx, \qquad 0 \le y \le +\infty.$$

Theorem 2. *Let $\Phi \in \Omega$, $\Phi(x + O(1)) = O(\Phi(x))$, $\Phi(x) = O(\Phi'(x))$ as $x \to +\infty$ and $\int_{r_0}^{\infty} \frac{r\Phi'(r)}{\Phi^2(r)} dr < +\infty$. Suppose that $F \in V$, the function $a(x)$ has regular variation with respect to F, $f'(x)/f(x) \nearrow K < +\infty$ as $x \to +\infty$, $\frac{1}{x}\ln\frac{1}{a(x)} \le \frac{1}{x} f^{-1}\left(\frac{1}{a(x)}\right) + c_1$ and $x - c_2 \le \Gamma_f(x) \le c_3 x$ for some $c_j > 0$ and all $x \ge x_0$. Also suppose that $\tau < +\infty$ and the function w is continuous and increasing to $+\infty$ on $[x_0,+\infty)$.*

In order that integral (1) belongs to the convergence Φ-class it is necessary and sufficient that

$$\int_{r_0}^{\infty} \Phi_1\left(\frac{1}{x} f^{-1}\left(\frac{1}{a(x)}\right)\right) dx < +\infty, \qquad \Phi_1(r) = \int_{r}^{\infty} \frac{dx}{\Phi(x)}. \tag{11}$$

P r o o f. Since $a(\nu(r))f(r\nu(r)) = \mu(r) \ge 1$ for $r \ge r_0$, we have $r \ge$ $\ge \frac{1}{\nu(r)} f^{-1}\left(\frac{1}{a(\nu(r))}\right)$ and $\Phi_1(r) \le \Phi_1\left(\frac{1}{\nu(r)} f^{-1}\left(\frac{1}{a(\nu(r))}\right)\right)$. Therefore, (8) holds if

$$\int_{r_0}^{\infty} \Phi_1\left(\frac{1}{\nu(r)} f^{-1}\left(\frac{1}{a(\nu(r))}\right)\right) d\nu(r) < +\infty,$$

i.e. if (11) holds. By Propositions 1 and 2, (11) implies (3). The sufficiency of condition (11) is proved.

Now we prove the necessity of condition (11). Since

$$(\ln a(x) + \ln f(xr))' = \frac{1}{x}\left(\frac{d \ln a(x)}{d \ln x} + \Gamma_f(xr)\right) = 0$$

for $r = w(x)$ and w is continuous and increasing to $+\infty$ function on $[x_0, +\infty)$, we obtain $r = w(\nu(r))$ and from (8) we get $\int_{r_0}^{\infty} \Phi_1(w(\nu(r)))\, d\nu(r) < +\infty$, i.e.

$$\int_{r_0}^{\infty} \Phi_1(w(x))\, dx < +\infty. \tag{12}$$

We choose $c(x) = w(x)$, $b(x) = 1$ and $\varphi(x) = \Phi_1(x)$. Then

$$(\varphi^{1/p}(x))'' = \frac{1}{p}(\Phi_1(x))^{(1/p)-2}\left(-\frac{p-1}{p}(\Phi_1'(x))^2 + \Phi_1(x)\Phi_1''(x)\right) =$$

$$= \frac{(\Phi_1(x))^{(1/p)-2}}{p(\Phi(x))^2}\left(\Phi_1(x)\Phi'(x) - \frac{p-1}{p}\right)$$

and in view of the conditions $\Phi(x + O(1)) = O(\Phi(x))$ and $\Phi(x) = O(\Phi'(x))$ as $x \to +\infty$ we have

$$\Phi_1(x)\,\Phi'(x) \ge \Phi'(x) \int_x^{x+1} \frac{dt}{\Phi(t)} \ge \frac{\Phi'(x)}{\Phi(x+1)} \ge \eta > 0.$$

Therefore, the function $\Phi_1^{1/p}$ is convex for $p > 1$ such that $\eta - \frac{p}{p-1} \ge 0$ and by Lemma 4 in view of (12) we have

$$\int_{x_0}^{\infty} \Phi_1\left(\frac{1}{x}\int_{x_0}^{x} w(t)\, dt\right) dx \le \left(\frac{p}{p-1}\right)^p \int_{x_0}^{+\infty} \Phi_1(w(x))\, dx < +\infty. \tag{13}$$

The condition $\Gamma_f(x) \ge x - c_2$ implies $\Gamma_f^{-1}(x) \le x + c_2$ and therefore,

$$\int_{x_0}^{x} w(t)\, dt = \int_{x_0}^{x} \frac{1}{t}\Gamma_f^{-1}\left(\frac{d \ln(1/a(t))}{d \ln t}\right) dt \le \int_{x_0}^{x}\left(\frac{d \ln(1/a(t))}{dt} + c_2\right) dt =$$

$$= \ln\frac{1}{a(x)} - \ln\frac{1}{a(x_0)} + c_2(\ln x - \ln x_0) \le \ln\frac{1}{a(x)} + c_2 x,$$

i.e. in view of the nonincreasing the function Φ_1 and of the condition

$$\frac{1}{x}\ln\frac{1}{a(x)} \le \frac{1}{x} f^{-1}\left(\frac{1}{a(x)}\right) + c_1,$$

we get

$$\int_{x_0}^{\infty} \Phi_1\left(\frac{1}{x}\int_{x_0}^{x} w(t)\,dt\right)dx \ge \int_{x_0}^{\infty} \Phi_1\left(\frac{1}{x}\ln\frac{1}{a(x)} + c_2\right)dx \ge$$

$$\ge \int_{x_0}^{\infty} \Phi_1\left(\frac{1}{x} f^{-1}\left(\frac{1}{a(x)}\right) + c_1 + c_2\right)dx \ge$$

$$\ge c_4 \int_{x_0}^{\infty} \Phi_1\left(\frac{1}{x} f^{-1}\left(\frac{1}{a(x)}\right)\right)dx\,, \qquad (14)$$

since in view of the condition $\Phi(x + O(1)) = O(\Phi(x))$ as $x \to +\infty$

$$\Phi_1(x + c_1 + c_2) = \int_{x+c_1+c_2}^{\infty} \frac{dt}{\Phi(t)} = \int_{x}^{\infty} \frac{dt}{\Phi(t - (c_1 + c_2))} \ge c_4 \int_{x}^{\infty} \frac{dt}{\Phi(t)} = c_4 \Phi_1(x).$$

From (13) and (14) we obtain (11). The proof of Theorem 2 is complete. ♦

2. Addition. Here we suppose that

$$c_1 p(r) p(x) \le \ln f(xr) \le c_2 p(r) p(x), \qquad (15)$$

where $0 < c_1 \le c_2 < +\infty$ and function p is a continuously differentiable and increasing to $+\infty$ on $[0, +\infty)$, $p(0) = 0$. Then

$$\int_{0}^{\infty} a(x) \exp\{c_1 p(r) p(x)\}\, dF(x) \le I(r) \le \int_{0}^{\infty} a(x) \exp\{c_2 p(r) p(x)\}\, dF(x)\,.$$

Therefore,

$$I(p^{-1}(r/c_2)) \le \int_{0}^{\infty} a(x) \exp\{r p(x)\}\, dF(x) =$$

$$= I^{**}(r) := \int_{0}^{\infty} a(p^{-1}(x)) \exp\{rx\}\, dF(p^{-1}(x)) \le I(p^{-1}(r/c_1))\,. \quad (16)$$

We apply Theorem 1 to the integral $I^{**}(r)$. In the proof of Theorem 1 in [2], the conditions imposed on Φ are used to prove the necessity and sufficiency of condition (6). Condition for regular variation of a function $a(x)$ with respect to $F(x)$ is used only in proof of the necessity of condition (6). In the proof of the sufficiency of condition (6) only condition (5) and a continuous increase of the function $v(x) := -(\ln a(x))'$ are used. Therefore, Theorem 1 implies the following assertion.

Proposition 3. *Let $F \in V$ and the function $\Phi \in \Omega$ satisfies the conditions of Theorem 1. If the function $a(p^{-1}(x))$ has regular variation with respect to $F(p^{-1}(x))$ and*

$$\int_{r_0}^{\infty} \frac{\Phi'(r) \ln I^{**}(r)}{\Phi^2(r)}\, dr < \infty\,, \qquad (17)$$

then

$$\int_{x_0}^{\infty} \frac{dx}{\Phi'\left(\frac{1}{x}\ln\frac{1}{a(p^{-1}(x))}\right)} < +\infty\,. \qquad (18)$$

If the function $-(\ln a(p^{-1}(x)))'$ is continuous and increasing to $+\infty$ on $[x_0, +\infty)$ and

$$\int_{x_0}^{\infty} \frac{\ln F(p^{-1}(x))}{x\Phi'(\Psi(\varphi(x)))} dx < \infty\,, \tag{19}$$

then (18) *implies* (17).

It is easy to check that the function $a(p^{-1}(x))$ has regular variation with respect to $F(p^{-1}(x))$ if $a(x)$ has regular variation with respect to $F(x)$, the function $-(\ln a(p^{-1}(x)))'$ is continuous and increasing if $-(\ln a(x))'/p'(x)$ is continuous and increasing, and (18) holds if and only if

$$\int_{x_0}^{\infty} \frac{p'(x)\,dx}{\Phi'\left(\dfrac{1}{p(x)}\ln\dfrac{1}{a(x)}\right)} < +\infty\,. \tag{20}$$

Therefore, in view of (16) the following theorem is true.

Theorem 3. *Let* $F \in V$, *the function* $\Phi \in \Omega$ *satisfies the conditions of Theorem 1 and condition* (15) *holds. If the function* $a(x)$ *has regular variation with respect to* $F(x)$ *and*

$$\int_{r_0}^{\infty} \frac{\Phi'(r)\ln I(p^{-1}(r/c_1))}{\Phi^2(r)} dr < \infty\,,$$

then (20) *holds. If the function* $-(\ln a(x))'/p'(x)$ *is continuous and increasing to* $+\infty$ *on* $[x_0,+\infty)$ *and the function* F *satisfies condition* (19), *then* (20) *implies*

$$\int_{r_0}^{\infty} \frac{\Phi'(r)\ln I(p^{-1}(r/c_2))}{\Phi^2(r)} dr < \infty\,.$$

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НАЛЕЖНІСТЬ ІНТЕГРАЛІВ ТИПУ ЛАПЛАСА – СТІЛТЬЄСА ДО Φ -КЛАСУ ЗБІЖНОСТІ

Для невід'ємної неспадної необмеженої неперервної справа на $[0,+\infty)$ *функції* F, *цілої трансцендентної функції* $f(z)=\sum_{k=0}^{\infty} f_k z^k$ *з* $f_k \ge 0$ *для всіх* $k \ge 0$ *і невід'ємної на* $[0,+\infty)$ *функції* $a(x)$ *інтеграл* $I(r)=\int_0^{\infty} a(x)f(xr)\,dF(x)$ *називається інтегралом типу Лапласа – Стілтьєса. Припустимо, що для додатної необмеженої на* $(-\infty,+\infty)$ *функції* Φ *похідна* Φ' *є додатною, неперервно диференційовною і зростає до* $+\infty$. *Знайдено умови, за яких* $\int_{r_0}^{\infty} \frac{\Phi'(r)\ln I(r)}{\Phi^2(r)}\,dr < +\infty$.

Ключові слова: *інтеграл типу Лапласа – Стілтьєса,* Φ *-клас збіжності.*

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Received
23.10.23