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ONE PROBLEM OF TORSION OF PIECEWISE HOMOGENEOUS ELASTIC BODIES

By means of method of hybrid integral transform of Legendre-Fourier-Fourier type integral representation of exact analytical solution of the problem of torsion of semi-bounded piecewise homogeneous elastic cylinder is obtained.

Key words: *Legendre equation, Fourier equation, Sturm-Liouville problem, hybrid integral transform, hybrid differential operator, the main solutions.*

Introduction. The problems of the theory of torsion of elastic bodies with different geometric structure are of considerable theoretical and practical interest [1–3]. One of the effective methods for solving such problems in the case of piecewise-homogeneous environments is a method of hybrid integral transforms. The hybrid integral transform of Legendre-Fourier-Fourier type is constructed in this paper, and this transform is applied for solving the problem of torsion of semi-bounded piecewise homogeneous elastic cylinder with different physical and mechanical characteristics.

Formulation of the problem. Let's consider a semi-bounded piecewise homogeneous elastic cylinder with radius R, which is composed of different materials. Physical and mechanical properties of this cylinder are changed according to the law

$$G(z) = G_1 shz\theta(z)\theta(l_1 - z) + G_2\theta(z - l_1)\theta(l_2 - z) + G_3\theta(z - l_2),$$

$$G_j = const;\ j = \overline{1,3},$$

here $\theta(x)$ is the Heaviside step function.

We consider inhomogeneous areas of cylinder be soldered together, and the bottom end $z = 0$ is free from stress. We consider that the movement is limited if $z = +\infty$, and lateral surface of the cylinder is loaded efforts $f(z)$.

The problem of torsion of such cylinder mathematically is reduced to a construction bounded on the set

$$D = \{(r,z) : r \in (0,R); z \in (0,l_1) \cup (l_1,l_2) \cup (l_2,+\infty)\}$$

solution of differential separate system of partial differential equations [1]

$$\left(B_1 + \Lambda_0 - \frac{1}{4}\right)u_1(r,z) = -F_1(r,z), \quad z \in (0,l_1),$$

$$\left(B_1+\frac{\partial^2}{\partial z^2}\right)u_2(r,z)=-F_2(r,z),\ z\in(l_1,l_2), \tag{1}$$

$$\left(B_1+\frac{\partial^2}{\partial z^2}\right)u_3(r,z)=-F_3(r,z),\quad z\in(l_2,+\infty),$$

with boundary conditions

$$\left.\frac{\partial u_1}{\partial z}\right|_{z=0}=0,\ \left.\frac{\partial u_j}{\partial r}\right|_{r=0}=0,\ \left.\left(\frac{\partial u_j}{\partial r}-\frac{1}{r}u_j\right)\right|_{r=R}=\frac{f(z)}{G_j(z)},\quad j=\overline{1,3}, \tag{2}$$

and conditions of mechanical contact

$$\begin{cases}\left.(u_1-u_2)\right|_{z=l_1}=0,\\ \left.\left(G_1 shz\frac{\partial u_1}{\partial z}-G_2\frac{\partial u_2}{\partial z}\right)\right|_{z=l_1}=0,\end{cases}\quad \begin{cases}\left.(u_2-u_3)\right|_{z=l_2}=0,\\ \left.\left(G_2\frac{\partial u_2}{\partial z}-G_3\frac{\partial u_3}{\partial z}\right)\right|_{z=l_2}=0,\end{cases} \tag{3}$$

here $B_1=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}$ is Bessel operator, $\Lambda_0=\frac{\partial^2}{\partial z^2}+cthz\frac{\partial}{\partial z}+\frac{1}{4}$ is Legendre operator.

The main part. Let's construct the exact analytical solution of the boundary value problem of conjugation (1)–(3) by the method of hybrid integral transform of Legendre-Fourier-Fourier type.

1. The hybrid integral transform of Legendre-Fourier-Fourier type. Let's consider the singular spectral Sturm-Liouville problem of the structure of solution, which is limited on the set

$$I_2^+=\{r:r\in(0,R_1)\cup(R_1,R_2)\cup(R_2,+\infty)\}$$

of separate system of ordinary differential Legendre and Fourier equations of the 2-nd order

$$L_1[V_1]\equiv\left(\Lambda_\mu+b_1^2a_1^{-2}\right)V_1(r)=0,\ r\in(0,R_1), \tag{4}$$

$$L_m[V_m]\equiv\left(\frac{d^2}{dr^2}+b_m^2a_m^{-2}\right)V_m(r)=0,\quad r\in(R_{m-1},R_m);\quad m=2,3;\quad R_3=+\infty$$

with the conjugate conditions

$$\left.\left[\left(\alpha_{j1}^k\frac{d}{dr}+\beta_{j1}^k\right)V_k(r)-\left(\alpha_{j2}^k\frac{d}{dr}+\beta_{j2}^k\right)V_{k+1}(r)\right]\right|_{r=R_k}=0;\quad j,k=1,2, \tag{5}$$

here $a_j>0$; $\alpha_{jk}^m\ge0$; $\beta_{jk}^m\ge0$; $b_j=\left(\lambda^2+\gamma_j^2\right)^{1/2}$; $\gamma_j^2\ge0$; $c_{jk}=\alpha_{2j}^k\beta_{1j}^k-$ $-\alpha_{1j}^k\beta_{2j}^k\ne0$; $\Lambda_\mu=\frac{d^2}{dr^2}+cthr\frac{d}{dr}+\frac{1}{4}-\frac{\mu^2}{sh^2r}$; $\mu>-\frac{1}{2}$; Λ_μ is Legendre operator [4].

The fundamental system of solutions for the equation $L_1[V_1]=0$ is formed by attached Legendre functions $P^{\mu}_{-\frac{1}{2}+iq_1}(chr)$ and $L^{\mu}_{-\frac{1}{2}+iq_1}(chr)$ [4], and for equation $L_m[V_m]=0$ — by trigonometric functions $\cos q_m r$ and $\sin q_m r$ [5]; $q_j = a_j^{-1} b_j(\lambda^2)$.

It is directly verify that functions

$$V_{\mu,1}(r,\lambda) = c_{21}c_{22}q_2(\lambda)q_3(\lambda)P^{\mu}_{-\frac{1}{2}+iq_1}(chr),$$

$$V_{\mu,2}(r,\lambda) = c_{22}q_3(\lambda)\left[Z^{11,\mu}_{-\frac{1}{2}+iq_1;11}(chR_1)\varphi^1_{22}(q_2R_1,q_2r) - \right.$$
$$\left. -Z^{11,\mu}_{-\frac{1}{2}+iq_1;21}(chR_1)\varphi^1_{12}(q_2R_1,q_2r)\right], \qquad (6)$$

$$V_{\mu,3}(r,\lambda) = \omega_{\mu,2}(\lambda)\cos q_3 r - \omega_{\mu,1}(\lambda)\sin q_3 r$$

are the solution of the boundary value problem (4), (5).

We use such denotation in equalities (6):

$$\omega_{\mu,j}(\lambda) = v^{2j}_{22}(q_3R_2)\left[Z^{11,\mu}_{-\frac{1}{2}+iq_1;21}(chR_1)\delta_{11}(q_2R_1,q_2R_2) - \delta_{21}(q_2R_1,q_2R_2)\times\right.$$
$$\left.\times Z^{11,\mu}_{-\frac{1}{2}+iq_1;11}(chR_1)\right] - v^{2j}_{12}(q_3R_2)\left[\delta_{12}(q_2R_1,q_2R_2)Z^{11,\mu}_{-\frac{1}{2}+iq_1;21}(chR_1) - \right. \qquad (7)$$
$$\left. -\delta_{22}(q_2R_1,q_2R_2)Z^{11,\mu}_{-\frac{1}{2}+iq_1;11}(chR_1)\right], \quad j=1,2;$$

$$\delta_{jk}(x,y) = v^{11}_{j2}(x)v^{22}_{k1}(y) - v^{12}_{j2}(x)v^{21}_{k1}(y), \quad j,k=1,2;$$

$$v^{k1}_{mj}(q_sR_k) = -\alpha^k_{mj}q_s\sin q_sR_k + \beta^k_{jm}\sin q_sR_k;$$

$$v^{k2}_{mj}(q_sR_k) = \alpha^k_{mj}q_s\sin q_sR_k + \beta^k_{jm}\sin q_sR_k;$$

$$Z^{1,\mu}_{\nu 1,j1}(chq_1R_1) = \alpha^1_{j1}shR_1\cdot P^{\mu'}_{\nu 1}(chR_1) + \beta^1_{j1}P^{\mu}_{\nu 1}(chR_1), \ \ \nu_1 = -\frac{1}{2}+iq_1;$$

bar means the derivative of the argument.

Let's define values and functions:

$$\sigma_1 = \frac{c_{11}c_{12}}{c_{21}c_{22}}\frac{a_1^{-2}}{shR_1}, \ \sigma_2 = \frac{c_{12}}{c_{22}}a_2^{-2}, \ \sigma_3 = a_3^{-2};$$

$$+V_{\mu,3}(r,\lambda)\theta(r-R_2);\ \Omega_\mu(\lambda)=\lambda q_3^{-1}([\omega_{\mu,1}(\lambda)]^2+[\omega_{\mu,2}(\lambda)]^2)^{-1},$$

$$\sigma(r)=\sigma_1 shr\theta(r)\theta(R_1-r)+\sigma_2\theta(r-R_1)\theta(R_2-r)+\sigma_3\theta(r-R_2). \quad (8)$$

Theorem 1. If the function

$$g(r)=f(r)[\sqrt{shr}\theta(r)\theta(R_1-r)+\theta(r-R_1)\theta(R_2-r)+\theta(r-R_2)]$$

is piecewise continuous, absolutely summable and has bounded variation in the interval $(0;+\infty)$, then for $r\in I_2^+$ integral representation is true

$$\frac{1}{2}\left[f(r-0)+f(r+0)\right]=\frac{2}{\pi}\int_0^\infty V_\mu(r,\lambda)\Omega_\mu(\lambda)d\lambda\int_0^\infty f(\rho)V_\mu(\rho,\lambda)\sigma(\rho)d\rho. \quad (9)$$

Proof. Functions $V_{\mu,j}(r,\lambda)$ and $V_{\mu,j}(r,\beta)$ are the solutions of differential equations

$$\begin{cases}\left[\Lambda_\mu+a_1^{-2}\left(\lambda^2+\gamma_1^2\right)\right]V_{\mu,1}(r,\lambda)=0,\\ \left[\Lambda_\mu+a_1^{-2}\left(\beta^2+\gamma_1^2\right)\right]V_{\mu,1}(r,\beta)=0;\end{cases} \quad (10)\text{-}(11)$$

$$\begin{cases}\left[\frac{d^2}{dr^2}+a_j^{-2}\left(\lambda^2+\gamma_j^2\right)\right]V_{\mu,j}(r,\lambda)=0,\\ \left[\frac{d^2}{dr^2}+a_j^{-2}\left(\beta^2+\gamma_j^2\right)\right]V_{\mu,j}(r,\beta)=0, \quad j=2,3.\end{cases} \quad (12)\text{-}(13)$$

Let's multiply the equality (10) on the function $V_{\mu,1}(r,\beta)shr$, and equality (11) — on the function $V_{\mu,1}(r,\lambda)shr$ and subtract second from the first. We obtain:

$$V_{\mu,1}(r,\lambda)V_{\mu,1}(r,\beta)shr=$$
$$=\frac{a_1^2}{\lambda^2-\beta^2}\frac{d}{dr}\left[shr\left(V_{\mu,1}(r,\lambda)\frac{dV_{\mu,1}(r,\beta)}{dr}-V_{\mu,1}(r,\beta)\frac{dV_{\mu,1}(r,\lambda)}{dr}\right)\right]. \quad (14)$$

Let's multiply the equality (12) on the function $V_{\mu,j}(r,\beta)$, and equality (13) — on the function $V_{\mu,j}(r,\lambda)$ and subtract second from first. We obtain:

$$V_{\mu,j}(r,\lambda)V_{\mu,j}(r,\beta)=\frac{a_j^2}{\lambda^2-\beta^2}\times$$
$$\times\frac{d}{dr}\left[V_{\mu,j}(r,\lambda)\frac{dV_{\mu,j}(r,\beta)}{dr}-V_{\mu,j}(r,\beta)\frac{dV_{\mu,j}(r,\lambda)}{dr}\right]. \quad (15)$$

Let's set a fairly large number $R>R_2$. Let's multiply the equality (14) on $\sigma_1 dr$ and integrate from 0 to R_1, and equality (15) let's multiply on $\sigma_j dr$ and integrate from R_j to R_{j+1} $(j=1,2;\ R_3=+\infty)$. At the result of adding the integrals we have, that

$$\int_0^R V_\mu(r,\lambda)V_\mu(r,\beta)\sigma(r) = \frac{1}{\lambda^2-\beta^2}\left[V_{\mu,j}(r,\lambda)\frac{d}{dr}V_{\mu,3}(r,\beta)-\right. \\ \left.\left.-V_{\mu,3}(r,\beta)\frac{d}{dr}V_{\mu,j}(r,\lambda)\right]\right|_{r=R}. \tag{16}$$

Let's calculate the double integral

$$I=\frac{2}{\pi}\int_0^\infty\int_c^d g(\lambda)V_\mu(r,\lambda)\Omega_\mu(\lambda)d\lambda V_\mu(r,\beta)\sigma(r)dr \tag{17}$$

for arbitrary positive numbers c and d $(c<d)$ and arbitrary finite function $g(\lambda)$, which is defined on the segment $[c,d]$.

Due to the equation (16) double integral (17) can be rewritten as:

$$I=\lim_{R\to\infty}\frac{2}{\pi}\int_c^d\frac{g(\lambda)}{\lambda^2-\beta^2}\left[V_{\mu,3}(R,\lambda)\frac{d}{dr}V_{\mu,3}(R,\beta)-\right. \\ \left.-V_{\mu,3}(R,\beta)\frac{d}{dr}V_{\mu,3}(R,\lambda)\right]\Omega_\mu(\lambda)d\lambda. \tag{18}$$

As a result of elementary transformations we obtain that

$$2\left[V_{\mu,3}(R,\lambda)\frac{d}{dr}V_{\mu,3}(R,\beta)-V_{\mu,3}(R,\beta)\frac{d}{dr}V_{\mu,3}(R,\lambda)\right]=\left[q_3(\lambda)-q_3(\beta)\right]\times \\ \times\left\{\omega_{\mu,2}(\lambda)\omega_{\mu,2}(\beta)-\omega_{\mu,1}(\lambda)\omega_{\mu,1}(\beta)\right\}\sin R\left[q_3(\lambda)+q_3(\beta)\right]+\left[q_3(\lambda)+q_3(\beta)\right]\times \\ \times\left\{\omega_{\mu,1}(\lambda)\omega_{\mu,1}(\beta)+\omega_{\mu,2}(\lambda)\omega_{\mu,2}(\beta)\right\}\sin R\left[q_3(\lambda)-q_3(\beta)\right]+ \\ +\left[q_3(\lambda)-q_3(\beta)\right]\times\left\{\omega_{\mu,1}(\lambda)\omega_{\mu,2}(\beta)+\omega_{\mu,1}(\beta)\omega_{\mu,2}(\lambda)\right\}\cos R\left[q_3(\lambda)+q_3(\beta)\right]+ \\ +\left[q_3(\lambda)+q_3(\beta)\right]\times\left\{\omega_{\mu,1}(\lambda)\omega_{\mu,2}(\beta)-\omega_{\mu,1}(\beta)\omega_{\mu,2}(\lambda)\right\}\cos R\left[q_3(\lambda)-q_3(\beta)\right]. \tag{19}$$

If to assume that the function $g(\lambda)$ is continuous, absolutely integrable and has bounded variation on $[c,d]$, then substituting (19) into (18), with further using Dirichlet and Riemann lemmas [6] leads to the equality

$$I\equiv\frac{2}{\pi}\int_0^\infty\int_c^d g(\lambda)V_\mu(r,\lambda)\Omega_\mu(\lambda)d\lambda V_\mu(r,\beta)\sigma(r)dr=\begin{cases} g(\beta), & \beta\in[c,d]; \\ 0, & \beta\notin[c,d]. \end{cases} \tag{20}$$

If the function $g(\lambda)$ has properties on the interval $(0,+\infty)$, which discussed above, then we obtain that

$$\frac{2}{\pi}\int_0^\infty\int_0^\infty g(\lambda)V_\mu(r,\lambda)\Omega_\mu(\lambda)d\lambda V_\mu(r,\beta)\sigma(r)dr=\begin{cases} g(\beta), & \beta\in[c,d]; \\ 0, & \beta\notin[c,d]. \end{cases} \quad (21)$$

Let now the function

$$f(r)=\frac{2}{\pi}\int_0^\infty g(\lambda)V_\mu(r,\lambda)\Omega_\mu(\lambda)d\lambda. \quad (22)$$

Let's multiply the equality (22) on $V_\mu(r,\beta)\sigma(r)dr$, where β is arbitrary positive number and integrate by r from $r=0$ to $r=+\infty$. Due to equation (21) we have that

$$\int_0^\infty f(r)V_\mu(r,\beta)\sigma(r)dr=g(\beta). \quad (23)$$

Let's substitute the function $g(\lambda)=\int_0^\infty f(\rho)V_\mu(\rho,\lambda)\sigma(\rho)d\rho$ to equality (22). We obtain the integral representation

$$f(r)=\frac{2}{\pi}\int_0^\infty V_\mu(r,\lambda)\Omega_\mu(\lambda)d\lambda\int_0^\infty f(\rho)V_\mu(\rho,\lambda)\sigma(\rho)d\rho. \quad (24)$$

Rejection from continuity of the function $f(r)$ in the point r leads to the integral representation (9). **The theorem is proved.**

The integral representation (9) defines the direct

$$H_{\mu;2}[f(r)]=\int_0^\infty f(r)V_\mu(r,\lambda)\sigma(r)dr\equiv\tilde{f}(\lambda) \quad (25)$$

and inverse

$$H^{-1}_{\mu;2}[\tilde{f}(\lambda)]=\frac{2}{\pi}\int_0^\infty \tilde{f}(\lambda)V_\mu(r,\lambda)\Omega_\mu(\lambda)d\lambda\equiv\frac{1}{2}\left[f(r-0)+f(r+0)\right] \quad (26)$$

hybrid integral transform of Legendre-Fourier-Fourier type.

Algebra of hybrid differential operator

$$M_\mu=a_1^2\theta(r)\theta(R_1-r)\Lambda_\mu+a_2^2\theta(r-R_1)\theta(R_2-r)\frac{d}{dr^2}+a_3^2\theta(r-R_2)\frac{d^2}{dr^2}$$

can be constructed due to the main identity.

Theorem 2. If the function $f(r)$ is a twice continuously differentiable on the set I_2^+, satisfies the conjugation conditions and conditions of the limited

$$\lim_{r\to\infty}\left[shr\left(\frac{df}{dr}V_{\mu,1}(r,\lambda)-f(r)\frac{d}{dr}V_{\mu,1}(r,\lambda)\right)\right]=0,$$

$$\lim_{r\to\infty}\left(\frac{df}{dr}V_{\mu,3}-f\frac{dV_{\mu,3}}{dr}\right)=0, \tag{27}$$

then the basic identity of integral transform of hybrid differential operator M_μ is true:

$$H_{\mu;2}\left[M_\mu[f(r)]\right]=-\lambda^2\tilde{f}(\lambda)-\sum_{j=1}^{3}\gamma_j^2\int_{R_{j-1}}^{R_j}f(r)V_{\mu,j}(r,\lambda)\sigma_j\varphi_j(r)dr, \tag{28}$$

$$R_0=0,\ R_3=+\infty;\ \varphi_1(r)=shr;\ \varphi_2(r)=\varphi_3(r)=1.$$

Proof. Let's define the values:

$$f^-(R_k)=\lim_{r\to R_k-0}f(r),\ f^+(R_k)=\lim_{r\to R_k+0}f(r);$$

$$\alpha_{11}^k=\alpha_{11}^k\alpha_{22}^k-\alpha_{21}^k\alpha_{12}^k,\ \alpha_{12}^k=\alpha_{11}^k\beta_{22}^k-\alpha_{21}^k\beta_{12}^k,$$

$$\alpha_{21}^k=\beta_{11}^k\alpha_{22}^k-\beta_{21}^k\alpha_{12}^k,\ \alpha_{22}^k=\beta_{11}^k\beta_{22}^k-\beta_{12}^k\beta_{21}^k.$$

From the conjugate conditions we find the relations:

$$\begin{aligned}\frac{df^-(R_j)}{dr}&=\frac{1}{c_{1j}}\left[\alpha_{21}^j\frac{df^+}{dr}(R_j)+\alpha_{12}^j f^+(R_j)\right]\\ f^-(R_j)&=-\frac{1}{c_{1j}}\left[\alpha_{11}^j\frac{df^+}{dr}(R_j)+\alpha_{22}^j f^+(R_j)\right],\ j=1,2\end{aligned} \tag{29}$$

The components $V_{\mu,j}(r,\lambda)$ of the spectral function $V_\mu(r,\lambda)$ have the same connections:

$$\begin{aligned}V_{\mu,j}(R_j,\lambda)&=-\frac{1}{c_{1j}}\left[\alpha_{11}^j\frac{dV_{\mu,j+1}(R_j,\lambda)}{dr}+\alpha_{12}^jV_{\mu,j+1}(R_j,\lambda)\right],\\ \frac{dV_{\mu,j}(R_j,\lambda)}{dr}&=\frac{1}{c_{1j}}\left[\alpha_{21}^j\frac{dV_{\mu,j+1}(R_j,\lambda)}{dr}+\alpha_{22}^jV_{\mu,j+1}(R_j,\lambda)\right].\end{aligned} \tag{30}$$

From equations (29) and (30) the identity follows

$$\begin{aligned}&\frac{df^-(R_j)}{dr}V_{\mu,j}(R_j,\lambda)-f^-(R_j)\frac{dV_{\mu,j}(R_j,\lambda)}{dr}=\\ &=\frac{c_{2j}}{c_{1j}}\left[\frac{df^+(R_j)}{dr}V_{\mu,j+1}(R_j,\lambda)-f^+(R_j)\frac{dV_{\mu,j+1}(R_j,\lambda)}{dr}\right],\ j=1,2.\end{aligned} \tag{31}$$

The proof of the theorem is obtained by integration by parts under the integral with following using of the limited conditions (27), identity (31), the properties of functions $V_{\mu,1},V_{\mu,2},V_{\mu,3},f(r)$ and structures of $\sigma_1,\sigma_2,\sigma_3$. **The theorem is proved.**

The identity (28) makes it possible to apply the introduced hybrid integral transform of Legendre-Fourier-Fourier type to the solving of singular problems of mathematical physics of inhomogeneous structures.

2. The solution of the problem (1)–(3). Let's write the system (1) and boundary conditions (2) in matrix form:

$$\begin{bmatrix} (B_1+\Lambda_0-\frac{1}{4})u_1(r,z) \\ (B_1+\frac{\partial^2}{\partial z^2})u_2(r,z) \\ (B_1+\frac{\partial^2}{\partial z^2})u_3(r,z) \end{bmatrix} = -\begin{bmatrix} F_1(r,z) \\ F_2(r,z) \\ F_3(r,z) \end{bmatrix}, \tag{32}$$

$$\frac{\partial}{\partial r}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\Bigg|_{r=0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \left(\frac{\partial}{\partial r}-\frac{1}{r}\right)\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\Bigg|_{r=R} = \begin{bmatrix} f_1(z)(G_1 shz)^{-1} \\ f_2(z)G_2^{-1} \\ f_2(z)G_3^{-1} \end{bmatrix}. \tag{33}$$

Listed by equations (6)–(8) values and functions for this case ($\alpha_{11}^k=\beta_{21}^k=\alpha_{12}^k=\beta_{22}^k=0,\ \ \beta_{11}^k=\beta_{12}^k=1,\ \ k=1,2;\ \ \alpha_{21}^1=G_1 shl_1,\ \ \alpha_{22}^1=$ $=G_2=\alpha_{21}^2,\ \ \alpha_{22}^2=G_3,\ \ \mu=0$) we denote by $\bar{\sigma}_1,\bar{\sigma}_2,\bar{\sigma}_3,\ \ V_{11}(z,\lambda)$, $V_{21}(z,\lambda)$ and $V_{31}(z,\lambda)$. In this case $c_{11}=G_1 shl_1$, $c_{12}=c_{21}=G_2$, $c_{22}=G_3$, $G_i=const$. Spectral density for this case we denote by $\Omega_0(\lambda)$.

Let's represent the integral operator $H_{0;2}$, which acts by the formula (25) as an operator matrix-row

$$H_{0;2}[...]=\left[\int_0^{l_1}...V_{11}(z,\lambda)\bar{\sigma}_1 shzdz\quad \int_{l_1}^{l_2}...V_{21}(z,\lambda)\bar{\sigma}_2 dz\quad \int_{l_2}^{+\infty}...V_{31}(z,\lambda)dz\right]. \tag{34}$$

Let's apply the operator matrix-row (34) to the problem (32), (33) according to matrices multiplication rule. As a result of main identity (28) (when $a_1^2=a_2^2=a_3^2=1,\ \ \gamma_1^2=0,\ \ \gamma_2^2=\gamma_3^2=\frac{1}{4}$) we get a boundary value problem: to construct a limited in the interval $(0,R)$ solution of Bessel equation for modified functions

$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\left(q^2+\frac{1}{r^2}\right)\right]\tilde{u}(r,\lambda)=-\tilde{F}(r,\lambda);\ \ q^2=\lambda^2+\frac{1}{4} \tag{35}$$

with boundary conditions

$$\frac{d\tilde{u}}{dr}\bigg|_{r=0}=0,\ \ \left(\frac{d}{dr}-\frac{1}{r}\right)\tilde{u}\bigg|_{r=R}=\tilde{f}(\lambda). \tag{36}$$

It is possible to verify that the desired solution of the boundary value problem (35), (36) is a function

$$\tilde{u}(r,\lambda)=\tilde{W}(r,\lambda)\tilde{f}(\lambda)+\int_0^R \tilde{E}(r,\rho,\lambda)\tilde{F}(\rho,\lambda)\rho d\rho. \qquad (37)$$

In the formula (37) there are the Green's function

$$\tilde{W}(r,\lambda)=RI_1(qr)(q_0RI_0(qR)-2I_1(qR))^{-1}\equiv\frac{RI_1(qr)}{\Delta_1(\lambda)}$$

and fundamental function

$$\tilde{E}(r,\rho,\lambda)=\frac{1}{\Delta_1(\lambda)}\begin{cases} I_1(qr)[\Delta_2(\lambda)I_1(q\rho)+\Delta_1(\lambda)K_1(q\rho)], & 0<r<\rho<R; \\ I_1(qr)[\Delta_2(\lambda)I_1(qr)+\Delta_1(\lambda)K_1(qr)], & 0<\rho<r<R, \end{cases}$$

here $\Delta_2(\lambda)=qRK_0(qR)+2K_1(qR)$; $I_\nu(x)$, $K_\nu(x)$ are modified Bessel functions of the first and second kind.

For resuming the function $u(r,z)=\{u_1(r,z);u_2(r,z);u_3(r,z)\}$ by its image $\tilde{u}(r,\lambda)$ let’s apply the operator matrix column to the matrix-element $[\tilde{u}(r,\lambda)]$ (function $\tilde{u}(r,\lambda)$ is defined by the formula (37)) according to matrices multiplication rule

$$H_{0;2}[...]=\begin{bmatrix} \frac{2}{\pi}\int_0^\infty ...V_{11}(z,\lambda)\Omega_0(\lambda)d\lambda \\ \frac{2}{\pi}\int_0^\infty ...V_{21}(z,\lambda)\Omega_0(\lambda)d\lambda \\ \frac{2}{\pi}\int_0^\infty ...V_{31}(z,\lambda)\Omega_0(\lambda)d\lambda \end{bmatrix},$$

as the inverse operator of the operator which is defined by (34).

As a result of elementary transformations we obtain unique solution of the *conjugate boundary* value *problem* (1)–(3):

$$u_j(r,z)=\sum_{m=1}^{3}\left[\int_{l_{m-1}}^{l_m} W_{jm}(r,z,\xi)f_m(\xi)\varphi_m(\xi)d\xi+\right.$$

$$\left.+\int_0^R\int_{l_{m-1}}^{l_m} H_{jm}(r,\rho,z,\xi)F_m(\rho,\xi)\bar{\sigma}_m\varphi_m(\xi)d\xi\rho d\rho\right],$$

here $l_0=0$, $l_3=+\infty$, $\varphi_1(z)=shz$, $\varphi_2(z)=\varphi_3(z)=1$, $\bar{\sigma}_1=G_3^{-1}G_1shl_1$, $\bar{\sigma}_2=G_3^{-1}G_2$, $\bar{\sigma}_3=1$, Green's functions

$$W_{jm}(r,z,\xi) = \int_0^\infty \tilde{W}(r,\lambda)V_{j1}(z,\lambda)V_{m1}(\xi,\lambda)\Omega_0(\lambda)d\lambda$$

and the influence functions

$$H_{jm}(r,\rho,z,\xi) = \int_0^\infty \tilde{E}(r,\rho,\lambda)V_{j1}(z,\lambda)V_{m1}(\xi,\lambda)\Omega_0(\lambda)d\lambda$$

of the boundary value problem (1)–(3).

If $f_j(z)$ and $F_j(r,z)$ are given then the position of cylinder which is discussed becomes known.

Conclusion. By means of method of hybrid integral transform of Legendre-Fourier-Fourier type integral representation of solution of the problem of torsion of semi-bounded piecewise homogeneous elastic cylinder is obtained.

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Методом гібридного інтегрального перетворення типу Лежандра-Фур’є-Фур’є одержано інтегральне зображення точного аналітичного розв’язку задачі кручення напівобмеженого кусково-однорідного пружного циліндра.

Ключові слова: *рівняння Лежандра, рівняння Фур’є, задача Штурма-Ліувілля, гібридне інтегральне перетворення, гібридний диференціальний оператор, головні розв’язки.*

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